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Uniform distribution and Voronoĭ convergence

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Abstract. There is a broad generalization of a uniformly distributed sequence according to Weyl where the frequency of elements of this sequence falling into an interval is defined by using a matrix summation method of a general form. In the present paper conditions for uniform distribution are found in the case where a regular Voronoĭ method is chosen as the summation method. The proofs are based on estimates of trigonometric sums of a certain special type. It is shown that the sequence of the fractional parts of the logarithms of positive integers is not uniformly distributed for any choice of a regular Voronoĭ method.

Bibliography: 6 titles.

§ 1. Voronoĭ means and types of uniform distribution

Let $q_0 > 0$, $q_n \geqslant 0$. For a sequence s_n, $n = 0, 1, \dots$, we set

$$u_n = \frac{q_n s_0 + q_{n-1} s_1 + \dots + q_0 s_n}{q_0 + q_1 + \dots + q_n}.$$

It is clear that if $s_j = s_0$, then $u_n = s_0$. The sequence q_n defines a *Voronoĭ summation method*: if $u_n \to s$ as $n \to \infty$, then by definition

$$s_n \to s \quad (W, q_n).$$

The criterion of *regularity* of a W-method is

$$\frac{q_n}{q_0 + \dots + q_n} \to 0. \tag{1.1}$$

It turns out that any two regular Voronoĭ methods are *compatible*, that is, if $s_n \to s(W)$ and $s_n \to s'(W')$, then $s = s'$.

For $q_0 = q_1 = \cdots$ the Voronoĭ method becomes the Cesàro method. If a W-method is regular and the sequence q_n is non-decreasing, then (W, q_n) *includes* the Cesàro method C: it follows from $s_n \to s(C)$ that $s_n \to s(W, q_n)$. Conditions

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for the reverse inclusion are of greater interest. Let $q_0 = 1$ and $q_n \geqslant 0$. If the W-method is regular, q_n is non-increasing, and

$$\frac{q_{n+1}}{q_n} \geqslant \frac{q_n}{q_{n-1}}$$

for all $n \geqslant 1$, then C includes (W, q_n).

The Voronoĭ summation theory is expounded in detail in [1].

Let x_n, $n = 0, 1, 2, \dots$, be a sequence of points of the unit segment $[0,1]$, $L \subset [0,1]$ an arbitrary segment of length l, and f the characteristic function of the segment L.

Consider some Voronoĭ summation method (W, q_n). We say that a sequence of points x_n is (W, q_n)-*uniformly distributed* mod 1 (for short: (W, q_n)-*ud* or W-*ud*) if for each segment L

$$f(x_n) \to l \quad (W, q_n). \tag{1.2}$$

This definition is different from the classical definition of Weyl in that each event $x_n \in L$ is taken into account with its weight. For $q_0 = q_1 = \cdots$ we obtain the definition of a sequence uniformly distributed according to Weyl. It is possible to give a more general definition of uniform distribution by replacing a Voronoĭ method by a linear summation method (see, for example, [2]).

It is easy to prove that relation (1.2) is valid for all Riemann-integrable functions f, only l must of course be replaced by the integral of f over the segment $0 \leqslant x \leqslant 1$. As in the classical case, the converse of this result is true: if

$$e^{2\pi i m x_n} \to 0 \quad (W, q_n) \tag{1.3}$$

for all integers $m \neq 0$, then the sequence x_n is (W, q_n)-*ud*.

It is clear that every sequence uniformly distributed in the generalized sense is everywhere dense on the segment $[0,1]$. The converse is of course not true. For example, as we shall show in § 4, the sequence $\{\alpha \ln n\}$, $\alpha \neq 0$, is everywhere dense but, generally speaking, is not W-*ud* for any regular W-method. Henceforth we denote by $\{x\}$ the fractional part of a number x. Note also that the regularity condition (1.1) is necessary for the existence of at least one W-*ud* sequence [3]. Apparently, this condition is also sufficient.

§ 2. Conditions for *W*-uniform distribution

Theorem 1. *Suppose that continuously differentiable functions $f(x)$, $g(x)$, $x \geqslant 0$, satisfy the following condition*:

1)
$$f' \neq 0 \quad \textit{and} \quad f'(x) \to 0 \quad \textit{as } x \to \infty;$$

2) *the function g is positive and monotonic*, *and*

$$\int_0^\infty g(x)\,dx = \infty, \tag{2.1}$$

$$\frac{g(x)}{\int_0^x g(t)\,dt} \to 0 \quad \textit{as } x \to \infty; \tag{2.2}$$

3) *the ratio* $g(n-x)/f'(x)$ *is non-decreasing or non-increasing on the segment* $0 \leqslant x \leqslant n$;

4)
$$|f'(x)| \int_0^x g(t)\,dt \to \infty \quad \text{as } x \to \infty.$$

Then the sequence $\{f(n)\}$ *is* $(W, g(n))$*-ud.*

Condition (2.2) is the regularity condition of the Voronoĭ method $(W, g(n))$. From this theorem we can derive several corollaries.

Corollary 1. *Under the hypotheses of Theorem* 1 *the sequence* $\{f(n)\}$ *is everywhere dense on the unit interval.*

Corollary 2 (Fejér's theorem [4]). *If* $f'(x)$ *tends monotonically to zero and* $x|f'(x)| \to \infty$ *as* $x \to \infty$, *then the sequence* $\{f(n)\}$ *is uniformly distributed according to Weyl.*

For the proof one must set $g(x) \equiv 1$.

Corollary 3. *Suppose that* $f' > 0$, $f'(x) \to 0$ *monotonically,* $g(x) > 0$ *is non-increasing, and*
$$f'(x) \int_0^x g(t)\,dt \to \infty \tag{2.3}$$
as $x \to \infty$. *Then* $\{f(n)\}$ *is* $(W, g(n))$*-ud.*

Indeed, condition 1) of Theorem 1 is obviously satisfied. Furthermore, the integral (2.1) diverges, for otherwise condition (2.3) is violated (since $f' \to 0$). Condition (2.2) is also satisfied since g is bounded and property (2.1) holds, as already established. Furthermore, the ratio $g(n-x)/f'(x)$ is non-decreasing on the interval $[0, n]$ as a product of two non-decreasing functions. Finally, condition 4) of Theorem 1 coincides with (2.3).

We set, for example, $g(x) = (x+1)^{-\alpha}$, $0 \leqslant \alpha < 1$. Then condition (2.3) takes the form
$$x^{1-\alpha} f'(x) \to \infty. \tag{2.4}$$
If $f'(x) > 0$ and monotonically tends to zero, then this condition guarantees that the sequence $\{f(n)\}$ is $(W, (n+1)^{-\alpha})$-*ud*. It is easy to verify that the $(W, (n+1)^{-\alpha})$-methods for all $0 \leqslant \alpha < 1$ are included in the Cesàro method. In particular, the sequence $\{f(n)\}$ is uniformly distributed according to Weyl. For $\alpha \to 0$ condition (2.4) becomes Fejér's condition (Corollary 2). We now set $\alpha = 1$. Then (2.4) is replaced by the condition
$$(\ln(x+1)) f'(x) \to \infty.$$
In this case the sequence $\{f(n)\}$ is $(W, 1/(n+1))$-*ud*.

The case where the function g increases is of special interest. Then the regular $(W, g(n))$-method includes the Cesàro method. But here condition (1.3) of Theorem 1, as a rule, is not satisfied.

We indicate one of the possible modifications of Theorem 1. In what follows we assume that the function f is twice continuously differentiable. We set $\varphi(x) = 1/f'(x)$. If we differentiate the product $\varphi(x)g(n-x)$ with respect to x and divide the result by this product, then we obtain

$$\frac{\varphi'(x)}{\varphi(x)} - \frac{g'(n-x)}{g(n-x)}\,. \tag{2.5}$$

Suppose that the functions φ'/φ and g'/g are *monotonically* decreasing to zero as $x \to \infty$. In particular, $f''/f' \to 0$ monotonically. Then for sufficiently large n, in the interval $[0, n]$ the function (2.5) has exactly one zero, which we denote by x_n.

Theorem 2. *Suppose that the following conditions are satisfied*:

1) $f > 0$, $f'(x) \to 0$, *and* $f''(x)/f'(x)$ *monotonically tends to zero as* $x \to \infty$;
2) $g > 0$, g *monotonically increases*, *and* $g'(x)/g(x) \to 0$ *monotonically as* $x \to \infty$;
3)

$$\frac{g(n)}{f'(x_n)\int_0^n g(t)\,dt} \to 0, \qquad \frac{g(x_n)}{f'(n)\int_0^n g(t)\,dt} \to 0 \tag{2.6}$$

as $n \to \infty$.

Then the sequence $\{f(n)\}$ *is* $(W, g(n))$*-ud*.

Corollary 4. *Suppose that conditions* 1), 2) *of Theorem* 2 *are satisfied*, *and*

$$\frac{g(x)}{f'(x)\int_0^x g(t)\,dt} \to 0 \tag{2.7}$$

as $x \to \infty$. *Then* $\{f(n)\}$ *is* $(W, g(n))$*-ud*.

Indeed, taking into account assumptions 1) and 2) we obtain condition (2.6) from (2.7).

Corollary 5. *Suppose that* $f' > 0$, *the functions* $f'(x)$ *and* $f''(x)/f'(x)$ *monotonically tend to zero as* $x \to \infty$, *and*

$$f'\left(\frac{x}{2}\right) f'(x) \int_0^x \frac{dt}{f'(t)} \to \infty. \tag{2.8}$$

Then $\{f(n)\}$ *is* $(W, 1/f'(n))$*-ud*.

It is easy to verify that in this case all the conditions of Theorem 2 are satisfied, and $x_n = n/2$. Here conditions (2.6) become (2.8).

Condition (2.8) can be replaced by the stronger condition

$$(f'(x))^2 \int_0^x \frac{dt}{f'(t)} \to \infty.$$

By L'Hôpital's rule this condition amounts to the condition

$$\frac{f''(x)}{(f'(x))^2} \to 0. \tag{2.9}$$

Example. We set $f(x) = \ln^\alpha(x+1)$, $\alpha > 0$. Condition (2.9) is satisfied only for $\alpha > 1$. In this case the sequence $\{\ln^\alpha(n+1)\}$ is W-*ud* for a suitable choice of a regular W-method. But, as we shall show in § 4, this property fails for $\alpha = 1$. Thus, the roughened sufficient condition for W-*ud* is in fact close to being necessary.

§ 3. Proofs

First we prove Theorem 1. According to (1.3) we need to show that each of the sums

$$g(n)\sin 2\pi f(0)+\dots+g(0)\sin 2\pi f(n), \qquad (3.1)$$
$$g(n)\cos 2\pi f(0)+\dots+g(0)\cos 2\pi f(n)$$

is

$$o\biggl(\sum_{s=0}^{n} g(s)\biggr). \qquad (3.2)$$

For definiteness we estimate the first sum (3.1); the second sum can be estimated in similar fashion. We set $g(x)=G(n-x)$ or, which is the same, $G(x)=g(n-x)$. For convenience we denote $2\pi f$ shortly by f.

First we show that (3.2) can be replaced by

$$o\biggl(\int_0^n g(x)\,dx\biggr).$$

For that we use the Euler–Maclaurin summation formula:

$$g(0)+\dots+g(n)=\int_0^n g(x)\,dx+\frac{1}{2}(g(0)+g(n))+O\biggl(\int_0^n |g'(x)|\,dx\biggr). \qquad (3.3)$$

By hypothesis the function g is monotonic. Consequently,

$$\int_0^n |g'|\,dx=\biggl|\int_0^n g'\,dx\biggr|=|g(n)-g(0)|.$$

We divide both parts of the equality (3.3) by the integral of the function g:

$$\frac{\sum_0^n g(s)}{\int_0^n g\,dx}=1+O\biggl(\frac{g(0)}{\int_0^n g\,dx}\biggr)+O\biggl(\frac{g(n)}{\int_0^n g\,dx}\biggr). \qquad (3.4)$$

Each of the ratios under the sign O tends to zero as $n\to\infty$: the first by condition (2.1), and the second by condition (2.2). Consequently, as $n\to\infty$ the left-hand side of (3.4) tends to 1. Hence the required result.

To estimate the first sum (3.1) we again use the Euler–Maclaurin formula:

$$\sum_{s=0}^{n} G(s)\sin f(s)=\int_0^n G(x)\sin f(x)\,dx$$
$$+\frac{1}{2}\bigl(G(0)\sin f(0)+G(n)\sin f(n)\bigr)+O\biggl(\int_0^n |(G(x)\sin f(x))'|\,dx\biggr). \qquad (3.5)$$

First we estimate the integral on the right-hand side of (3.5). Since $f'\neq 0$ (condition 1)), we can perform the change of variable $f(x)=t$. Then

$$\int_0^n G(x)\sin f(x)\,dx=\int_{f(0)}^{f(n)} \frac{G}{f'}\sin t\,dt. \qquad (3.6)$$

According to condition 3) the ratio G/f' is a monotonic function of the new variable t. By the second mean value theorem the integral (3.6) is equal to

$$\frac{G}{f'}\bigg|_{f(0)} \int_{f(0)}^{\xi} \sin t\, dt + \frac{G}{f'}\bigg|_{f(n)} \int_{\xi}^{f(n)} \sin t\, dt, \tag{3.7}$$

and $f(0) \leqslant \xi \leqslant f(n)$. It is clear that

$$\frac{G}{f'}\bigg|_{f(0)} = \frac{G(f^{-1}(t))}{f'(f^{-1}(t))}\bigg|_{t=f(0)} = \frac{G(0)}{f'(0)}\,.$$

Similarly,

$$\frac{G}{f'}\bigg|_{f(n)} = \frac{G(n)}{f'(n)}\,.$$

Since the integral of the absolute value of the sine is at most 2, the integral (3.6) is estimated by the quantity

$$2\frac{G(0)}{|f'(0)|} + 2\frac{G(n)}{|f'(n)|} = 2\frac{g(n)}{|f'(0)|} + 2\frac{g(0)}{|f'(n)|}\,.$$

After division by the integral

$$\int_0^n g(x)\, dx \tag{3.8}$$

both these summands tend to zero as $n \to \infty$: the first summand by the regularity condition (2.2), and the second by condition 4).

For the same reasons,

$$\frac{G(0)\sin f(0) + G(n)\sin f(n)}{\int_0^n g\, dx} \to 0$$

as $n \to \infty$.

Finally we estimate

$$\int_0^n |(G\sin f)'|\, dx \leqslant \int_0^n |G'|\, dx + \int_0^n G|f'|\, dx.$$

Since G is monotonic, we have

$$\int_0^n |G'|\, dx = |g(n) - g(0)|.$$

Consequently,

$$\frac{\int_0^n |G'|\, dx}{\int_0^n g\, dx} \to 0.$$

Since $|f'(x)| \to 0$ as $x \to \infty$ (condition 1)), by L'Hôpital's rule we have

$$\frac{\int_0^n G|f'|\, dx}{\int_0^n G\, dx} \to 0.$$

Thus, the proof of Theorem 1 is complete.

Theorem 2 can be proved in exactly the same way as Theorem 1. Only now the interval $[0, n]$ is divided into two intervals $[0, x_n]$ and $[x_n, n]$, on each of which the ratio $g(n-x)/f'(x)$ is monotonic. Applying the mean value theorem on each of these intervals we obtain the following estimate of the integral (3.7):

$$2\frac{G(x_n)}{|f'(0)|} + 2\frac{G(0)}{|f'(x_n)|} + 2\frac{G(n)}{|f'(x_n)|} + 2\frac{G(x_n)}{|f'(n)|}.$$

By the hypotheses of Theorem 2, after division by the integral (3.8) each of these summands tends to zero as $n \to \infty$.

§ 4. On the sequence of the fractional parts of logarithms

Theorem 3. *Let $a \geqslant 2$ be an integer and let W be a regular Voronoĭ summation method. Then the sequence*

$$\left\{\frac{1}{2}\log_a n\right\} \tag{4.1}$$

is not W-ud.

Remark. Apparently, this result is valid for any sequence $\{\alpha \ln n\}$, $\alpha \neq 0$.

Proof of Theorem 3. Consider the sequence s_n, $n \geqslant 1$, defined by the following rule: $s_n = 1$ if

$$a^{2m} \leqslant n < a^{2m+1}, \tag{4.2}$$

where $m \geqslant 0$ is an integer; $s_n = 0$ in all other cases. We claim that if the sequence (4.1) is W-*ud*, then

$$s_n \to \frac{1}{2} \quad (W). \tag{4.3}$$

Indeed, taking the logarithm of the inequality (4.2) we obtain

$$m \leqslant \frac{1}{2}\log_a n < m + \frac{1}{2},$$

whence

$$0 \leqslant \left\{\frac{1}{2}\log_a n\right\} < \frac{1}{2}.$$

Consequently, $s_n = 1$ if and only if $\{(\log_a n)/2\}$ belongs to the interval $L = [0, 1/2)$ of length $l = 1/2$. Applying (1.2) we obtain the required result.

We now show that the sequence s_n is not W-convergent. Its elements are the partial sums of the series

$$u_1 + u_2 + \cdots, \tag{4.4}$$

where $u_{a^{2m}} = 1$, $u_{a^{2m+1}} = -1$, and all the other terms are equal to zero. We associate with the numerical series (4.4) the power series

$$\sum u_k x^k = x - x^a + x^{a^2} - x^{a^3} + \cdots. \tag{4.5}$$

We now use a general result going back to Voronoĭ (see [1], Theorem 18). If a W-method is regular and (4.3) holds, then the function defined by the series (4.5) tends to $1/2$ as $x \to 1$ over real values smaller than 1. But it is well known (see, for example, [1], § 4.10) that for $a > 1$ the function (4.5) does not tend to any limit at all as $x \to 1$.

Remark. As shown in [5], the sequence of the fractional parts of the logarithms is $(R, 1/(n+1))$-*ud*. Here (R, p_n) is the Riesz summation method: $s_n \to s(R, p_n)$ if

$$\frac{p_0 s_0 + p_1 s_1 + \dots + p_n s_n}{p_0 + p_1 + \dots + p_n} \to s.$$

Theorem 3 leads to the conjecture that the $(R, 1/(n+1))$-method includes all the regular Voronoĭ methods.

In [5] sufficient conditions of a general nature on differentiable functions f and g were found under which the sequence $\{f(n)\}$ proves to be $(R, g(n))$-*ud*. In [6] somewhat different conditions were given. Theorem 1 is an analogue of the results of [5], [6] for Voronoĭ's method.

Bibliography

[1] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford 1949.
[2] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience, New York 1974.
[3] E. Gordeliĭ, "On the von Neumann theorem on perturbations of everywhere dense sequences", *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **2004**:6 (2004), 18–24; English transl. in *Moscow Univ. Math. Bull.* **59** (2005).
[4] S. B. Gashkov and V. N. Chubarikov, *Arithmetic. Algorithms. Complexity of calculations. A popular introduction to number theory and arithmetic complexity theory*, Nauka, Moscow 1996. (Russian)
[5] M. Tsuji, "On the uniform distribution of numbers mod 1", *J. Math. Soc. Japan* **4** (1952), 313–322.
[6] V. V. Kozlov, "On uniform distribution", *Izv. VUZ. Severo-Kavkaz. Reg. Ser. Estestv. Nauki* **2001** (2001), 96–99.

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