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SHORT COMMUNICATIONS

Remarks on a Lie Theorem on the Integrability of Differential Equations in Closed Form

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1. Let $v_1, v_2, \ldots, v_n$ be a set of vector fields linearly independent at all points in $\mathbb{R}^n = \{x\}$ (or in some domain in $\mathbb{R}^n$ in which these fields are considered). Suppose that these fields generate a solvable Lie algebra g with respect to the ordinary commutator $[\,,\,]$:

$$[v_1, v_j] = c_{1,j}^1 v_1, \qquad (1_1)$$

$$[v_2, v_j] = c_{2,j}^1 v_1 + c_{2,j}^2 v_2, \qquad \ldots, \qquad (1_2)$$

$$[v_n, v_j] = c_{1,j}^1 v_1 + c_{2,j}^2 v_2 + \cdots + c_{n,j}^n v_n. \qquad (1_n)$$

Here the $c_{i,j}^k$ are the structural constants of g.

A classical Lie theorem says that the differential equation

$$\dot{x} = v_1(x), \qquad x \in \mathbb{R}^n, \qquad (2)$$

is integrable by quadratures [1] (for details, see also [2, 3]). More precisely, all of its solutions can be found by "algebraic operations" (including inversion of functions) and "quadratures," that is, computation of integrals of known functions of one variable. This definition has a local nature. The algebra g generates an n-dimensional solvable Lie group G that acts freely on $\mathbb{R}^n$. Relations (1) imply that the transformations in G take the trajectories of system (2) to trajectories of the same system. In other words, the solvable group G "permutes" the trajectories of the differential equation (2). Lie treated his theorem as an analog of Galois theory for ordinary differential equations. However, there is a more natural analog of Galois theory, namely, Picard–Vessiot theory, which studies extensions of differential fields by solutions of linear differential equations (e.g., see [4]).

Nevertheless, the Lie theorem is one of the key results in the problem of closed-form integrability of nonlinear differential equations. It suffices to note that, in the simplest case, this theorem readily implies the well-known Liouville theorem on the integrability of Hamiltonian differential equations with a complete set of independent first integrals in involution [3].

The following assertion is the main result of the present paper.

Theorem 1. *Each of the n differential equations*

$$\dot{x} = v_j(x), \qquad x \in \mathbb{R}^n, \qquad (3)$$

is integrable by quadratures.

Let us indicate a corollary to Theorem 1. Let $w_1, \ldots, w_{n-1}$ be a set of commuting vector fields independent almost everywhere in $\mathbb{R}^n$; moreover, let

$$[v, w_j] = \lambda_j w_j, \qquad \lambda_j = \text{const}. \qquad (4)$$

Then the equation

$$\dot{x} = v(x), \qquad x \in \mathbb{R}^n, \qquad (5)$$

is integrable by quadratures. Indeed, if we set $v_1 = w_1, \ldots, v_{n-1} = w_{n-1}, v_n = v$, then the commutation relations (1) are valid.

Note that condition (4) implies that the integral curves of the field w_j are frozen in the phase flow of system (5). Therefore, a system of equations admits integration in closed form provided that there are $n-1$ independent commuting vector fields whose integral curves are frozen in the phase flow of the system. However, one should have in mind that the property of being frozen does not necessarily imply that the coefficients λ_j are constant. However, it is necessary to assume that the λ_j are constant for integrability in closed form.

For $j=1$, Theorem 1 coincides with the Lie theorem, and in the other extreme case $j=n$, condition (1) has a rather different geometric interpretation. Consider the k-dimensional distribution of planes in $\mathbb{R}^n=\{x\}$ generated by linear combinations of the linearly independent vectors $v_1(x),\ldots,v_k(x)$. By virtue of the commutation relations (1), this distribution is integrable (by the Frobenius theorem). Therefore, the entire $\mathbb{R}^n$ is foliated by k-dimensional integral manifolds σ^k of this distribution. Let $\sigma^k(x)$ be the manifold containing a point $x\in\mathbb{R}^n$. Obviously, $\sigma^1(x)\subset\sigma^2(x)\subset\cdots\subset\sigma^n(x)=\mathbb{R}^n$. Relations (1) imply that each integral manifold σ^k is frozen in the phase flow of system (3) (if $j=n$). Therefore, system (3) integrable by quadratures (for $j=n$) admits a flag of frozen integrable distributions.

2. The proof of Theorem 1 is based on an inductive application of the theorem on the rectification of a vector field. Let $v_j^1,\ldots,v_j^n$ be the components of the field v_j. Since (by assumption) $v_1,\ldots,v_n$ are linearly independent vectors, we have $v_j\neq 0$ everywhere in $\mathbb{R}^n$. Therefore, in a small neighborhood of each point $\mathbb{R}^n$, one can choose local coordinates $x_1,\ldots,x_n$ such that the field v_1 has the components $1,0,\ldots,0$. It follows from (1_1) that, in these coordinates, the components $v_j^2,\ldots,v_j^n$ are independent of x_1 and $v_j^1=c_{1,j}^1x_1+\tilde{v}_j^1\left(x_2,\ldots,x_n\right)$.

Now consider the $n-1$ vector fields $\tilde{v}_2,\ldots,\tilde{v}_n$ in $\mathbb{R}^{n-1}=\{x_2,\ldots,x_n\}$ with components $\tilde{v}_j=\left(v_j^2,\ldots,v_j^n\right)$. Their commutators satisfy relations (1) with $c_{2,j}^1=\cdots=c_{n,j}^1=0$. We apply the rectification theorem to the field $\tilde{v}_2$. In some local coordinates (we again denote them by $x_2,\ldots,x_n$), its components have the form $1,0,\ldots,0$. The relations $[\tilde{v}_2,\tilde{v}_j]=c_{2,j}^2\tilde{v}_2$ imply that $v_j^3,\ldots,v_j^n$ are independent of x_2 (as well as of x_1) and $v_j^2=c_{2,j}^2x_2+\tilde{v}_j^2\left(x_3,\ldots,x_n\right)$, and so on.

As a result, in the new local coordinates, system (3) acquires the form

$$\dot{x}_1=c_{1,j}^1x_1+\tilde{v}_j^1\left(x_2,\ldots,x_n\right),\qquad \dot{x}_2=c_{2,j}^2x_2+\tilde{v}_j^2\left(x_3,\ldots,x_n\right),\qquad \ldots,\qquad \dot{x}_n=v_j^n\left(x_n\right). \tag{6}$$

This system can readily be integrated in closed form. Indeed, the solutions of the last equation are obtained by the inversion of the formula

$$t=\int\frac{dx_n}{v_j^n},$$

and the remaining equations in system (6) (from bottom to top) acquire the form $\dot{x}=\lambda x+f(t)$, $\lambda=\text{const}$, and can be integrated by quadratures.

It remains to show that the vector field v_1 (as well as $\tilde{v}_2,\ldots$) can be rectified in closed form by quadratures. Indeed, the field v_1 satisfies the classical Lie theorem on integration by quadratures, whose proof provides $n-1$ independent first integrals in closed form (e.g., see [3]). However, if these integrals are known, then the rectification of trajectories can be carried out constructively with the use of simple changes of variables. The possibility of rectifying the fields $\tilde{v}_2,\ldots$ (defined in $\mathbb{R}^{n-1},\ldots$) can be proved in a similar way, since each of them also satisfies the Lie theorem.

3. Theorem 1 can be applied to Hamiltonian systems, which results in a generalization of the Liouville theorem on complete integrability. Let $\mathbb{R}^{2n}=\{p_1,\ldots,p_n,q_1,\ldots,q_n\}$ be the phase space, and let $\{\,,\,\}$ be the standard Poisson bracket. Consider n smooth functions

$$F_1,\ldots,F_n \tag{7}$$

defined in $\mathbb{R}^{2n}$ whose gradients are independent at all points in $\mathbb{R}^{2n}$ (or in some subdomain). Suppose that the linear combinations of the functions (7) span an n-dimensional solvable Lie algebra with respect to the commutator $\{\,,\,\}$:

$$\{F_1,F_j\}=c_{1,j}^1F_1,\qquad \{F_2,F_j\}=c_{2,j}^1F_1+c_{2,j}^2F_2,\qquad \ldots,$$
$$\{F_n,F_j\}=c_{n,j}^1F_1+c_{n,j}^2F_2+\cdots+c_{n,j}^nF_n.$$

Let $M_c = \{p, q: \ F_1(p, q) = c_1, \ \ldots, \ F_n(p, q) = c_n\}$ be an n-dimensional integral manifold in $\mathbb{R}^{2n}$, where $c = (c_1, \ldots, c_n)$.

Theorem 2. *If*

$$c^1_{1,j} c_1 = 0, \qquad c^1_{2,j} c_1 + c^2_{2,j} c_2 = 0, \qquad \ldots, \qquad c^1_{n,j} c_1 + c^2_{n,j} c_2 + \cdots + c^n_{n,j} c_n = 0 \tag{8}$$

for all j, then the following assertions are valid:

(1) *M_c is an invariant manifold of the Hamiltonian systems with Hamiltonians $F_1, \ldots, F_n$;*

(2) *the trajectories of each of these Hamiltonian systems on M_c can be found by quadratures.*

If the algebra of integrals is commutative ($c^k_{i,j} = 0$), then the sets of constants $c_1, \ldots, c_n$ can be arbitrary. In this case, Theorem 2 becomes the classical Liouville theorem on the integrability of Hamiltonian systems with a complete set of independent first integrals.

Theorem 2 was proved in [5] for the special case in which the function F_1 is chosen as the Hamiltonian. In the general case, the proof can be performed in a similar way. It follows from condition (8) that the Poisson brackets $\{F_i, F_j\}$ vanish on M_c. Consequently, the independent Hamiltonian fields $V_1, \ldots, V_n$ generated by the Hamiltonian functions $F_1, \ldots, F_n$ (a) are tangent to M_c; (b) satisfy the commutation relations (1) on M_c.

It remains to apply the local Theorem 1 to these fields and introduce local coordinates on M_c.

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