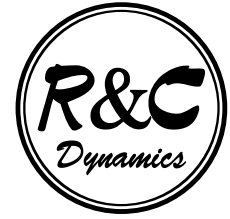


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BILLIARDS, INVARIANT MEASURES, AND EQUILIBRIUM THERMODYNAMICS. II

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The kinetics of collisionless continuous medium is studied in a bounded region on a curved manifold. We have assumed that in statistical equilibrium, the probability distribution density depends only on the total energy. It is shown that in this case, all the fundamental relations for a multi-dimensional ideal gas in thermal equilibrium hold true.

According to Gibbs, the basic object of statistical mechanics is an ensemble of identical Hamiltonian systems. The systems do not interact with each other and the assembly of them makes essentially a collisionless continuous medium. From the viewpoint of kinetics, the Hamiltonian systems with elastic impacts, i. e. *billiards*, are especially interesting. These are systems where particles move inertially inside a bounded region and bounce elastically against the boundaries of the region. As it is shown in Refs. [1, 2], in a billiards, the probability distribution density as a function of time t (this function satisfies the classic Liouville equation) necessarily has the *weak* limit as $t \rightarrow \pm\infty$. This result justifies the *Zeroth* Law of Thermodynamics in the Gibbs theory. The weak limit is a first integral of the Hamilton equations and depends, in the ergodic case, only on the system's energy. It is noted in Ref. [3] that this is very often justified even without any ergodic hypothesis: it is important here to keep in mind the function class, to which the probability distribution density function belongs.

In Ref. [3], we developed the thermodynamics of billiards in the Euclidean space. It turns out to be possible to extend these observations to the general case of a curved configurational space.

Let M^n be a compact configurational space of a natural mechanical system with n degrees of freedom, $x = (x_1, \dots, x_n)$ be local coordinates on M , and $y = (y_1, \dots, y_n)$ be the conjugate canonical momenta. The motion to be considered is inertial. So the Hamiltonian is a positively defined quadratic form with respect to the momenta:

$$H = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) y_i y_j. \quad (1)$$

If we denote the matrix of coefficients $\|a_{i,j}\|$ as A , then $H = (Ay, y)/2$.

Let $f(\cdot)$ be a nonnegative summable function of one variable. By analogy with the Gibbs canonical distribution, we introduce the density of the stationary probability distribution in the phase space $\Gamma = T^*M$:

$$\rho(x, y) = \frac{f(\beta H)}{\int_{\mathbb{R}^n} \int_M f(\beta H) d^n y d^n x}. \quad (2)$$

Here, the factor β is introduced to non-dimensionalize the argument of f . It is customary to take $\beta = 1/k\tau$, where k is the Boltzmann constant, and τ is the absolute temperature.

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The denominator in (2) is referred to in Ref. [3] as the *generalized* statistic integral. It can be easily expressed in terms of τ , the only external thermodynamical parameter here being M , Riemann's volume of the manifold. To this effect, we perform the linear change of variables $y \mapsto p$:

$$y = C(x)p, \quad C^T AC = E.$$

Thus,

$$F = \int_{\mathbb{R}^n} \int_M f(\beta H) d^n y d^n x = \int_M \int_{\mathbb{R}^n} f\left(\frac{\beta}{2} \sum p_i^2\right) (\det A)^{-\frac{1}{2}} d^n x d^n p = \frac{bv}{(\sqrt{\beta})^n},$$

where

$$v = \int_M (\det A^{-1})^{\frac{1}{2}} d^n x$$

is the volume M with respect to Riemannian metric (1),

$$b = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty r^{n-1} f\left(\frac{r^2}{2}\right) dr = \text{const},$$

where Γ is the Euler gamma function.

Now we calculate the average kinetic energy:

$$\begin{aligned} E &= \frac{1}{F} \int_{\mathbb{R}^n} \int_M \frac{1}{2} (Ay, y) f\left(\frac{\beta}{2} (Ay, y)\right) d^n x d^n y = \\ &= \frac{1}{F} \int_{\mathbb{R}^n} \int_M \frac{1}{2} \sum p_j^2 f\left(\frac{\beta}{2} \sum p_j^2\right) (\det A)^{-\frac{1}{2}} d^n x d^n p = \\ &= \frac{a}{\beta b}, \end{aligned}$$

where

$$a = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty r^{n+1} f\left(\frac{r^2}{2}\right) dr = \text{const}.$$

It is interesting to note that the average internal energy of a collisionless medium does not depend on the volume, which correlates with *Joule's law* for ideal gas.

As it is shown in Ref. [4], if a Hamiltonian is a homogeneous function with respect to momenta, then the quantities calculated using the general routines of statistical mechanics and density (2) satisfy the First and the Second Laws of Thermodynamics. Let us calculate, for example, the thermodynamic entropy. For this, we should first (according to Ref. [3]) write down the following relation:

$$E = \varkappa \frac{\partial F}{\partial \beta},$$

which gives the coefficient \varkappa . According to the general theory, this coefficient must be a function of the statistical integral F . In the case in question,

$$\varkappa = -\frac{2a}{bn} \frac{1}{F}.$$

Let $\Phi(F)$ be the antiderivative of $\varkappa(F)$. Then, as shown in Ref. [3], the thermodynamic entropy is given by the equation

$$S = \beta \frac{\partial \Phi}{\partial \beta} - \Phi.$$

Hence,

$$S = \frac{a}{b} + \frac{2a}{bn} \ln F = \text{const} + \frac{2a}{bn} \left(\ln v + \frac{n}{2} \ln \tau \right). \quad (3)$$

In the case of the Gibbs canonical distribution ($f(z) = e^{-z}$), one can easily show, upon integration by parts, that $2a = nb$.

Usually, the entropy of ideal gas in a *three-dimensional* vessel Π with volume w is

$$N \ln w + \frac{3N}{2} \ln \tau + \text{const}, \quad (4)$$

where N is the number of gas particles (this expression is sometimes multiplied by the Boltzmann constant k , but we do without it). To compare (3) and (4), let us consider the Boltzmann–Gibbs gas consisting of N identical small balls, moving in a vessel Π . The balls collide elastically with each other and with the walls of the vessel. Then, obviously, $n = 3N$, while the volume v is approximately equal to w^N (for, in the case of non-interacting balls, the configurational space M of the system is the direct product of N copies of Π). Having made these remarks, we see that (3) and (4) become identical up to the insignificant constant factor $2a/nb$ that depends on the type of the function $f(\cdot)$ and on the number of degrees of freedom in the system.

On the other hand, the entropy in statistical mechanics is given by the integral

$$S = - \int_{\Gamma} \rho \ln \rho d^n x d^n y. \quad (5)$$

In the case of the canonical distribution, this integral coincides with the thermodynamical entropy. Of course, for more general distributions of the form (2), this remarkable Gibbs' result is not valid. However, the Gibbs entropy (5) looks as follows:

$$\frac{\gamma}{b} + \ln F, \quad (6)$$

where

$$\gamma = - \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} r^{n-1} f\left(\frac{r^2}{2}\right) \ln f\left(\frac{r^2}{2}\right) dr = \text{const}.$$

We see that (3) and (6) coincide up to an insignificant additive constant and a somewhat less insignificant constant positive factor.

The latter remark is very important for the kinetics of a collisionless medium, especially for the validation of the Second Law in the case of irreversible processes. The matter is that (as proved in Refs. [1] and [5]) if we replace the density ρ in (5) with its weak limit, then the Gibbs entropy gets a nonnegative increment. If the entropies from (3) and (5) were not so closely related, this general result would not allow a natural thermodynamical interpretation.

In the case of *ergodic* billiards, we can make not only general conclusions on increase in entropy in irreversible processes, but we can also calculate these increments. As a simple example, let us consider the case where a collisionless medium is initially enclosed in the portion $M_- \subset M$ (regions M_- and $M \setminus M_-$ are separated with a wall), being in statistical equilibrium. After removal of the wall, the medium expands irreversibly, tending to fill the whole region M . During this, its internal energy (and, consequently, its temperature) does not change. According to (3), the entropy gets a positive

increment, proportional to logarithm of the ratio of volumes M_-/M_+ , where $M_+ = M$. This result is in a good agreement with the predictions of the phenomenological thermodynamics. We should, probably, also mention that ergodicity of the Boltzmann–Gibbs gas for a vessel shaped as a rectangular parallelepiped was ascertained by Ya. G. Sinai [6].

According to Ref. [3], the thermodynamical variable P , conjugate to the volume v , is given as

$$P = -\frac{1}{\beta} \frac{\partial \Phi}{\partial v}. \quad (7)$$

Hence,

$$P = \frac{2a}{nb} \frac{k\tau}{v}. \quad (8)$$

This is the equation of state for the considered system in statistical equilibrium. This equation is identical in form with the classical Clapeyron equation. If $f(z) = e^{-z}$, then $2a = nb$, and (8) exactly fits the Clapeyron equation for a mole of ideal gas. The physical meaning of the variable P is pressure.

Let us return to the Boltzmann–Gibbs gas of N small balls in a three-dimensional vessel with volume w . Then $n = 3N$, and we can assume that $v = w^N$. Substituting this expression into (8), we obtain an equation, which is different from the Clapeyron equation. However, there is no contradiction here, for P stands for the pressure of $3N$ -dimensional gas. The pressure p in ordinary gas, as a thermodynamical quantity, conjugate to the volume w , is given by (7), only Φ should first be presented as a function of τ and w :

$$p = -\frac{1}{\beta} \frac{\partial \Phi}{\partial w} = -\frac{1}{\beta} \frac{\partial \Phi}{\partial v} \frac{dv}{dw} = \frac{2a}{3b} \frac{k\tau}{w}.$$

For the Maxwell distribution (where $f(z) = e^{-z}$), $2a/3b = N$, and we obtain the classical ideal gas equations:

$$E = \frac{3}{2} Nk\tau, \quad pw = Nk\tau. \quad (9)$$

For non-Maxwellian distributions, the value of $2a/3b$, surely, differs from N . However, within a wide range of distributions, for large N , this value is approximately equal to N :

$$\lim_{N \rightarrow \infty} \frac{2a(N)}{3Nb(N)} = 1. \quad (10)$$

For example, this range includes distributions with densities

$$f\left(\frac{r^2}{2}\right) = g(r)e^{-r^2/2}, \quad (11)$$

where $g(r)$ is an arbitrary non-negative polynomial in r .

Indeed,

$$\int_0^\infty r^{n+\alpha-1} e^{-r^2/2} dz = \frac{1}{n+\alpha} \int_0^\infty r^{n+\alpha+1} e^{-r^2/2} dz.$$

Since $\frac{n}{n+\alpha} \rightarrow 1$ when α is fixed, this results in the limit relation (10). Recall that functions of the form (11) are referred to as partial sums of *Gram-Charlier series*, and are commonly used to approximate the distribution densities of arbitrary random variables. In the case in question, such an approximation is possible due to the well-known observation (traced as far back as to Boltzmann) that in the major portion of a high-dimensional space, any distribution is close to normal (strict formulations and discussion can be found, for example, in Ref. [7]). Since N , as a rule, is extremely large (of the order of 10^{23}) and not precisely known, we can as well use the classical equations (9) instead of $E = (a/b)k\tau$ and $pw = (2a/3b)k\tau$.

Thus, we have built a complete (nonequilibrium) theory of ideal gas within the framework of Gibbs' general approach, using the concept of weak limits of probability distributions and the result concerning the ergodic behaviour of the Boltzmann–Gibbs gas. As distinct from Boltzmann's approach, we do not use any additional assumptions (like the condition of statistical independence of double collisions). The substantial difference from Boltzmann's approach is that in our theory the gas reaches statistical (thermal) equilibrium both as $t \rightarrow +\infty$ and as $t \rightarrow -\infty$, these equilibriums being identical. This fact is fully consistent with the invertibility property of the equations of motion.

Note also that by virtue of (10), the entropy equation (3) yields the classical formula (4) for monatomic ideal gas. Besides, the statistical entropy (5) coincides (up to an additive constant) with the thermodynamical entropy (3) as $t \rightarrow \infty$.

A supplement. Particle distribution functions

Let

$$\rho_N(x_1, \dots, x_N, t) \tag{12}$$

be the distribution density of the Boltzmann–Gibbs gas, which is a system of N small identical balls enclosed in a rectangular box; x_j denotes the coordinates and momenta of the j -th ball. The function (12) satisfies the Liouville equation and the initial condition $\rho_N(x, 0)$ at $t = 0$. According to Bogolyubov (see Ref. [8]), it is useful to introduce s -particle distribution functions $\rho_s(x_1, \dots, x_s, t)$, averaging density (12) over x_{s+1}, \dots, x_N . The particle distribution functions satisfy the infinite chain of “hooked” equations, a so-called BBGKY (Bogolyubov, Born, Green, Kirkwood and Yvon) chain. Under some *additional* assumptions (specifically, that of molecular chaos “in the past”), in the case of rarefied Boltzmann–Gibbs gas, one derives the kinetic Boltzmann equation for one-particle distribution function ρ_1 . These assumptions are not self-evident, do not follow from the principles of Gibbs' statistical mechanics, and are to certain extent similar to Boltzmann's assumption of statistical independence of the balls' velocities before a double collision. Two important facts follow from the Boltzmann equation:

- 1) Boltzmann's entropy

$$- \int \rho_1 \ln \rho_1 d^6 x_1$$

monotonously increases with time, and

- 2) as $t \rightarrow +\infty$, the distribution ρ_1 tends to the Maxwell distribution.

However, these conclusions (at least, the former) cannot be *directly* verified by experiment. The matter is that the thermodynamical entropy is introduced only for equilibrium states. The ideas of determination of entropy for nonequilibrium states (like those given, for example, in Refs. [9,10]) are of methodical nature, proposing to introduce an infinite number of additional internal thermodynamical parameters. Using these ideas, the reader can find, for example, the entropy of ideal gas as a function of time as the gas is adiabatically expanding into vacuum (Joule's classical experiment).

We develop a different approach in the nonequilibrium statistical mechanics of the Boltzmann–Gibbs gas. It has nothing to do with the analysis of *additional assumptions* that can be used to close Bogolyubov's chain of equations. We evolve Gibbs' classical principles and try to avoid entirely additional assumptions of *conceptual* nature. The crucial idea of our approach is: transition to thermodynamical (statistical) equilibrium is equal to replacement of the distribution (12) with its weak limit. This idea arises very naturally when one proceeds from microscopic to macroscopic description of a dynamical system. The weak limit (as $t \rightarrow +\infty$ and $t \rightarrow -\infty$) of density (12) (if it exists) coincides with Birkhoff's average

$$\bar{\rho}(x_1, \dots, x_N).$$

Let $\varphi(x_1)$ be a test function. Then

$$\begin{aligned} \int \varphi \rho_1 d^6 x_1 &= \int \varphi \rho_N d^6 x_1 \dots d^6 x_N \xrightarrow{t \rightarrow \infty} \\ &\rightarrow \int \varphi \bar{\rho}_N d^6 x_1 \dots d^6 x_N = \int \varphi \bar{\rho}_1 d^6 x_1, \end{aligned}$$

where

$$\bar{\rho}_1(x_1) = \int \bar{\rho}_N d^6 x_2 \dots d^6 x_N. \quad (13)$$

If the initial system with $3N$ degrees of freedom is ergodic, then $\bar{\rho}_N$ is a summable function that depends only on the total energy:

$$\bar{\rho}_N = f\left(\frac{\beta m}{2}(v_1^2 + \dots + v_N^2)\right)/v \int_{\mathbb{R}^{3N}} f d^3 v_1 \dots d^3 v_N. \quad (14)$$

Here m is the mass of the points, v_j^2 is the squared velocity of the j -th ball, the parameter β has the dimension of the inverse of energy (it is introduced to non-dimensionilize the argument of f) and v is the volume of the $3N$ -dimensional configurational space of the system of N balls. The denominator in (14) makes the integral of ρ_N over the *whole* phase space equal to unity.

Formula (14) shows that every possible position of the N balls is equiprobable. This fact, noted earlier in Ref. [2], means that a *homogeneous* distribution is established in the state of thermal equilibrium. In fact, a similar conclusion follows Boltzmann's theory: the density of gas particles gets equalized and at the same time the Maxwell velocity distribution is established.

Hence, the limit one-particle distribution $\bar{\rho}_1$ does not depend on the coordinates, and therefore, the averaging in (13) can be replaced with merely averaging over the velocities v_2, \dots, v_N . As a result, the following simple expression is obtained:

$$\bar{\rho}_1(u) = \frac{\int_{\mathbb{R}^{3N-3}} f\left(\frac{1}{2\kappa}(u^2 + v_2^2 + \dots + v_N^2)\right) d^3 v_2 \dots d^3 v_N}{\int_{\mathbb{R}^{3N}} f\left(\frac{1}{2\kappa}(v_1^2 + v_2^2 + \dots + v_N^2)\right) d^3 v_1 \dots d^3 v_N} \quad (15)$$

where $\kappa = \beta/m$, $u \in \mathbb{R}^3$.

It turns out that when certain additional constraints (of *analytical* nature, not statistical) are imposed on the function f , the limit one-particle distribution function $\bar{\rho}_1$ tends to the Maxwell distribution as $N \rightarrow \infty$. That is, for nearly *any* initial distribution $\rho_N(x_1, \dots, x_N, 0)$ (even without assuming that ρ_N is symmetrical relative to x_1, \dots, x_N), the balls' velocity distribution in the state of thermal equilibrium is, to all practical purpose, normal (if, as usual, N is sufficiently large).

It should be underlined that, in such an approach, it is meaningless to speak of the rate of convergence of ρ_1 (as a function of time) to the limit distribution $\bar{\rho}_1$, since the ρ_1 itself does not tend anywhere at all. One can only speak of the rate of convergence of the *average values* of the dynamic quantities. If, for example, one takes the characteristic function of certain region inside the vessel as a test function φ , then it will be just reasonable to consider the rate of equalization of the number of balls in this region.

To derive an expression for the limit distribution (as $N \rightarrow \infty$), we put $3N = m + 2$ and transform (15):

$$\bar{\rho}_1(u) = \frac{\Gamma\left(1 + \frac{m}{2}\right) \int_0^\infty r^{m-2} f\left(\frac{u_1^2 + u_2^2 + u_3^2 + r^2}{2\kappa}\right) dr}{\pi^{3/2} \Gamma\left(1 + \frac{m-3}{2}\right) \int_0^\infty r^{m+1} f\left(\frac{r^2}{2\kappa}\right) dr}. \quad (16)$$

Here $u = (u_1, u_2, u_3)$ and Γ is the gamma function. The density, actually, depends on the velocity $|u|$. Therefore, its k -th moment

$$\int_{\mathbb{R}^3} \bar{\rho}_1(|u|)|u|^k du_1 du_2 du_3$$

is equal to

$$\begin{aligned} & 4\pi \int_0^\infty \bar{\rho}_1(x)x^{k+2} dx = \\ & = \frac{2\Gamma\left(1 + \frac{m}{2}\right) \Gamma\left(\frac{k+3}{2}\right) \int_0^\infty \xi^{m+k+1} f\left(\frac{\xi^2}{2\kappa}\right) d\xi}{\sqrt{\pi} \Gamma\left(\frac{k+m}{2} + 1\right) \int_0^\infty \xi^{m+1} f\left(\frac{\xi^2}{2\kappa}\right) d\xi}. \end{aligned} \tag{17}$$

When deriving this expression, we used (16) and the elementary properties of the gamma function.

Assume that the variable x takes all real values; then, it is natural to consider the density $\bar{\rho}_1(x)$ an even function. To simplify the notation, we put $\kappa = 1$ (or replace the function $f(z)$ with $f(\kappa z)$). Our goal is to show that, as $m \rightarrow \infty$, a distribution with the density

$$2\pi x^2 \bar{\rho}_1(x), x \in \mathbb{R} \tag{18}$$

tends to the normal one.

To this end, we accept two assumptions:

- (a) the limit (as $m \rightarrow \infty$) density (18) has a finite positive variance (the second moment, $k = 2$), and
- (b) the function f has a summable derivative.

Condition (a) is the condition of *non-degeneracy* of the limit distribution, while condition (b) is of technical nature and can probably be weakened. Besides, f should decay at infinity faster than any power function : otherwise the integrals in (16) and (17) are not defined for every m .

Putting $k = 2$ in (17), integrating by parts and using assumption (a), we obtain :

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_{-\infty}^\infty 2\pi x^4 \bar{\rho}_1(x) dx = \\ & = \lim_{m \rightarrow \infty} \frac{3 \int_0^\infty \xi^{m+3} f\left(\frac{\xi^2}{2}\right) d\xi}{m+2 \int_0^\infty \xi^{m+1} f\left(\frac{\xi^2}{2}\right) d\xi} = \\ & = -3 \lim_{m \rightarrow \infty} \frac{\int_0^\infty \xi^{m+3} f\left(\frac{\xi^2}{2}\right) d\xi}{\int_0^\infty \xi^{m+3} f'\left(\frac{\xi^2}{2}\right) d\xi} = 3c > 0. \end{aligned} \tag{19}$$

Here $3c$ is the variance, which exists due to (a).

Let us put $f + cf' = g$. Then, according to (19),

$$\int_0^\infty \xi^{m+3} g\left(\frac{\xi^2}{2}\right) d\xi \Big/ \int_0^\infty \xi^{m+3} f\left(\frac{\xi^2}{2}\right) d\xi \rightarrow 0 \tag{20}$$

as $m \rightarrow \infty$.

Now, we calculate the limit of the fourth moments ($k = 4$):

$$\lim_{m \rightarrow \infty} \frac{2\Gamma\left(1 + \frac{m}{2}\right) \Gamma\left(\frac{7}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{m}{2} + 3\right)} \frac{\int_0^\infty \xi^{m+5} f\left(\frac{\xi^2}{2}\right) d\xi}{\int_0^\infty \xi^{m+1} f\left(\frac{\xi^2}{2}\right) d\xi}. \tag{21}$$

According to (20), the integral in the numerator of (21) can be replaced with the integral

$$-c \int_0^\infty \xi^{m+5} f'\left(\frac{\xi^2}{2}\right) d\xi = c(m+4) \int_0^\infty \xi^{m+3} f\left(\frac{\xi^2}{2}\right) d\xi.$$

Using (19), it is easy to calculate the limit (21). It is equal to $1 \cdot 3 \cdot 5c^5$.

In the similar way, we prove that when $k = 2n$, the limit (17), as $m \rightarrow \infty$, is

$$(2n + 1)!!c^n. \tag{22}$$

All the odd moments are, obviously, equal to zero.

Now, let $\hat{\rho}_1$ be the density of the normal distribution in the three-dimensional Euclidean space:

$$\frac{1}{(\sqrt{2\pi}\sigma)^3} e^{-\frac{u_1^2 + u_2^2 + u_3^2}{2\sigma^2}}.$$

Then

$$2\pi x^2 \hat{\rho}_1(x) = \frac{1}{\sqrt{2\pi}\sigma^3} x^2 e^{-\frac{x^2}{2\sigma^2}}.$$

Let us calculate the variance of this distribution:

$$\int_{-\infty}^\infty \frac{x^4}{\sqrt{2\pi}\sigma^3} e^{-\frac{x^2}{2\sigma^2}} dx = 3\sigma^2.$$

The fourth moment is equal to $1 \cdot 3 \cdot 5\sigma^4$; more generally, the $2n$ -th moment is equal to

$$(2n + 1)!!\sigma^{2n}. \tag{23}$$

Formulas (22) and (23) coincide if we put $c = \sigma^2$. Hence, according to the Chebyshev–Markov moment theorem (see Ref. [11]),

$$\lim_{m \rightarrow \infty} \bar{\rho}_1(u) = \hat{\rho}(u), \quad u \in \mathbb{R}^3; \quad \sigma = \sqrt{c} \tag{24}$$

Up to now, we have been using the assumption that the function f does not depend on the number of particles. In general, of course, this is not the case, and instead of a single function f , we have a sequence of functions, f_{m+2} . Nevertheless, we can again put $g_{m+2} = f_{m+2} + cf'_{m+2}$ (provided that the

limit distribution has a finite positive variance). It is easy to show that (24) remains valid if the limit expression (20) is replaced with a more general one:

$$\int_0^\infty \xi^{m+2k+1} g_{m+2}\left(\frac{\xi^2}{2}\right) d\xi \Big/ \int_0^\infty \xi^{m+2k+1} f_{m+2}\left(\frac{\xi^2}{2}\right) d\xi \rightarrow 0$$

as $m \rightarrow \infty$ for any integer $k \geq 1$.

Recall that, for a fixed value of the total energy, the distribution (14) is called *microcanonical*. According to Maxwell and Borel, it can be transformed to the *canonical* Gibbs distribution if one assumes that the total energy is equal to NE , the average energy E of a single particle being independent of N (see Ref. [12]). Of course, this is an important assumption. A more general implementation of this idea, applied to an ensemble of weakly interacting identical subsystems, can be found, for example, in Ref. [13]. In essence, the author specifies the conditions, under which the microcanonical distribution *weakly* converges to the canonical distribution as the number of subsystems increases indefinitely. Besides, as test functions, the author uses so called *adders*, symmetrical functions of some specially chosen canonical variables. Our construction of the normal distribution is based on different ideas.

One should bear in mind that Boltzmann's and Bogolyubov's theories are not free of all these problems, either. Suppose that, at the initial time $t = 0$, the distribution ρ_N coincides with the distribution (14). The distribution (14) is stationary, and corresponds to the state of thermodynamical equilibrium (in Gibbs' approach). In Boltzmann's theory, however, density $\bar{\rho}_1(u)$ (which is given by (15)) corresponds, in the general case, to the initial nonstationary distribution and should tend, in the course of time, to the Maxwell distribution. In Bogolyubov's theory, we have a similar case: not every summable function can be readily used as the density of the initial distribution. It was supposed that the velocities in a particle system should in some remote past (when the particles were far from each other) be independent (so that every s -particle distribution function was reduced to a product of one-particle functions). It should be underlined that this remote *past* cannot be replaced with the remote *future* (for discussion, see [8]). However, in our approach, the tendency to statistical equilibrium is invariant under the time reversal.

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