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On the Stability Degree

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1. STABILITY DEGREE AND INDICES OF INERTIA

Consider a linear Hamiltonian system in $\mathbb{R}^{2n} = \{x\}$ with the standard symplectic structure. Let $x = (p_1, \dots, p_n, q_1, \dots, q_n)$ be the canonical variables (the set of momenta and coordinates), and let the Hamiltonian have the form

$$H = \frac{1}{2}(Bx, x), \tag{1}$$

where B is a symmetric linear operator. Then the canonical equations can be represented in the form

$$\dot{x} = Ax, \tag{2}$$

where

$$A = IB, \qquad I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

In the variables p and q, Eq. (2) acquires the usual form of Hamiltonian equations

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \qquad \dot{q}_k = \frac{\partial H}{\partial p_k} \qquad (1 \le k \le n).$$

Let A be a nondegenerate operator: $|A| \ne 0$ (This is equivalent to the condition $|B| \ne 0$.) Then the eigenvalues of A can be of three types: real pairs $\pm a$, pure imaginary pairs $\pm ib$, and quadruples $\pm a \pm ib$.

It was shown in [1] that the linear system (2) with nondegenerate operator A that admits an integral in the form of a nondegenerate quadratic form (1) can always be reduced to a system of Hamiltonian equations. Therefore, the theory developed below applies to this (formally, more general) case.

The *instability degree* u of system (2) is defined as the number of roots (counted according to their multiplicities) of the characteristic equation of A in the right half-plane, and the *stability degree* s is defined as the number of pure imaginary roots (counted according to their multiplicities) of the characteristic equation of A. By i^+ and i^- we denote the positive and negative indices of inertia of the quadratic form (1).

By the Thomson theorem [2],

$$u \equiv i^- \bmod 2. \tag{3}$$

In particular, if i^- (or i^+) is odd, then so is u. Consequently, the characteristic polynomial of A necessarily has a positive real root in this case; therefore, $x = 0$ is an unstable equilibrium. More precisely, the classical Thomson theorem holds for linear mechanical systems subjected to additional gyroscopic and dissipative forces. The congruence (3) was proved in [3] for systems of general form (with $\dot{H} \le 0$). If $\dot{H} = 0$, then the Thomson theorem is equivalent to the following assertion on the stability degree: s is even if and only if

$$i^+ \equiv i^- \bmod 4. \tag{4}$$

The following assertion is the main result of the present paper.

Theorem 1. *One has*

$$\left|i^+ - i^-\right| \le 2l,$$

where l is the number of pairs of pure imaginary eigenvalues of A with Jordan blocks of odd order.

Corollary. *One has*

$$\left|i^+ - i^-\right| \le 2s. \tag{5}$$

Inequality (5) was indicated in [4] for the typical case in which all roots of the characteristic polynomial are simple. In what follows, we prove Theorem 1 in the general case of multiple roots with the use of the Williamson theory of real normal forms of linear Hamiltonian equations [5]. The Thomson theorem can readily be proved by the same method.

Consider an illustrative example. Let $i^- = 1$. Then, by the Thomson theorem, there exists a real pair $\pm a$ of eigenvalues. Furthermore, all other eigenvalues are pure imaginary by inequality (5). Moreover, in this case, the matrix A can be reduced to a diagonal form by Theorem 1. This situation is typical for the case in which the total energy H lacks positive definiteness.

2. PROOF OF THEOREM 1

By the Williamson theorem, the space $\mathbb{R}^{2n}$ can be decomposed into a direct sum of skew-orthogonal subspaces such that the form (1) is equal to the sum of quadratic forms (partial Hamiltonians) on these subspaces; moreover, the following assertions are valid:

(a) to a real pair $\pm a$ of eigenvalues of order k, there corresponds a partial Hamiltonian

$$H = -a\sum_{j=1}^{k} p_j q_j + \sum_{j=1}^{k-1} p_j q_{j+1};$$

(b) to a quadruple $\pm a \pm ib$ of order k, there corresponds a Hamiltonian

$$H = -a\sum_{j=1}^{2k} p_j q_j + b\sum_{j=1}^{k}\left(p_{2j-1}q_{2j} - p_{2j}q_{2j-1}\right) + \sum_{j=1}^{2k-2} p_j q_{j+2};$$

(c) to a pure imaginary pair $\pm ib$ of order $2k+1$, there corresponds a Hamiltonian

$$H = \pm\frac{1}{2}\left[\sum_{j=1}^{k}\left(b^2 p_{2j}p_{2k-2j+2} + q_{2j}q_{2k-2j+2}\right) - \sum_{j=1}^{k+1}\left(b^2 p_{2j-1}p_{2k-2j+3} + q_{2j-1}q_{2k-2j+3}\right)\right] - \sum_{j=1}^{2k} p_j q_{j+1};$$

(d) to a pure imaginary pair $\pm ib$ of order $2k$, there corresponds a Hamiltonian

$$H = \pm\frac{1}{2}\left[\sum_{j=1}^{k}\left(b^{-2} q_{2j-1}q_{2k-2j+1} + q_{2j}q_{2k-2j+2}\right) - \sum_{j=1}^{k-1}\left(b^2 p_{2j+1}p_{2k-2j+1} + p_{2j+2}p_{2k-2j+2}\right)\right] - b^2\sum_{j=1}^{k} p_{2j-1}q_{2j} + \sum_{j=1}^{k} p_{2j}q_{2j-1}.$$

Let us find the signatures of each of these Hamiltonians.

In case (a), we have

$$H = -ap_1q_1 + \sum_{j=2}^{k}\left(p_{j-1} - ap_j\right)q_j = \sum_{j=1}^{k}\tilde{p}_j\tilde{q}_j,$$

where $\tilde{p} = Cp$, $\tilde{q} = q$, and C is the bidiagonal matrix with main diagonal $(-a, \dots, -a)$ and unit subdiagonal, $|C| \neq 0$. Hence we find that the pair $\pm a$ of order k gives the signature $\underbrace{+\cdots+}_{k}\underbrace{-\cdots-}_{k}$.

Note that here and throughout the following the new variables $\tilde{p}$ and $\tilde{q}$ are not necessarily canonical. However, this makes no difference for the computation of the signature.

In case (b), the Hamiltonian H acquires the form

$$H = -a\sum_{j=1}^{2k} p_j q_j + b\sum_{j=1}^{k}\left(p_{2j-1}q_{2j} - p_{2j}q_{2j-1}\right) + \sum_{j=1}^{2k-2} p_j q_{j+2} = \sum_{j=1}^{2k} \tilde{p}_j \tilde{q}_j,$$

where $\tilde{p} = Cp$, $\tilde{q} = q$, and

$$C = \begin{pmatrix} -a & -b & 0 & 0 & \dots & 0 \\ b & -a & 0 & 0 & \dots & 0 \\ 1 & 0 & -a & -b & \dots & 0 \\ 0 & 1 & b & -a & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & b & -a \end{pmatrix}, \qquad |C| \neq 0.$$

Hence it follows that the quadruple $\pm a \pm ib$ of order k gives the signature $\underbrace{+\cdots+}_{2k}\underbrace{-\cdots-}_{2k}$.

In case (c), we have

$$\begin{aligned} H = \pm\frac{1}{2}\Bigg[&\sum_{j=1}^{k}\left(b^2 p_{2j}p_{2k-2j+2} + q_{2j}q_{2k-2j+2}\right) \\ &- \sum_{j=1}^{k+1}\left(b^2 p_{2j-1}p_{2k-2j+3} + q_{2j-1}q_{2k-2j+3}\right)\Bigg] - \sum_{j=1}^{2k} p_j q_{j+1} \\ = \pm&\frac{1}{2}\sum_{j=1}^{2k+1}(-1)^j\left(b^2 p_j p_{2k+2-j} + q_j q_{2k+2-j}\right) - \sum_{j=1}^{2k} p_j q_{j+1} \\ = \pm&\sum_{j=1}^{k}(-1)^j\left(b^2 p_j p_{2k+2-j} + q_j q_{2k+2-j}\right) \pm \frac{1}{2}(-1)^{k+1}\left(b^2 p_{k+1}^2 + q_{k+1}^2\right) - \sum_{j=1}^{2k} p_j q_{j+1} \\ = \pm&\sum_{j=1}^{k} p_j\left((-1)^j b^2 p_{2k+2-j} \mp q_{j+1}\right) \pm \sum_{j=k+2}^{2k+1} q_j\left((-1)^j q_{2k+2-j} \mp p_{j-1}\right) \\ &\pm\frac{1}{2}(-1)^{k+1}\left(b^2 p_{k+1}^2 + q_{k+1}^2\right) \\ = \pm&\sum_{j=1,\, j\neq k+1}^{2k+1} \tilde{p}_j\tilde{q}_j \pm (-1)^{k+1}\left(\tilde{p}_{k+1}^2 + \tilde{q}_{k+1}^2\right), \end{aligned}$$

where

$$\begin{aligned} \tilde{p}_j &= p_j, \qquad j = 1, \dots, k, \\ \tilde{p}_{k+1} &= 2^{-1/2}q_{k+1}, \qquad \tilde{p}_j = (-1)^j q_{2k+2-j} \mp p_{j-1}, \qquad j = k+2, \dots, 2k+1; \\ \tilde{q}_j &= (-1)^j b^2 p_{2k+2-j} \mp q_{j+1}, \qquad j = 1, \dots, k, \\ \tilde{q}_{k+1} &= 2^{-1/2} b p_{k+1}, \qquad \tilde{q}_j = q_j, \qquad j = k+2, \dots, 2k+1. \end{aligned}$$

One can readily see that the transition matrix is nondegenerate. Consequently, the pair $\pm ib$ of order $2k+1$ gives the signature $\underbrace{+\cdots+}_{2k+2}\underbrace{-\cdots-}_{2k}$ or $\underbrace{+\cdots+}_{2k}\underbrace{-\cdots-}_{2k+2}$.

(d) Finally, for the pure imaginary pair $\pm ib$ of order $2k$, we consider two cases. If $k = 2l+1$ (k is odd), then we have

$$
\begin{aligned}
H &= \pm \sum_{j=1}^{l} \left(b^{-2} q_{2j-1} q_{4l-2j+3} + q_{2j} q_{4l-2j+4}\right) \pm \frac{1}{2}\left(b^{-2} q_{2l+1}^2 + q_{2l+2}^2\right) \\
&\quad \mp \sum_{j=1}^{l} \left(b^2 p_{2j+1} p_{4l-2j+3} + p_{2j+2} p_{4l-2j+4}\right) - b^2 \sum_{j=1}^{2l+1} p_{2j-1} q_{2j} + \sum_{j=1}^{2l+1} p_{2j} q_{2j-1} \\
&= \sum_{j=1}^{l} \left[q_{2j-1}\left(p_{2j} \pm b^{-2} q_{4l-2j+3}\right) + q_{2j}\left(-b^2 p_{2j-1} \pm q_{4l-2j+4}\right)\right] \\
&\quad + q_{2l+1}\left(p_{2l+2} \pm \frac{1}{2} b^{-2} q_{2l+1}\right) + q_{2l+2}\left(-b^2 p_{2l+1} \pm \frac{1}{2} q_{2l+2}\right) \\
&\quad \mp \sum_{j=l+2}^{2l+1} \left(b^2 p_{2j-1} p_{4l-2j+5} + p_{2j} p_{4l-2j+6}\right) - b^2 \sum_{j=l+2}^{2l+1} p_{2j-1} q_{2j} + \sum_{j=l+2}^{2l+1} p_{2j} q_{2j-1} \\
&= \sum_{j=1}^{l} \left[q_{2j-1}\left(p_{2j} \pm b^{-2} q_{4l-2j+3}\right) + q_{2j}\left(-b^2 p_{2j-1} \pm q_{4l-2j+4}\right)\right] \\
&\quad + q_{2l+1}\left(p_{2l+2} \pm \frac{1}{2} b^{-2} q_{2l+1}\right) + q_{2l+2}\left(-b^2 p_{2l+1} \pm \frac{1}{2} q_{2l+2}\right) \\
&\quad + \sum_{j=l+2}^{2l+1} \left[p_{2j-1}\left(\mp b^2 p_{4l-2j+5} - b^2 q_{2j}\right) + p_{2j}\left(\mp p_{4l-2j+6} + q_{2j-1}\right)\right] \\
&= \sum_{j=1}^{4l+2} \tilde{p}_j \tilde{q}_j = \sum_{j=1}^{2k} \tilde{p}_j \tilde{q}_j,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{p}_{2j-1} &= p_{2j} \pm b^{-2} q_{4l-2j+3}, \qquad \tilde{p}_{2j} = -b^2 p_{2j-1} \pm q_{4l-2j+4}, \qquad j = 1, \ldots, l, \\
\tilde{p}_{2l+1} &= p_{2l+2} \pm \frac{1}{2} b^{-2} q_{2l+1}, \qquad \tilde{p}_{2l+2} = -b^2 p_{2l+1} \pm \frac{1}{2} q_{2l+2}, \\
\tilde{p}_j &= p_j, \qquad j = 2l+3, \ldots, 4l+2; \\
\tilde{q}_j &= q_j, \qquad j = 1, \ldots, 2l+2, \\
\tilde{q}_{2l+2j+1} &= \mp b^2 p_{2l-2j+3} - b^2 q_{2l+2j+2}, \qquad \tilde{q}_{2l+2j+2} = \mp p_{2l-2j+4} - q_{2l+2j+1}, \qquad j = 1, \ldots, l.
\end{aligned}
$$

But if $k = 2l$ (k is even), then

$$
\begin{aligned}
H &= \pm \sum_{j=1}^{l} \left(b^{-2} q_{2j-1} q_{4l-2j+1} + q_{2j} q_{4l-2j+2}\right) \mp \frac{1}{2}\left(b^2 p_{2l+1}^2 + p_{2l+2}^2\right) \\
&\quad \mp \sum_{j=1}^{l-1} \left(b^2 p_{2j+1} p_{4l-2j+1} + p_{2j+2} q_{4l-2j+2}\right) - b^2 \sum_{j=1}^{2l} p_{2j-1} q_{2j} + \sum_{j=1}^{2l} p_{2j} q_{2j-1} \\
&= \sum_{j=1}^{l} \left[q_{2j-1}\left(p_{2j} \pm b^{-2} q_{4l-2j+1}\right) + q_{2j}\left(-b^2 p_{2j-1} \pm q_{4l-2j+2}\right)\right] \\
&\quad + p_{2l+1}\left(\mp \frac{1}{2} b^2 p_{2l+1} - b^2 q_{2l+2}\right) + p_{2l+2}\left(\mp \frac{1}{2} p_{2l+2} + q_{2l+1}\right) \\
&\quad \mp \sum_{j=l+2}^{2l} \left(b^2 p_{2j-1} p_{4l-2j+3} + p_{2j} p_{4l-2j+4}\right) - b^2 \sum_{j=l+2}^{2l} p_{2j-1} q_{2j} + \sum_{j=l+2}^{2l} p_{2j} q_{2j-1}
\end{aligned}
$$

Table

Eigenvalue	Order	Signature
$\pm a$	k	$\underbrace{+\cdots+}_{k}\underbrace{-\cdots-}_{k}$
$\pm a \pm ib$	k	$\underbrace{+\cdots+}_{2k}\underbrace{-\cdots-}_{2k}$
$\pm ib$	odd k	$\underbrace{+\cdots+}_{k+1/k-1}\ \underbrace{-\cdots-}_{k-1/k+1}$
$\pm ib$	even k	$\underbrace{+\cdots+}_{k}\underbrace{-\cdots-}_{k}$

$$= \sum_{j=1}^{l} \left[q_{2j-1}\left(p_{2j} \pm b^{-2} q_{4l-2j+1}\right) + q_{2j}\left(-b^2 p_{2j-1} \pm q_{4l-2j+2}\right)\right]$$
$$+ p_{2l+1}\left(\mp\frac{1}{2}b^2 p_{2l+1} - b^2 q_{2l+2}\right) + p_{2l+2}\left(\mp\frac{1}{2}p_{2l+2} + q_{2l+1}\right)$$
$$+ \sum_{j=l+2}^{2l}\left[p_{2j-1}\left(\mp b^2 p_{4l-2j+3} - b^2 q_{2j}\right) + p_{2j}\left(\mp p_{4l-2j+4} + q_{2j-1}\right)\right] = \sum_{j=1}^{4l}\tilde{p}_j\tilde{q}_j = \sum_{j=1}^{2k}\tilde{p}_j\tilde{q}_j,$$

where

$$\tilde{p}_{2j-1} = p_{2j} \pm b^{-2} q_{4l-2j+1}, \qquad \tilde{p}_{2j} = -b^2 p_{2j-1} \pm q_{4l-2j+2}, \qquad j = 1, \ldots, l,$$
$$\tilde{p}_j = p_j, \qquad j = 2l+1, \ldots, 4l;$$
$$\tilde{q}_j = q_j, \qquad j = 1, \ldots, 2l,$$
$$\tilde{q}_{2l+1} = \mp\frac{1}{2}b^2 p_{2l+1} - b^2 q_{2l+2}, \qquad \tilde{q}_{2l+2} = \mp\frac{1}{2}p_{2l+2} + q_{2l+1},$$
$$\tilde{q}_{2l+2j+1} = \mp b^2 p_{2l-2j+1} - b^2 q_{2l+2j+2}, \qquad j = 1, \ldots, l-1,$$
$$\tilde{q}_{2l+2j+2} = \mp p_{2l-2j+2} + q_{2l+2j+1}, \qquad j = 1, \ldots, l-1.$$

One can show that the transition matrix is nondegenerate in both cases. Therefore, the pair $\pm ib$ of order $2k$ gives the signature $\underbrace{+\cdots+}_{2k}\underbrace{-\cdots-}_{2k}$.

The results of computations are represented in the table.

Let l be the number of pairs of pure imaginary eigenvalues of A with odd multiplicities, and let $k_1, \ldots, k_l$ be their multiplicities; moreover, let $k_1, \ldots, k_{l_1}$ be the multiplicities of eigenvalues corresponding to signatures of the form $\underbrace{+\cdots+}_{k+1}\underbrace{-\cdots-}_{k-1}$, and let $k_{l_1+1}, \ldots, k_l$ be multiplicities of eigenvalues corresponding to signatures of the form $\underbrace{+\cdots+}_{k-1}\underbrace{-\cdots-}_{k+1}$; let m be the number of pairs of pure imaginary eigenvalues of A with even multiplicities, and let $k'_1, \ldots, k'_m$ be their multiplicities. Then the positive and negative indices i^+ and i^- of inertia of the quadratic form (1) are equal to

$$i^{\pm} = u + \sum_{i=1}^{l_1}(k_i \pm 1) + \sum_{i=l_1+1}^{l}(k_i \mp 1) + \sum_{i=1}^{m} k'_i,$$

and the stability degree is equal to $s = \sum_{i=1}^{l} k_i + \sum_{i=1}^{m} k'_i$. Obviously, the stability degree s is even if and only if so is l. Consider the difference

$$i^+ - i^- = 2l_1 - 2(l - l_1) = 4l_1 - 2l$$

of indices of inertia, whence we find that first, $|i^+ - i^-| \le 2l \le 2s$ (since $l_1 \le l \le s$), which completes the proof of Theorem 1, and second, $i^+ - i^- \equiv 0 \bmod 4$ if and only if l is even. Since $s \equiv l \bmod 2$, it follows that the stability degree s is even if and only if condition (4) is satisfied.

Note also that the formulas for the indices of inertia readily imply the Thomson theorem (3). Indeed, it suffices to note that the integers k_i (respectively, k_i') are odd (respectively, even).

3. AN APPLICATION TO THE GYROSCOPIC STABILIZATION PROBLEM

Consider the equation of motion of a mechanical system subjected to gyroscopic and potential forces:

$$\ddot{z} + \Gamma\dot{z} + Pz = 0, \qquad z \in \mathbb{R}^n, \tag{6}$$

where $\Gamma^t = -\Gamma$ and $P^t = P$. The term $-\Gamma\dot{z}$ is referred to as the gyroscopic force, and $-Pz$ is referred to as the potential force acting on the system with n degrees of freedom. The signature of the corresponding Hamiltonian [the total energy $H = (1/2)(\dot{z}, \dot{z}) + (1/2)(Pz, z)$] has the form $\underbrace{+\cdots+}_{n+n-u}\underbrace{-\cdots-}_{u}$. Here u is the instability degree of the linear system in the absence of the gyroscopic force (the Poincraé instability degree u is determined by the matrix P), and accordingly, $n-u$ is the stability degree in the absence of the gyroscopic force. By (5), $|i^+ - i^-| = |2n - 2u| \le 2s$, which implies that $n - u \le s$, where s is the stability degree of the linear system described by Eq. (6). Therefore, we obtain the following important assertion.

Theorem 2. *The addition of gyroscopic forces does not diminish the stability degree of a system.*

This result supplements the classical Thomson theorem on the possibility of a gyroscopic stabilization of a system with even Poincaré instability degree [2].

Now in Eq. (6), we replace the matrix Γ of gyroscopic forces by $\mu\Gamma$, $\mu \in \mathbb{R}$. We consider the case in which $P < 0$, so that the potential energy $(1/2)(Pz, z)$ attains the maximum value at the equilibrium $z = 0$. If $|\Gamma| \ne 0$, then for sufficiently large μ, the equilibrium $z = 0$ becomes stable [6, 7]. Since the matrix Γ is skew-symmetric, it follows that n is even.

Theorem 3. *Let n be odd, and let* $\operatorname{rank}\Gamma = n - 1$. *If $\mu \ge \mu_0$, then the instability degree of system* (6) *is equal to* 1.

Since n is odd and $P < 0$, it follows from the Thomson theorem that $u \ge 1$. Theorem 3 implies that, for nonsingular matrices of gyroscopic forces and for sufficiently large values of the parameter μ, the instability degree attains the minimum possible value $u = 1$.

Proof of Theorem 3. In addition to the energy integral H, system (6) admits the integral

$$F = \frac{1}{2}\left(P^{-1}\dot{z}, \dot{z}\right) - \left(\Gamma P^{-1}\dot{z}, z\right) + \frac{1}{2}\left(\left(E - \Gamma P^{-1}\Gamma\right) z, z\right).$$

Here E is the identity $n \times n$ matrix. We replace Γ by $\mu\Gamma$ and consider the quadratic integral

$$\Phi = 2H - 2F/\mu^{3/2} = (\dot{z}, \dot{z}) + (Pz, z) - \mu^{1/2}\left(P^{-1}\Gamma z, \Gamma z\right) + O\left(\mu^{-1/2}\right).$$

First, we note that, for large μ, this quadratic form is nondegenerate. Indeed, the determinant $\left|\sqrt{\mu}\,\Gamma P^{-1}\Gamma + P\right|$ is a polynomial in $\sqrt{\mu}$, which is nonzero for $\mu = 0$. Consequently, this determinant does not vanish for $\mu \ge \mu_0$. Further, by assumption, the quadratic form $-\left(P^{-1}\Gamma z, \Gamma z\right)$ is nonnegative and vanishes only on the one-dimensional subspace $\ker\Gamma$, where (Pz, z) is a negative definite form. Therefore, for large μ, the negative index of inertia of the nondegenerate quadratic integral Φ is equal to 1. Now we use the following result in [1]: by a linear change of variables, one can reduce system (6) to linear Hamiltonian equations with the nondegenerate quadratic integral Φ playing the role of the Hamiltonian. It remains to use inequality (5).

Remark. The inequality

$$u \le i^- \tag{7}$$

was proved in [8, 9] for a Hamiltonian system of the form (6); here i^- is the negative index of inertia of the nondegenerate potential energy $(1/2)(Pz, z)$. Using the relations $i^+ + i^- = 2n$ and $u + s = n$, we find that inequality (7) is equivalent to (5). Inequality (7) was refined in [10]. This refinement was based on a generalization of the Pontryagin theorem on self-adjoint operators in spaces with indefinite metric. Our approach, applied to linear Hamiltonian systems of general form, is based on the Williamson theory of normal forms.

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