Mathematical Notes, vol. 78, no. 3, 2005, pp. 329–337.
Translated from Matematicheskie Zametki, vol. 78, no. 3, 2005, pp. 358–367.

Weighted Means, Strict Ergodicity, and Uniform Distributions

V. V. Kozlov

Received Mart 31, 2005

Abstract—We strengthen the well-known Oxtoby theorem for strictly ergodic transformations by replacing the standard Cesáro convergence by the weaker Riesz or Voronoi convergence with monotonically increasing or decreasing weight coefficients. This general result allows, in particular, to strengthen the classical Weyl theorem on the uniform distribution of fractional parts of polynomials with irrational coefficients.

Key words: *Weyl theorem, Cesáro and Voronoi convergences, Borel measure, Oxtoby's theorem, strictly ergodic transformation, Riesz summation method.*

1. DEFINITIONS

Suppose that s_n, $n \geq 0$, is a sequence of numbers. Consider the weighted means of the following form:

$$r_n = \frac{p_0 s_0 + \dots + p_n s_n}{p_0 + \dots + p_n}, \qquad w_n = \frac{p_0 s_n + \dots + p_n s_0}{p_0 + \dots + p_n},$$

where $p_0 > 0$, $p_n \geq 0$, and $\sum p_n = \infty$. The numbers p_n are called *weight coefficients* or simply *weights*.

The sequence r_n is related to the *Riesz summation method*: if $r_n \to s$, then, by definition,

$$s_n \to s \qquad (R, p_n).$$

This method is *regular*, i.e., if $s_n \to s$, then $r_n \to s$. By Cesáro's theorem [1, Theorem 14], if $p_{n+1} \geq p_n$ and $\sum p_n = \infty$, then the Cesáro method (all weights p_n are equal) *includes* the Riesz method (R, p_n). If the growth of the weight coefficients p_n is exponentially fast, then (R, p_n)-method is equivalent to ordinary convergence [1, Theorem 15]. In order to extend the Riesz method (R, p_n) to the summation of divergent sequences, we assume that

$$\frac{p_n}{p_0 + \dots + p_n} \to 0 \qquad \text{as} \quad n \to \infty. \tag{1.1}$$

The sequence w_n is related to the *Voronoi method of summation*: if $w_n \to s$, then, by definition, we set

$$s_n \to s \qquad (W, p_n).$$

Condition (1.1) is the regularity criterion for the Voronoi method. Any two regular Voronoi methods are *compatible*: if $s_n \to s$ (W, p_n) and $s_n \to s'$ (W, p'_n), then $s = s'$. By Hardy's theorem [1, Theorem 23], if Eq. (1.1) holds, then the sequence of weight coefficients p_n does not increase, and

$$p_0 = 1, \quad p_n > 0, \quad \frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}}, \qquad n > 0,$$

then the Cesáro method includes the Voronoi method (W, p_n).

0001-4346/2005/7834-0329

In ergodic theory, one ordinarily uses Cesáro convergence. In this connection, the problem in question is the following one: Is it possible to replace the Cesáro summation method by the weaker Riesz or Voronoi methods? A positive answer would yield a stronger form of the ergodic theorem.

In [2], assumptions on the general methods of linear summation for which the generalized *statistical* ergodic theorem holds are formulated. These assumptions are certainly fulfilled for the Riesz and Voronoi methods with monotone sequences of weight coefficients (see [3], where, in particular, a simple proof of the statistical ergodic theorem is given for the monotone case).

The case of the individual ergodic theorem is much more complicated. In [4], the Birkhoff–Khintchin theorem is generalized to the Riesz means when $p_n \le 1$ and the sequence p_n converges to a finite limit. A simpler proof is given in [5]. However (from the point of view of Riesz summation theory), this case is not of special interest: if $\lim p_n \ne 0$, then the (R, p_n)-method is equivalent to the Cesáro method, but if p_n converges *monotonically* to zero, then the (R, p_n)-method certainly includes the Cesáro method. In [3], the problem of strengthening the individual ergodic theorem for the Bernoulli transformation $x \to \{2x\}$ is considered by using the Riesz method with monotonically increasing weights. This problem is closely related to the strong form of the law of large numbers. By using the Khintchin–Kolmogorov law of the iterated logarithm, it was shown that Cesáro convergence can be replaced by (R, p_n)-convergence if the sequence p_n increases as $\exp[n/(\ln n)^\alpha]$, $\alpha > 1$. It is still unknown whether this result holds for $\alpha = 1$ or not.

In this paper, we study a simpler averaging problem for *strictly ergodic* transformations. Suppose that M is a compact metric space and T is a homeomorphism of M. The Krylov–Bogolyubov theorem implies that T preserves a certain Borel measure on M. If the normed T-*invariant* Borel measure is unique, then the continuous transformation T is said to be *strictly ergodic*.

It is clear that a strictly ergodic homeomorphism T is ergodic with respect to its unique invariant Borel measure μ. However, an arbitrary continuous ergodic transformation is not necessarily strictly ergodic. For example, a strictly ergodic transformation has no periodic points. Indeed, otherwise, there would exist a complementary invariant Borel measure concentrated on the periodic trajectory of the homeomorphism T.

Let us recall the characteristic property of strictly ergodic transformations (Oxtoby's theorem [6]): the convergence

$$f(T^n x) \rightrightarrows \int_M f\, d\mu \qquad (C) \tag{1.2}$$

holds for any continuous function $f\colon M \to \mathbb{R}$. The symbol $\rightrightarrows$ stands for uniform convergence. Since the transformation T is invertible, relation (1.2) also holds as $n \to -\infty$.

For example, let T be an orientation preserving homeomorphism of the circle $M = \{x \bmod 2\pi\}$. It is obvious that $Tx = x + f(x)$, where f is a continuous 2π-periodic function. As proved by Poincaré, the convergence

$$f(T^n x) \longrightarrow 2\pi\lambda \qquad (C) \tag{1.3}$$

holds for all x. The number λ is called the *rotation number* of the homeomorphism T. If T has no periodic points, it is strictly ergodic (see, for example, [7]). In this case, according to (1.2),

$$\lambda = \frac{1}{2\pi}\int_M f\, d\mu.$$

Relation (1.2) can be extended to a wider class of functions. We say that a function $f\colon M \to \mathbb{R}$ is $\mathcal{R}$-*integrable*, if for any $\varepsilon > 0$ there exist two continuous functions f_1 and f_2 such that

$$f_1(x) \le f(x) \le f_2(x) \qquad \forall\, x \in M, \tag{1.4}$$

$$\int_M (f_2 - f_1)\, d\mu < \varepsilon. \tag{1.5}$$

Obviously, any $\mathcal{R}$-integrable function is bounded and all continuous functions are $\mathcal{R}$-integrable.

The integral of an $\mathcal{R}$-integrable function is defined in a natural way. Consider a sequence ε_n converging to zero. The corresponding sequences of continuous functions satisfying conditions (1.4) and (1.5) will be denoted by $f_1^{(n)}$ and $f_2^{(n)}$. It can readily be proved that the limits of the sequences of integrals of $f_1^{(n)}$ and $f_2^{(n)}$ as $n \to \infty$ exist and coincide; this number is called the $\mathcal{R}$-*integral* of f with respect to the Borel measure μ; it will be denoted by $\int_M f\,d\mu$. The fact that the $\mathcal{R}$-integral is well defined (is independent of the choice of sequences ε_n, $f_1^{(n)}$, and $f_2^{(n)}$) is obvious.

Suppose, for example, that M is a multidimensional torus $\mathbb{T}^k$ and μ is the standard Borel measure on $\mathbb{T}^k$. In this case, the class of $\mathcal{R}$-integrable function coincides with the class of functions integrable in the sense of Riemann.

Oxtoby's theorem can be slightly generalized in the following way.

Theorem 1. *The transformation T is strictly ergodic if and only if, for any $\mathcal{R}$-integrable function f, the following convergence holds*:

$$f(T^n x) \rightrightarrows \int_M f\,d\mu \qquad (C).$$

A subset $D \subset M$ is said to be $\mathcal{R}$-*measurable* if its characteristic function $f = \mathbf{1}_D$ is $\mathcal{R}$-integrable. Set

$$\operatorname{mes} D = \int_M \mathbf{1}_D\,d\mu.$$

It is clear that $\operatorname{mes} M = 1$ because of the normalization assumption for the measure μ.

In the role of the function f from Theorem 1, we can take the characteristic function of *any* $\mathcal{R}$-measurable domain D. Let us define the sequence s_n, $n \ge 0$, by the rule: $s_n = 1$, if $x_k = T^k x \in D$, and $s_n = 0$ otherwise. Suppose that $\nu(n) = \sum_0^n s_k$. Then, for any strictly ergodic transformation, we have

$$\lim_{n\to\infty} \frac{\nu(n)}{n} = \operatorname{mes} D.$$

Thus, the trajectory x_k, $k \ge 0$, of any point $x \in M$ stays in the domain D, on average, for a period of time proportional to the measure of this domain. This is the general definition of a *uniformly distributed* sequence in the *sense of Weyl* (see, for example, [8]).

If the previously defined sequence s_k converges to $\operatorname{mes} D$ in the sense of the (R, p_n)-convergence for any choice of an $\mathcal{R}$-measurable domain D, then the sequence $x_k \in M$, $k \ge 0$, is called (R, p_n)-*uniformly distributed* sequence of points; the set of (W, p_n)-uniformly distributed sequences is introduced similarly.

2. THEOREMS

First, we show that Oxtoby's theorem can be strengthened by using the Riesz and Voronoi summation methods *included* in the Cesáro method.

Theorem 2. *Suppose that T is a strictly ergodic homeomorphism of a compact metric space M and f is an $\mathcal{R}$-integrable function on M. Then*

(1) *if $q_{n+1} \le q_n$ and $\sum q_n = \infty$, then*

$$f(T^n x) \rightrightarrows \int_M f\,d\mu \qquad (W, q_n); \tag{2.1}$$

(2) *if $p_{n+1} \ge p_n$ and Eq.* (1.1) *holds, then*

$$f(T^n x) \rightrightarrows \int_M f\,d\mu \qquad (R, p_n). \tag{2.2}$$

Suppose, for example, that T is an orientation preserving homeomorphism of the circle without periodic points. Then the limit relation (1.3) can be strengthened by replacing the Cesáro method by arbitrarily weak Riesz or Voronoi methods. This fact was established in [9] in the general case, in which the homeomorphism T may possess a periodic point.

As another example, consider the *compound skew shift* on the k-dimensional torus

$$\mathbb{T}^k = \{x_1, \dots, x_k \bmod 1\}$$

defined by the formula

$$Tx = ((x_1 + \alpha) \bmod 1, (x_2 + p_{2,1}x_1) \bmod 1, \dots, (x_k + p_{k,1}x_1, \dots, p_{k,k-1}x_{k-1}) \bmod 1), \qquad (2.3)$$

where α, $p_{i,j}$ are real numbers. This transformation clearly preserves the standard measure on $\mathbb{T}^k$ (the Haar measure on $\mathbb{T}^k$ regarded as a commutative Lie group). As was shown in [7], if α is irrational, and $p_{j,j-1} \neq 0$ for all $2 \leq j \leq k$, then the transformation (2.3) is strictly ergodic. Hence, for each point $x \in \mathbb{T}^k$, its trajectory (the sequence of points $T^n x$, $n \geq 0$) is uniformly distributed in the sense of Weyl.

Suppose that $\pi_j \colon \mathbb{T}^k \to \{x_j \bmod 1\}$ is the natural projection of k-dimensional torus on the circle:

$$\pi_j(x_1, \dots, x_k) = x_j.$$

It is obvious that the π_j-projection of each trajectory $T^n x$, $n \geq 0$, is a sequence of points uniformly distributed $\bmod 1$ (in the sense of the classical definition). In order to prove this fact, it suffices to take any periodic function depending on the only variable x_j for the function f.

Now, consider the case $x = 0$. One can prove (see [7, 10]) that

$$\pi_k(T^n x)\big|_{x=0} = \{P(n)\}, \qquad (2.4)$$

where $P(z)$ is a polynomial of degree k in z, and the coefficient at z^k is irrational if α is irrational. The converse statement holds as well: for any polynomial P of degree k with irrational coefficient in the leading term, there exists a compound skew shift of the k-dimensional torus of the form (2.3) whose number α is irrational, $p_{j,j-1} \neq 0$, and formula (2.4) is valid. These facts (according to Fürstenberg [7]) imply Weyl's remarkable result on the uniform distribution of the fractional parts of a polynomial with irrational leading coefficient. The original idea of Weyl's proof was completely different (see [11]). The general case, in which there exists an irrational coefficient at z^k, $k \geq 1$, can be reduced readily to this case.

By using Theorem 2, the Fürstenberg reduction, and the strict ergodicity of the skew shift (2.3), we obtain the following strengthening of Weyl's theorem.

Theorem 3. *Suppose that $P(z) = a_0 z^k + a_1 z^{k-1} + \dots + a_k$ is a polynomial one of whose coefficients $a_0, \dots, a_{k-1}$ is irrational. Then*

(1) *if $q_{n+1} \leq q_n$ and $\sum q_n = \infty$, then the sequence $\{P(n)\}$ is (W, q_n)-uniformly distributed*;
(2) *if $p_{n+1} \geq p_n$ and* (1.1) *holds, then the sequence $\{P(n)\}$ is (R, p_n)-uniformly distributed.*

Remark. H. Weyl proved the generalization of his theorem on the uniform distribution of fractional parts of polynomials by using the Riesz (R, p_n)-method with weight coefficients p_n which either increase or decrease monotonically as powers of n [11, Theorem 10]. However, this result says nothing new, because, in the first case, the (R, p_n)-method includes the Cesáro method, and, in the second case, the (R, p_n)-method is equivalent to the Cesáro method.

By using the generalized Weyl characterization of uniform distributions (see [8]), we obtain the following statement.

Corollary. *Under the assumptions of Theorem 3 on the polynomial $P(z)$ and weight coefficients,*

$$e^{2\pi i P(n)} \to 0 \quad (W, q_n), \qquad e^{2\pi i P(n)} \to 0 \quad (R, p_n). \tag{2.5}$$

It is quite interesting to prove these limit relations directly. In the continuous case, the proof is rather simple [12]. In fact, relations (2.5) for polynomials of degree 1 were proved in [13, 14].

To conclude this section, we extend Oxtoby's theorem by an assertion on the behavior of the sum

$$\sigma_n(x) = f(x) + f(Tx) + \cdots + f(T^{n-1}x), \qquad x \in M. \tag{2.6}$$

Theorem 4. *Suppose that T is a strictly ergodic homeomorphism of the compact metric space M and f is a continuous function on M. Then there exists a point x_+ (respectively, x_-) on M such that*

$$\sigma_n(x_+) - n\int_M f\,d\mu \ge 0 \qquad \left(\textit{respectively,} \quad \sigma_n(x_-) - n\int_M f\,d\mu \le 0\right)$$

for all integer n.

This proposition is a generalization of an old theorem of Bohl on integrals of quasiperiodic functions [15]. The problem considered by Bohl can clearly be reduced to the analysis of the ergodic shifts of the multidimensional torus. Bohl assumed additionally that the sum (2.6) is unbounded as a function of n. However, as was shown in [16], this assumption is superfluous.

Corollary. *Suppose that the assumptions of Theorem 4 hold. Then*

(1) *if $q_{k+1} \ge q_k$, then*

$$\sum_{k=0}^{n-1} q_k f(T^{n-k}x_\pm) - \left(\sum_{k=0}^{n-1} q_k\right)\int_M f\,d\mu \ge 0 \quad (\le 0)$$

for all n;

(2) *if $p_{k+1} \le p_k$, then*

$$\sum_{k=0}^{n-1} p_k f(T^k x_\pm) - \left(\sum_{k=0}^{n-1} p_k\right)\int_M f\,d\mu \ge 0 \quad (\le 0) \tag{2.7}$$

for all n.

For example, let us prove Eq. (2.7).

Proof. According to the Abel transformation and Theorem 4, we have

$$\begin{aligned}\sum_{k=0}^{n-1} p_k f(T^k x_\pm) &= (p_0 - p_1)\sigma_1(x_+) + \cdots + (p_{n-2} - p_{n-1})\sigma_{n-1}(x_+) + p_{n-1}\sigma_n(x_+) \\ &\ge [(p_0 - p_1) + \cdots + (n-1)(p_{n-2} - p_{n-1}) + np_{n-1}]\int_M f\,d\mu \\ &= (p_0 + \cdots + p_{n-1})\int_M f\,d\mu.\end{aligned}$$

This completes the proof. □

On the other hand, if the continuous function f is not constant, then, for almost of all $x \in M$, the difference

$$\sigma_n(x) - n\int f\,d\mu \tag{2.8}$$

changes sign infinitely many times as $n \to \infty$. This is the consequence of a general result proved in [17] for general ergodic transformations. It is interesting to note that the set of points $x \in M$ for which the number of sign changes of the difference (2.8) is finite can be everywhere dense in M. The corresponding example of an ergodic rotation of the circle is given in [16]. In that example, f is a continuous, but nowhere differentiable, function.

The theorem on the oscillations of the difference (2.8) was generalized in [18] for the Riesz (Voronoi) means with nondecreasing (nonincreasing) weights. A continuous analog of this result was proved earlier in [12].

3. PROOFS

First, let us prove Theorem 2 for continuous functions.

Proof. We will use the following well-known result: the set of functions of the form

$$f(x) = g(Tx) - g(x), \tag{3.1}$$

where g is a continuous function on M, is dense (in the metric of uniform convergence) in the linear space of continuous functions with zero mean, i.e., for functions f such that

$$\int_M f\, d\mu = 0$$

(see, for example, [10]). To be definite, here we consider Riesz summation. The substitution (3.1) and the monotonicity of the weight coefficients imply that

$$\frac{\left|\sum_0^n p_k f(T^k x)\right|}{\sum_0^n p_k} = \frac{\left|\sum_0^n p_k (g(T^{k+1}x) - g(T^k x))\right|}{\sum_0^n p_k} \le \frac{2p_n\|g\|}{\sum_0^n p_k}, \tag{3.2}$$

where $\|g\| = \max|g(x)|$ is the norm in the space of continuous functions. According to condition (1.1), it is clear that the right-hand side of Eq. (3.2) converges to zero uniformly as $n \to \infty$.

Next, if $f = \mathrm{const}$, then Eq. (2.2) is clearly fulfilled. Subtracting from an arbitrary continuous function f its mean value with respect to the measure μ, we approximate the difference by functions of the form (3.1): for a fixed $\varepsilon > 0$, we find a continuous function $f_\varepsilon(x) = g_\varepsilon(Tx) - g_\varepsilon(x)$ such that

$$\|f - f_\varepsilon\| < \varepsilon. \tag{3.3}$$

Hence

$$\left\|\frac{\sum_0^n p_k f(T^k x)}{\sum_0^n p_k}\right\| = \left\|\frac{\sum_0^n p_k f(T^k x)}{\sum_0^n p_k} - \frac{\sum_0^n p_k f_\varepsilon(T^k x)}{\sum_0^n p_k}\right\| + \left\|\frac{\sum_0^n p_k f_\varepsilon(T^k x)}{\sum_0^n p_k}\right\|. \tag{3.4}$$

The first summand on the right-hand side is clearly less than ε due to inequality (3.3). According to what was previously proved, for $n \ge N(\varepsilon)$ the second summand is not greater than ε as well. Hence, for $n \ge N(\varepsilon)$, the right-hand side of Eq. (3.4) is not greater than 2ε for all $x \in M$.

Now, suppose that the function f is $\mathcal{R}$-integrable. Then, according to Eq. (1.4), we have

$$\frac{\sum_0^n p_k f_1(T^k x)}{\sum_0^n p_k} \le \frac{\sum_0^n p_k f(T^k x)}{\sum_0^n p_k} \le \frac{\sum_0^n p_k f_2(T^k x)}{\sum_0^n p_k}$$

for all n and x. According to what was previously proved, the differences between the left- and right-hand sides of this inequality and $\bar{f}_1$ and $\bar{f}_2$, respectively, are less than or equal to ε for $n \ge n_0(\varepsilon)$; for the sake of brevity, here and further, the bar stands for the integral with respect to the measure μ over the entire set M.

Next, $\bar{f}_1 \le \bar{f} \le \bar{f}_2$ and $\bar{f}_2 - \bar{f}_1 \le \varepsilon$ (according to (1.5)). Hence $\bar{f}_2 - \bar{f} \le \varepsilon$, $\bar{f} - \bar{f}_1 \le \varepsilon$. Therefore, for $n \ge n_0$, we have

$$\left| \frac{\sum_0^n p_k f(T^k x)}{\sum_0^n p_k} - \int_M f\,d\mu \right| \le 3\varepsilon,$$

as required. □

These arguments are based on the ideas of Riesz's well-known proof of the statistical ergodic theorem and Weyl's proof of the uniform distribution of fractional parts of a linear function. Let us indicate an alternative approach to the proof of Theorem 2 outlined in the short communication [9].

Let us prove, for example, relation (2.1):

$$\frac{q_n f(x) + \dots + q_0 f(T^n x)}{q_0 + \dots + q_n} \rightrightarrows \int_M f\,d\mu. \tag{3.5}$$

Set $T^n x = z$ or equivalently, $x = T^{-n} z$. Then Eq. (3.5) takes the following form:

$$f(T^{-n} z) \rightrightarrows \int_M f\,d\mu \qquad (R, q_n). \tag{3.6}$$

Since the sequence q_n does not increase and $\sum q_n = \infty$, it follows that the (R, q_n)-method includes the Cesáro method. Hence Eq. (3.6) is a consequence of Oxtoby's theorem (taking into account that T^{-1} is also a strictly ergodic transformation of M).

However, z depends not only on x, but also on n. Thus, a simple reference to the theorem on the inclusion of the Riesz method is insufficient. We need to prove more: if the sequence of continuous functions $s_n(x)$, $x \in M$, converges to $s(x)$ *uniformly* in the Cesáro sense, then $s_n(x) \rightrightarrows s(x) \quad (R, q_n)$.

Let us prove this in the case we are interested in, i.e., for $s_n(x) = f(T^n x)$. Suppose that

$$t_m = \frac{s_0 + s_1 + \dots + s_m}{m}, \qquad u_m = \frac{q_0 s_0 + q_1 s_1 + \dots + q_m s_m}{q_0 + q_1 + \dots + q_m}. \tag{3.7}$$

Then

$$u_m = \sum c_{m,n} t_n, \tag{3.8}$$

where

$$c_{m,n} = \begin{cases} \dfrac{n(q_n - q_{n+1})}{\sum_0^m q_j} & \text{if } n < m, \\ \dfrac{m q_m}{\sum_0^m q_j} & \text{if } n = m, \\ 0 & \text{if } n > m. \end{cases}$$

Since $\sum_0^m q_j \to 0$, it follows that

$$c_{m,n} \to 0 \tag{3.9}$$

for fixed n as $m \to \infty$. Next, $c_{m,n} \ge 0$ (because the weights q_n do not increase) and

$$\sum_{n \ge 0} c_{m,n} = 1. \tag{3.10}$$

In order to prove Eq. (3.10), it suffices to put $s_j = 1$ in (3.7). Then $t_m = 1$ and $u_m = 1$; by substituting these values into Eq. (3.8), we obtain Eq. (3.10).

Suppose that $a = \max_M |f(x)|$. Then $|t_m| \le a$ for all $m \ge 0$. Now, we know that $t_m(x) \rightrightarrows s(x)$ and that $u_m(x) \to s(x)$ for each $x \in M$. By subtracting $s(x)$ from both sides of Eq. (3.8) and using Eq. (3.10), we conclude that it remains to prove the assertion in the case $s(x) \equiv 0$.

Since $t_n(x) \rightrightarrows 0$, it follows that $|t_n(x)| < \varepsilon$ for $n > N(\varepsilon)$. Consider the following estimate of the right-hand side of Eq. (3.8):

$$|u_m| \le \left|\sum_{n=0}^{N} c_{m,n} t_n\right| + \left|\sum_{n=N}^{\infty} c_{m,n} t_n\right| \le \left|\sum_{n=0}^{N} c_{m,n}\right| a + \left|\sum_{n=0}^{\infty} c_{m,n}\right| \varepsilon = \left|\sum_{n=0}^{N} c_{m,n}\right| a + \varepsilon.$$

According to Eq. (3.9) for the previously chosen ε, the sum in the right-hand side (denoted by σ) tends to zero as $m \to \infty$. Therefore, $|\sigma| < \varepsilon$ for all $m \ge M(\varepsilon, N) = M(\varepsilon)$. Hence, for $m \ge M(\varepsilon)$,

$$|t_m(x)| < 2\varepsilon,$$

as required. □

Finally, let us prove Theorem 4. Without loss of generality, we will consider only continuous functions of zero mean. First, suppose that the function f has the form (3.1). Then it is obvious that

$$\sigma_n(x) = g(x) - g(T^n x).$$

If x_+ is the point of maximum of the continuous function g on the compact set M, then we have $\sigma_n(x_+) \ge 0$ for all integers n. In the general case, there exists a sequence of continuous functions $f_m \colon M \to \mathbb{R}$, $m \ge 1$, of the form (3.1) uniformly converging to the function f. According to what was previously proved, there exists a sequence of points $x_m \in M$ such that

$$\sigma_n^m(x_m) = f_m(x_m) + \dots + f_m(T^{n-1} x_m) \ge 0$$

for all n. Since M is compact, we can extract a subsequence x_{m_k} of x_m converging to a certain point x_+. For any fixed n, we have

$$0 \le \sigma_n^{m_k}(x_{m_k}) = \sum_{j=0}^{n-1} \Big(f_{m_k}(T^j x_{m_k}) - f(T^j x_{m_k}) \Big) + \sum_{j=0}^{n-1} f(T^j x_{m_k}) \to \sum_{j=0}^{n-1} f(T^j x_+) = \sigma_n(x_+)$$

as $k \to \infty$. This proves Theorem 4.

ACKNOWLEDGMENTS

This research was supported by the Russian Foundation for Basic Research by the program "Leading Scientific Schools" under grant no. NSh-136.2003.1.

REFERENCES

1. G. Hardy, *Divergent Series*, Oxford, 1949.
2. D. L. Hanson and G. Pledger, "On the mean ergodic theorem for weighted averages," *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **13** (1969), no. 13, 141–149.
3. V. V. Kozlov, "Summation of divergent series and ergodic theorems," *Trudy Sem. Petrovsk.*, **22** (2002), 142–168.
4. G. Baxter, "An ergodic theorem with weighted averages," *J. Math. Mech.*, **13** (1964), no. 3, 481–488.
5. R. V. Chacon, "Ordinary means imply recurrent means," *Bull. Amer. Math. Soc.*, **70** (1964), 796–797.
6. J. C. Oxtoby, "Ergodic sets," *Bull. Amer. Math. Soc.*, **58** (1952), no. 2, 116–136.

7. H. Fürstenberg, "Strict ergodicity and transformation of the torus," *Amer. J. Math.*, **83** (1961), no. 4, 573–601.
8. L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley–Interscience, New York, 1974.
9. V. V. Kozlov and T. Madsen, "The Poincaré rotation numbers and the Riesz and Voronoi means," *Mat. Zametki* [*Math. Notes*], **74** (2003), no. 2, 314–315.
10. I. P. Kornfel′d, Ya. G. Sinai, and S. V. Fomin, *Ergodic Theory* [in Russian], Nauka, Moscow, 1980.
11. H. Weyl, "Über die Gleichverteilung von Zahlen mod Eins," *Math. Ann.*, **77** (1915/16), 313–352.
12. V. V. Kozlov, "On uniform distributions on the torus," *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* [*Moscow Univ. Math. Bull.*] (2004), no. 2, 22–29.
13. J. Cigler, "Methods of summability and uniform distribution mod 1," *Compos. Math.*, **16** (1964), 44–51.
14. A. F. Doroidar, "A note on the generalized uniform distribution (mod 1)," *J. Natur. Sci. Math.*, **11** (1971), 185–189.
15. P. Bohl, "Uber eine Differentialgleichung der Störungstheorie," *J. Reine Angew. Math.*, **131** (1906), no. 4, 268–321.
16. V. V. Kozlov, "On integrals of quasiperiodic functions," *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* [*Moscow Univ. Math. Bull.*] (1978), no. 1, 106–115.
17. G. Halász, "Remarks on the remainder in Birkhoff's ergodic theorem," *Acta Math. Acad. Sci. Hungar.*, **28** (1976), nos. 3–4, 289–395.
18. A. A. Sorokin, "On oscillations of Riesz and Voronoi means," *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* [*Moscow Univ. Math. Bull.*] (2005), no. 2, 13–17.

V. A. Steklov Mathematics Institute,
Russian Academy of Sciences, Moscow