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MATHEMATICAL PHYSICS

Topological Obstructions to the Existence of Quantum Conservation Laws

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1. TOPOLOGICAL OBSTRUCTIONS

For a number of reasons, it is useful to know whether there exist differential operators commuting with a given Hamiltonian operator. It is assumed that these differential operators are defined on the entire configuration space and are polynomial with respect to differentiations. The point is that all the known commuting operators are polynomials in differentiations (or functions of polynomials).

This problem is especially important because, as is known, Hermitian differential operators commuting with a Hamiltonian operator generate conservation laws for quantum systems. Moreover, the presence of commuting operators provides evidence for the regularity of the behavior of a quantum system. In particular, the condition of the absence of global commuting operators plays an important role in the rigorous substantiation of the canonical Gibbs distribution in quantum mechanics. These questions were considered in [1] from the point of view of classical statistical mechanics.

We consider the existence of nontrivial polynomial differential Hamiltonian operators for a multidimensional quantum system. An operator is said to be nontrivial if cannot be expressed in terms of a Hamiltonian operator. Suppose that M^n is the n-dimensional compact configuration space of a quantum system, $x = (x^1, x^2, \dots, x^n)$ is the vector of local coordinates, and $y = (y_1, y_2, \dots, y_n)$ is the vector of conjugate momenta. Our point of departure is the classical Hamiltonian

$$H = \frac{1}{2}\sum g^{kj}(x)y_k y_j + V(x), \qquad (1)$$

where the first term is kinetic energy and V is potential energy. The coefficients g^{kj} and the potential V are assumed to be smooth (infinitely differentiable) functions. Applying the usual quantization rules, we associate Hamiltonian (1) with the Hamiltonian operator

$$\hat{H} = \frac{\Delta}{2} + \hat{V}, \qquad (2)$$

where

$$\Delta = -\frac{1}{\sqrt{g}}\sum \frac{\partial}{\partial x^j}\left(\sqrt{g}g^{kj}\frac{\partial}{\partial x^k}\right)$$

is the Laplace–Beltrami operator of the Riemannian metric on M determined by the kinetic energy and $\hat{V}$ is the operator of multiplication by the function V. Operator (2) is often corrected by adding a term proportional to the curvature of the Riemannian metric

$$\sum g_{kj}(x)dx^k dx^j$$

(see, e.g., [2]). However, as can be seen from what follows, such a complication of operator (2) is inessential. In any case, this additional term, which does not depend on differentiations, can be included in the potential energy.

We set $\partial = (\partial_1, \partial_2, \dots, \partial_n)$, where $\partial_k = \frac{\partial}{\partial x^k}$, and let $\hat{F}(x, \partial)$ be a polynomial differential operator defined everywhere on M and commuting with the Hamiltonian operator. It is assumed that $\hat{F}$ is smooth with respect to x. We do not assume that the operator $\hat{F}$ is Hermitian.

Taking all the differentiations in the expressions for the operator to the right, we reduce the polynomial differential operator to the form

$$\hat{F} = \sum_{|k| \le N} f_{k_1, \dots, k_n}(x)\partial_1^{k_1}\cdots\partial_n^{k_n},$$

where $k = (k_1, k_2, \dots, k_n)$, $k_j \ge 0$ are integers, and $|k| = k_1 + k_2 + \dots + k_n$.

The number N is called the degree of the polynomial operator and is denoted by $\deg\hat{F}$. We set

$$\mathrm{Symb}\hat{F} = \sum_{|k| \le N} f_{k_1, k_2, \dots, k_n}(x)p_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}.$$

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This homogeneous polynomial in the n auxiliary variables $p_1, p_2, \ldots, p_n$ is called the principal symbol of the operator $\hat{F}$.

It turns out that the presence of nontrivial commuting operators imposes significant constraints on the topology of the configuration manifold. First, we state the corresponding result for systems with two degrees of freedom. In this case, M^2 is a two-dimensional surface; we assume it to be orientable. As is known, the topological structure of such a surface is determined by its genus (the number handles attached to the sphere).

Theorem 1. *If the genus of the surface M^2 is larger than* 1, *then the operator $\hat{F}$ is a polynomial in $\hat{H}$ with constant coefficients.*

In other words, if M is not homeomorphic to the sphere of the torus, then a quantum system admits no nontrivial polynomial differential operators commuting with the Hamiltonian operator. There are simple examples of quantum systems with nontrivial conservation laws whose configuration spaces are homeomorphic to the sphere and the torus. If M is not assumed to be oriented, then we should add the projective plane and the Klein to the sphere and the torus. This can be easily derived from the well known fact that any nonorientable surface admits a nonbranched double covering by orientable surfaces. Theorem 1 is a quantum analog of the theorem on topological obstructions to the integrability of classical mechanical systems given in [3] (see, also, [4]).

Theorem 1 can be generalized to the multidimensional case. Although it is then concerned with the existence of $n = \dim N$ independent polynomial operators commuting with the Hamiltonian operator. Polynomial differential operators are independent if their principal symbols are independent as functions of x and p. To simplify the exposition, we assume that all the objects under consideration are analytic.

Theorem 2. *Suppose that a quantum system admits n independent polynomial differential operators commuting with the Hamiltonian operator. Then*,

$$b_k \le C_n^k, \tag{3}$$

where b_k is the kth Betti number of the manifold M.

This assertion is an analog of Taimanov's theorem [5] on topological obstructions to the complete integrability of classical Hamiltonian systems. For $n = 2$, condition (3) coincides with the condition of Theorem 1.

We say that a polynomial differential operator $\hat{F}$ of minimal degree commuting with $\hat{H}$ and independent of $\hat{H}$ is irreducible (the degree of any other such operator is at least $\deg \hat{F}$). It turns out that not only the existence of a nontrivial commuting operator but also the degree of an irreducible operator is related to the topology of the configuration space. Let us demonstrate this for systems whose configuration spaces are the two-dimensional torus. As is known, any metric on the two-dimensional torus $\mathbb{T}^2$ can be reduced to the form

$$\frac{a\,dx^2 + 2b\,dx\,dy + c\,dy^2}{g(x, y)}, \tag{4}$$

where x, y mod2π are angular coordinates on $\mathbb{T}^2$; a, b, and c are constants (such that $a > 0$ and $b^2 - ac < 0$); and g is a smooth function on $\mathbb{T}^2$ (2π-periodic in x and y). The corresponding Laplace–Beltrami operator in (2) is constructed from Riemannian metric (4).

Theorem 3. *Suppose that g is a trigonometric polynomial and the corresponding quantum system admits an irreducible polynomial differential operator $\hat{F}$ commuting with $\hat{F}$.*

Then, $\deg \hat{F} \le 2$.

This result was obtained for classical systems, in [6, 7]. Apparently, Theorem 3 remains valid in the general case, where g is an arbitrary smooth function on $\mathbb{T}^2$. At the very least, the inequality $\deg \hat{F} \le 2$ holds for a set of quantum systems that is dense everywhere. Possibly, if the configuration space is homeomorphic to the two-dimensional sphere, the degree of an irreducible commuting operator is at most 4.

2. PRINCIPAL SYMBOLS

Suppose that $[\hat{H}, \hat{F}] = \hat{H}\hat{F} - \hat{F}\hat{H}$ is the commutator of polynomial differential operators and

$$\{H, F\} = \sum\left(\frac{\partial F}{\partial x_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial H}{\partial x_j}\right)$$

is the Poisson bracket of functions defined on the cotangent bundle of the configuration manifold M.

Principal symbol lemma. *The following relation holds*:

$$\mathrm{Symb}[\hat{H}, \hat{F}] = \{\mathrm{Symb}\hat{H}, \mathrm{Symb}\hat{F}\}.$$

This assertion can be proved by simple calculations.

Corollary. *If a polynomial differential operator $\hat{F}$ commutes with the Hamiltonian operator $\hat{H}$, then the function* $\mathrm{Symb}\,\hat{F}$ *is a first integral of the classical system with the Hamiltonian function*

$$H = \frac{1}{2}\sum g^{ij}(x)p_i p_j \tag{5}$$

homogeneous in the momenta p.

The system with Hamiltonian (5) describes motion along the geodesic lines of the Riemannian manifold (M, ds), where $ds^2 = \sum g_{jk}(x)dx^j dx^k$. The problem of the existence of nontrivial integrals for such systems has been considered for a long time, starting with the classical works of Birkhoff, Whittaker, and Darboux (see [8]). However, in these works, the local point of view prevailed. The modern setting of the problem, which goes back to Poincaré, is concerned with the existence of integrals defined on the entire phase space of a classical system. The known results are surveyed in [9].

Theorem 2 is a direct corollary to the result of [5] and the principal symbol lemma. Theorems 1 and 3 are proved in a similar way with the use of the results obtained in [3, 6]. Suppose, for example, that $\hat{F}$ is a polynomial differential operator commuting with the Hamiltonian operator of a quantum system on a two-dimensional surface of genus >1. Then, its principal symbol is an integer power of the kinetic energy of the corresponding classical system with some constant multiplier: Symb $\hat{F} = c(\text{Symb}\,\hat{H})^m$, where $m \in N$. The polynomial differential operator $\hat{\Phi} = \hat{F} - c\hat{H}^m$ also commutes with $\hat{H}$, and its degree is smaller than deg $\hat{F}$. Applying this procedure to the operator $\hat{F}$, we reduce the degree of $\hat{\Phi}$ again. As a result, we obtain an operator $\hat{F}$ that is a polynomial in the Hamiltonian $\hat{H}$ with constant coefficients.

A method based on a reduction to equations of geodesic lines gives a general estimate for the possible number of independent polynomial differential operators commuting with the Hamiltonian operator. The Birkhoff number β of a Hamiltonian system with Hamiltonian (5) is defined as the maximum number of functionally independent integrals of this system that have the form of homogeneous polynomials in momenta. This characteristic was introduced in [10]. It was also mentioned in [10] that the addition of potential energy can only decrease the number of polynomial integrals. The Birkhoff number depends on the topological characteristics of the manifold M. In particular, for a two-dimensional oriented surface M of genus g, we have $\beta = 1$ if $g > 1$, $\beta \le 2$ if $g = 1$, and $\beta \le 3$ if $g = 0$.

Theorem 4. *The number of polynomial differential operators with independent principal symbols commuting with Hamiltonian operator* (2) *is, at most,* β.

This simple assertion has important corollaries. One of them is as follows. If a geodesic flow on M is ergodic, then the corresponding quantum system does not admit any nontrivial polynomial differential operators commuting with the Hamiltonian operator. This conclusion is also valid for quantum billiards.

3. SOME RELATED PROBLEMS

Theorems 1–4 give rise to problems concerning polynomial conservation laws for the equations of quantum mechanics. The classical analogs of some of these problems have already been solved.

(a) Is it true that, if there exists a general operator commuting with the Hamiltonian operator $\hat{H}$ but independent of $\hat{H}$, then there exists a nontrivial polynomial differential operator commuting with $\hat{H}$?

(b) Is it true that there exist local irreducible differential operators of arbitrarily high degree commuting with the Hamiltonian operator? In the classical case, the answer is positive (see [9]).

(c) Consider a classical Hamiltonian system with the toroidal configuration space $\mathbb{T}^n = \{x_1, x_2, \ldots, x_n \bmod 2\pi\}$ and Hamiltonian function

$$H = \frac{1}{2}\sum a_{ij}y_iy_j + V(x),$$

where $\|a_{ij}\|$ is a positive definite matrix with constant elements and V is an analytic function on $\mathbb{T}^n$. Is it true that, after a typical quantization, the degree of an irreducible polynomial differential operator commuting with $\hat{H}$ does not exceed 2? For the case where V is a trigonometric polynomial, this assertion was proved recently in [11].

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