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MATHEMATICAL PHYSICS

To the Piston Problem

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1. INTRODUCTION

In this paper, we consider the evolution of a dynamical system consisting of an ideal gas in a cylindrical vessel partitioned by a massive movable piston. Special attention is given to the asymptotic behavior as $t \to \infty$, when the system asymptotically approaches a state of statistical equilibrium. We also consider the related problem about a piston attached to one end of an elastic spring in an ideal gas.

Experience shows that, in the end, the piston stops, and the pressures of the gas in both parts of the vessel become equal. From the point of view of classical mechanics, this fact is amazing. If, in the spirit of the Euler field approach, the gas is treated as an ideal continuous medium, then, obviously, in the system under consideration, continuous oscillations must settle. If the gas is modeled by a large (but finite) number of elastically colliding balls, then such a final behavior contradicts, e.g., the Poincaré return theorem. For this reason, the final behavior is usually explained and described by using approximate statistical models in which irreversibility is inherent from the very beginning. For example, in [1–3], the Boltzmann kinetic equation was used, and, in [4–6], probability approaches (including random processes) were suggested. In [1–6], the accent was put on the more nontrivial problem about the asymptotic equalization of temperatures on the left and right of a movable piston.

In this paper, an ideal gas is treated as a collision-free solid medium whose evolution is described by the classical Liouville equation. This assumption completely agrees with the general statistical approach of Gibbs based on the introduction of continual ensembles of noninteracting mechanical systems. This assumption makes it possible to rigorously formulate and partially prove statements about the asymptotic behavior of a piston in an ideal gas. For simplicity, we consider a one-dimensional gas, in which particles move along a straight line. In an elastic collision of two equal particles, a simple interchange of their velocities occurs; therefore, a one-dimensional gas can be represented as a continuous medium whose particles constantly elastically collide. Such a medium is the natural limit of a very large system of small equal balls that elastically collide with each other and with the walls of the vessel.

Thus, suppose that a one-dimensional ideal gas is enclosed in a vessel with the form of the unit interval $0 \le x \le 1$ and partitioned by a piston of mass M (see Fig. 1). Let X and $\dot{X}$ be the coordinate and velocity of the piston, respectively. Suppose that $m_1(m_2)$ is the mass of the gas on the left (right) of the piston and $\rho_1(\omega, x)$ $[\rho_2(\omega, x)]$ is the initial distribution density of the gas particles over the coordinates x and velocities ω. Clearly, the functions ρ_1 and ρ_2 are defined for x from the intervals $[0, X]$ and $[X, 1]$.

Let $\rho_1^t(\omega, x)$ and $\rho_2^t(\omega, x)$ be the densities of the gas on the left and right of the piston at time t. They are solutions to the Liouville equation with the initial (at $t = 0$) conditions ρ_1 and ρ_2. Naturally, the movability of the boundary condition of particle reflection at $x = X$ must be taken into account.

Thus, the evolution of the distribution density in the problem under consideration is described as follows:

(i) until no collisions occur [i.e., $x - \omega t \in (0, X)$], we have

$$\rho_1^t(\omega, x) = \rho_1(\omega, x - \omega t); \quad (1)$$

(ii) if $x - \omega t = 0$, then ω is replaced by $-\omega$ and $x - \omega t$ is replaced by $x + \omega t$;

(iii) if $x - \omega t = X$, then ω is replaced by $-\omega + \dot{X}$ (this corresponds to an elastic collision of a particle of zero

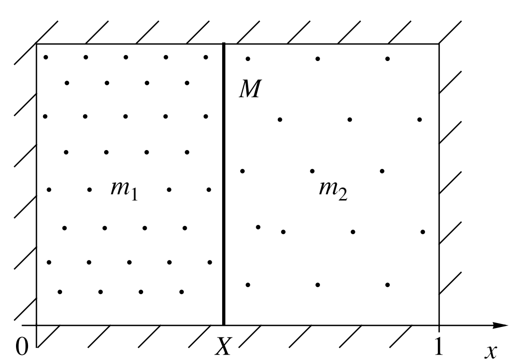

Fig. 1. A piston in a gas.

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mass and the moving slab) and $x - \omega t$ is replaced by $x - (-\omega + \dot{X})t$; the velocity $\dot{X}$ is calculated at the moment of collision.

The evolution of the distribution density ρ_2^t is similar.

The integro-differential equation describing the motion of the piston is

$$M\ddot{X} = 2m_1 \int_{\dot{X}}^{+\infty} (\omega - \dot{X})^2 \rho_1^t(\omega, X)\,d\omega - 2m_2 \int_{-\infty}^{\dot{X}} (\omega - \dot{X})^2 \rho_2^t(\omega, X)\,d\omega. \quad (2)$$

The right-hand side of this formula is the difference between the pressures on the piston moving with velocity $\dot{X}$ of the gas on the two sides of the piston. The formula for the pressure of an ideal gas on the wall is quite classical; it is derived in, e.g., [7]. Thus, Eqs. (1) and (2) (supplemented by the equations for ρ_1^t and ρ_2^t) completely describe the evolution of the system under consideration.

Remark. An equation similar to (2) was obtained in [8]. The heuristic derivation of this equation uses the BBGKY chain of equations. We emphasize that, in the framework of the approach used in this paper, equation (2) is exact.

2. A PISTON ATTACHED TO A SPRING

First, consider the similar but simpler problem about a piston attached to an elastic spring (see Fig. 2). Its equation of motion has the form

$$M\ddot{X} + kX = -2m \int_{-\infty}^{\dot{X}} (\omega - \dot{X})^2 \rho^t(\omega, X)\,d\omega. \quad (3)$$

The right-hand side is the pressure of the gas on the piston at time t; $\rho^t(\omega, x)$, where $X \le x \le 1$, is the distribution density of the gas particles; m is the mass of the gas; and k is the elasticity coefficient of the spring. The density ρ^t is determined from the Liouville equation for a billiard with a moving left boundary, which is similar to (1). The initial density must be such that the integral

$$\int_{X(0)}^{1} \int_{-\infty}^{\infty} \omega^2 \rho \, d\omega \, dx$$

converges. Multiplying this integral by $\frac{m}{2}$, we obtain the kinetic energy of the ideal gas.

The sum of the complete mechanical energy of the piston and the kinetic energy of the gas does not change

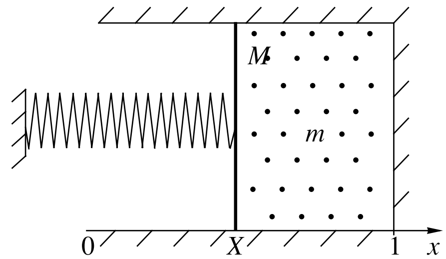

Fig. 2. A piston attached to a spring.

with time. This fact is useful for controlling the accuracy of numerical calculations.

It is easy to show that the stationary states (fixed point) of the system are determined by

$$\rho(\omega, X) = \bar{\rho}(\omega),$$

which is an arbitrary function even with respect to ω,

$$X = \Delta = -\frac{2m}{k} \int_{-\infty}^{0} \omega^2 \bar{\rho}(\omega)\,d\omega = \text{const}. \quad (4)$$

Numerical calculations show that, as $t \to \infty$, the system asymptotically approaches some stationary state of form (4); i.e., the system tends to a statistical equilibrium, and

(i) $X(t) \to \Delta = \text{const}$ as $\dot{X}(t) \to 0$;

(ii) the density ρ^t weakly converges to a stationary density $\bar{\rho}(\omega)$ that is even with respect to the velocity ω.

It was shown in [9] that, for billiards with fixed boundaries, the weak limit of the distribution density always exists and is a first integral of the equation of motion. It is desirable to give a rigorous proof of properties (i) and (ii).

Figure 3 shows the typical form of the phase trajectory of the piston (here, $M = 1$; $k = 1$; $m = 1$; $\rho_0(\omega, x) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2}$; $X(0) = 0$; and $\dot{X}(0) = 0$).

The limit expansion of the spring, which is given by (4), can be written as

$$\Delta = -\frac{2E_+}{kl_+},$$

where E_+ is the energy of the gas in the limit state and $l_+ = 1 - \Delta$ is the distance from the piston to the wall of the vessel (the "volume" of the vessel in the equilibrium state).

Setting $X = \Delta + \xi$, we linearize Eq. (3) and obtain

$$M\ddot{\xi} + \kappa\dot{\xi} + k\xi = 0, \quad (5)$$

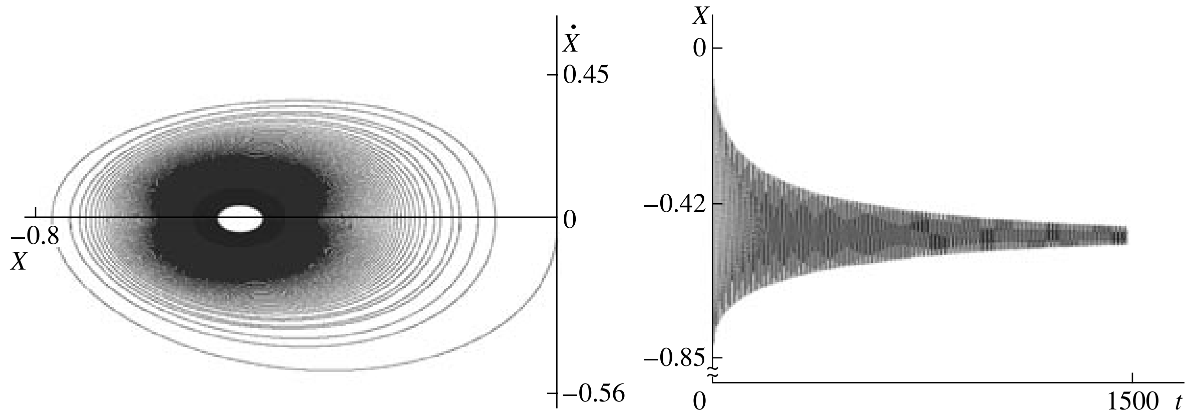

Fig. 3. A piston and a spring.

where

$$\kappa = -4m\int_{-\infty}^{0}\omega\bar{\rho}(\omega)d\omega > 0,$$

because $\bar{\rho}$ is an even nonnegative function with a positive integral over the real line. Equation (5) describes small damped oscillations with viscous friction κ. For example, setting

$$\bar{\rho}(\omega) = \frac{e^{-\omega^2/2\sigma}}{l_+\sqrt{2\pi\sigma}}, \qquad (6)$$

we obtain

$$\kappa = \frac{4m\sqrt{\sigma}}{\sqrt{2\pi}l_+}.$$

For the Maxwell distribution, the variance is proportional to the absolute temperature; hence, the friction coefficient increases with the gas temperature in the equilibrium limit state.

Certainly, the limit distribution $\bar{\rho}$ must not be Maxwellian. However, averaging with an arbitrary even density $\bar{\rho}$ (ω) gives the usual state equations for an ideal gas (see [7]).

3. THE PISTON PROBLEM

Numerical calculations show that, as $t \to \infty$, the gas and the piston also tend to an equilibrium statistical state, i.e., $X(t) \to \bar{X} = \text{const}$ and $\dot{X}(t) \to 0$, and the densities ρ_1^t and ρ_2^t weakly converge to functions $\bar{\rho}_1$ and $\bar{\rho}_2$ depending only on the squared velocity ω^2 of the particles. Figure 4 shows the typical form of the phase trajectory of the piston for this case [here, $M = 1$; $m_1 = 1$; $\rho_0^{(1)} = \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}$; $m_2 = 2$; $\rho_0^{(2)} = \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}$; $X(0) = 0.5$; and $\dot{X}(0) = (0)$].

Setting $\dot{X} \equiv 0$ in Eq. (2) and replacing the densities ρ_1^t and ρ_2^t by their weak limits, we obtain

$$m_1\int_{-\infty}^{\infty}\omega^2\bar{\rho}_1 d\omega = m_2\int_{-\infty}^{\infty}\omega^2\bar{\rho}_2 d\omega. \qquad (7)$$

Thus, the point $x = \bar{X}$ divides the interval [0, 1] with one-dimensional gas into two halves with the same pressures. Equality (7) can be rewritten in the equivalent form

$$\frac{E_1}{l_1} = \frac{E_2}{l_2}, \qquad (8)$$

where E_k ($k = 1, 2$) are kinetic energies of the gases on the left and right of the piston and l_k ($k = 1, 2$) are their "volumes," i.e., the distances from the piston to the endpoints of the interval. The internal energies E_k are proportional to the absolute temperatures. However, equality (8) by no means implies the equality of the temperatures on the left and right of the piston.

If condition (7) holds and the velocity $\dot{X}$ of the piston is small, then equation (2) can be linearized as

$$M\ddot{X} + \kappa\dot{X} = 0, \qquad (9)$$

where

$$\kappa = 4m_1\int_{0}^{\infty}\omega\bar{\rho}_1(\omega)d\omega - 4m_2\int_{-\infty}^{0}\omega\bar{\rho}_2(\omega)d\omega > 0$$

is the effective coefficient of viscous friction. It is seen from (9) that the piston velocity exponentially decreases, and the path length of the piston (until it stops completely) is finite. Interestingly, in a box with

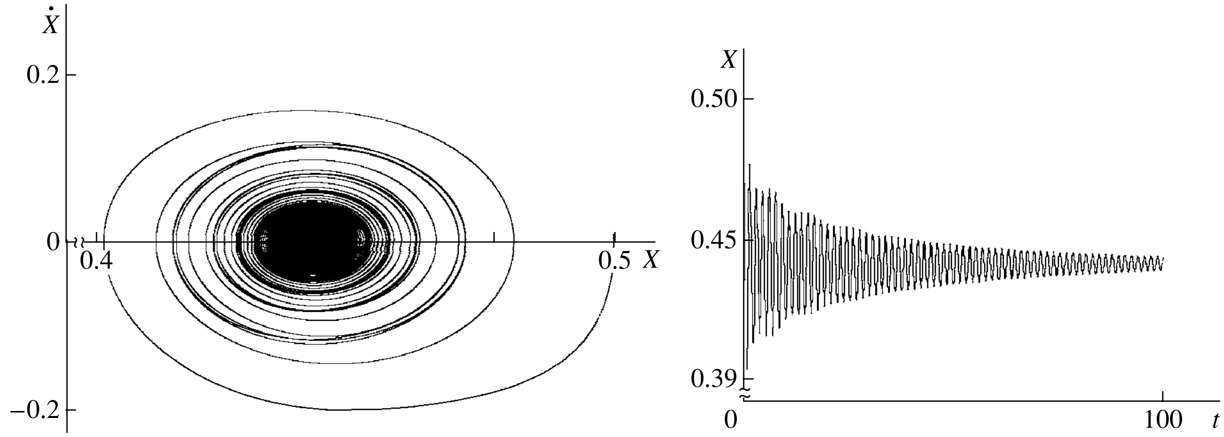

Fig. 4. A piston between two gases.

unmovable walls, the density of a collision-free medium evens even more rapidly [10].

Of special interest is the case where $m_1 = 0$ or $m_2 = 0$, i.e., all the gas on the right or left of the piston. It may seem that, in this case, the piston continuously oscillates, elastically colliding with the lateral wall of the cylindrical vessel. However, in this case, the piston oscillations damp as well, and, as $t \to \infty$, the piston stops at one of the vessel walls. This assertion has not been completely proved as yet. However, it becomes almost obvious under the assumption that $\dot{X}$ is small; it suffices to replace the density ρ^t by its weak limit $\bar{\rho}\,(\omega)$. In this case, the problem reduces to studying the jumps in a viscous medium of a heavy ball elastically colliding with a fixed horizontal slab. Clearly, the altitude of the jumps and the velocity of the ball decrease to zero with time.

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