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LINEAR SYSTEMS WITH A QUADRATIC INTEGRAL AND SYMPLECTIC GEOMETRY OF ARTIN SPACES†

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New relations are established between the spectrum of a linear system and the indices of inertia of its quadratic integral. A detailed investigation is made of the case in which the positive and negative indices of inertia of the quadratic integral are identical. Conditions are found under which the singular planes will be Lagrangian relative to some natural symplectic structure. They are closely related to the conditions for strong stability of a linear system. The general results are applied to the classical problem of gyroscopic stabilization. © 2004 Elsevier Ltd. All rights reserved.

1. LINEAR SYSTEMS WITH A QUADRATIC INTEGRAL AND ARTIN SPACES

Consider the linear system of differential equations

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n \tag{1.1}$$

with a non-degenerate operator $A(|A| \neq 0)$; the system is assumed to have a first integral which is a non-degenerate quadratic form

$$f = (Bx, x)/2, \quad |B| \neq 0 \tag{1.2}$$

It has been shown [1] that Eqs (1.1) are Hamiltonian. A symplectic structure ω is defined by the skew-symmetric matrix

$$\Omega = BA^{-1}(\omega(x', x'') = (\Omega x', x''))$$

and the Hamiltonian is identical with the quadratic form f:

$$i_\upsilon\omega = \omega(\upsilon, dx) = df, \quad \upsilon = Ax$$

In particular, n is even $(n = 2k)$ and, as pointed out in [2], the spectrum of the operation A is symmetrical about the real and imaginary axes.

The case when the inertia index of the quadratic form (1.2) equals $n/2 = k$ is of particular interest. If the form (1.2) is taken as a pseudo-Euclidean metric in ($\mathbb{R}^n$, then $(\mathbb{R}^n, f)$ will be an *Artin space* [3]. On the other hand, $\mathbb{R}^n$ has a natural symplectic structure ω. This enables us to generate the symplectic geometry of the Artin space. The first steps were carried out in [1], where, for $n = 4$, the question of the position of the completely singular planes relative to the three-dimensional family of Lagrangian planes was linked with the construction of the spectrum and eigenvectors of the operator A. Some results of [1] will be extended below to the case of arbitrary n.

We recall a k-dimensional plane Λ^k (containing the point $x = 0$) is said to be *Lagrangian* if $\omega(x', x'') = 0$ for all $x', x'' \in \Lambda$. A plane Λ^k is said to be *singular* if it lies entirely in the isotropic cone $\{f(x) = 0\}$. Finally, a plane Λ is said to be *invariant* if the trajectory of system (1.1) through each point of Λ lies entirely in Λ.

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Proposition 1. Singular Lagrangian planes are invariant.

Proof. We must prove that if $x \in \Lambda$, then $\dot{x} = Ax \in \Lambda$. This means that $\omega(Ax, z) = 0$ for all vectors $z \in \Lambda$. But

$$\omega(Ax, z) = (BA^{-1}(Ax), z) = (Bx, z)$$

On the other hand

$$2(Bx, z) = (B(x+z), x+z) - (Bx, x) - (Bz, z) = 0$$

by virtue of the singularity assumption, which it was required to prove.

One can prove in a similar fashion that invariant Lagrangian planes are singular.

Example. The linearized equations of motion of a mechanical system with k degrees of freedom to which potential and gyroscopic forces are applied are

$$\ddot{z} + \Gamma\dot{z} + Pz = 0, \quad z \in \mathbb{R}^k \tag{1.3}$$

where $\Gamma^T = -\Gamma$ is the matrix of gyroscopic forces and $V = (Pz, z)/2$ is the potential energy. Equations (1.3) may be written as Lagrange equations with Lagrangian

$$L = \frac{1}{2}(\dot{z}, \dot{z}) + \frac{1}{2}(\dot{z}, \Gamma z) - \frac{1}{2}(Pz, z)$$

Applying a Legendre transformation, one can change to Hamiltonian equations with a quadratic Hamiltonian

$$H = \frac{1}{2}(\dot{z}, \dot{z}) + V = \frac{1}{2}(y, y) - \frac{1}{2}(y, \Gamma z) + \frac{1}{2}(Pz, z) - \frac{1}{8}(z, \Gamma^2 z)$$

where $y = \dot{z} + \Gamma z/2$. Clearly, the inertial index of the integral H equals $k = n/2$ if the potential energy V has a strict maximum at the equilibrium position $z = 0$ (the matrix P is negative definite).

Let $\Lambda = \{y = Dz\}$ be a k-dimensional plane in $\mathbb{R}^{2k}$ containing the equilibrium state $z = y = 0$. This plane will be singular if

$$(Dz, Dz) - (Dz, \Gamma z) + (Pz, z) - (z, \Gamma^2 z)/4 = 0$$

In other words,

$$\frac{D^T D + DD^T}{2} - \frac{D^T\Gamma - \Gamma D}{2} + P - \frac{\Gamma^2}{4} = 0 \tag{1.4}$$

The plane Λ is Lagrangian (relative to the standard symplectic structure in $\mathbb{R}^{2k}$) if the matrix D is symmetric. In the case Eq. (1.4) is slightly simplified:

$$D^2 - \frac{D\Gamma - \Gamma D}{2} + P - \frac{\Gamma^2}{4} = 0 \tag{1.5}$$

As is well-known (see [4]) this is the criterion for the plane Λ to be invariant. In particular, a Lagrangian singular plane will be invariant (as stated in Proposition 1).

2. DEGREES OF STABILITY AND INDICES OF INERTIA

The *degree of stability* s of system (1.1) is the number of pairs of pure imaginary roots of the characteristic equation of the operator A (counting their multiplicities). The *degree of instability* u is the number of roots (with their multiplicities) of the characteristic equation of A that lie in the right complex half-plane. One can also define the *real degree of instability* r as the number of positive real roots of the characteristic equation. Since the spectrum of the other respect to reflection in the real axis, it follows that

$$u \equiv r \bmod 2 \tag{2.1}$$

Let $i^+(i^-)$ be the positive (negative) index of inertia of the quadratic form (1.2). Since the form is non-degenerate, $i^+ + i^- = n$. Obviously, $i^+ - i^-$ is always even. It has been shown [1] that

$$u \equiv i^- \bmod 2 \tag{2.2}$$

By (2.1), this congruence is equivalent to $r \equiv i^-$ (mod 2). In particular, if i^- is odd, the equilibrium $x = 0$ of system (1.1) is unstable. This statement generalizes a classical theorem of Thomson, which states that an equilibrium of system (1.3) with odd Poincaré degree of instability cannot be gyroscopically stabilized.

Example. Let system (1.1) and integral (1.2) depend on a parameter ε, and suppose that for small $\varepsilon < 0$ the form (1.2) is positive definite ($i^- = 0$); when $\varepsilon = 0$ it becomes degenerate, and for small $\varepsilon > 0$ its index of inertia i^- equals 1. Then system (1.1) becomes unstable as ε passes through the value zero. Note that this *stability exchange principle* is independent of the dimensionality of the phase space, and therefore (under suitable natural conditions) it also holds in the infinite-dimensional case.

We now add to the congruence (2.2) a simpler proposition regarding the degree of stability.

Theorem 1. The degree of stability is even if and only if $i^+ \equiv i^-$ (mod 4).

Corollary. If the difference between the indices of inertia $i^+ - i^-$ is not divisible by 4, there is at least one pair of pure imaginary roots.

The proof of Theorem 1 uses the fact that $|A||B| > 0$. Indeed, the matrix $\Omega = BA^{-1}$ is non-singular and skew-symmetric. Consequently, n is even and $|\Omega| > 0$. Since the spectrum of A is symmetrical about the real and imaginary axes, its characteristic polynomial $|A - \lambda E|$ is in fact a polynomial in $\mu = \lambda^2$ of degree $n/2 = k$. It has the form

$$g(\mu) = \mu^k + \ldots + g_k, \quad g_k = |A|$$

Since the quadratic form (1.2) is, by assumption, non-degenerate, $i^+ = k + m$, $i^- = k - m$, and consequently $i^+ - i^- = 2m$. Clearly, sign $|B| = (-1)i^- = (-1)^{k-m}$. Since $|A||B| > 0$, it follows that sign $g_k = (-1)^{k-m}$.

Let k be even. Then $\mu^k \to +\infty$ as $\mu \to -\infty$ and sign $g_k = (-1)^m$. Consequently, if m is even (odd), then the number s of negative roots (with multiplicities) of the polynomial g is even (odd).

Now let k be odd. Then $\mu^k \to -\infty$ as $\mu \to -\infty$ and sign $g_k = -(-1)^m$. Consequently, if m is even (odd), then s is also even (odd), which it was required to prove.

Example. Let system (1.3) have two degrees of freedom ($k = 2$) and Poincaré degree of instability one. Then $i^+ = 3$, $i^- = 1$, and so $i^+ - i^i$ is not divisible by 4. Thus, by Theorem 1, there is always a pair of pure imaginary roots. By Thomson's theorem, the other two roots will be real numbers of opposite signs.

In the typical case when the eigenvalues of the operator A are different, one can indicate simple relations among the degrees of stability and instability and indices of the quadratic integral, from which the propositions formulated above will follow. Since system (1.1) is Hamiltonian, it follows from Williamson's theorem that $\mathbb{R}^n$ is a direct sum of invariant subspaces which are skew-orthogonal (relative to the bilinear form ω), so that integral (1.2) may be represented as a sum of quadratic forms in these subspaces. These forms are usually called partial Hamiltonians. To a simple real pair of eigenvalues a, $-a$ there corresponds a partial Hamiltonian apq of signature $+-$, to a pure imaginary pair $\pm ib$ there corresponds a Hamiltonian $\pm b(p^2 + q^2)/2$ of signature $++$ or $--$, and to a quadruplet of eigenvalues $\pm a \pm ib$ there corresponds a Hamiltonian $-a(p_1q_1 + p_2q_2) + b(p_1q_2 - p_2q_1)$ of signature $++--$.

Let $s^+(s^-)$ be the number of pairs of pure imaginary eigenvalues to which correspond partial Hamiltonians of signature $++(--)$. Obviously, $s^+ + s^- = s$. Since f is non-degenerate,

$$u = 2s^+ = i^+, \quad u + 2s^- = i^- \tag{2.3}$$

This immediately implies the congruence (2.2). Subtracting the second relation of (2.3) from the first, we get

$$2(s^+ - s^-) = i^+ - i^- \tag{2.4}$$

Since the numbers $s^+ - s^-$ and $i^+ - i^-$ are of the same parity, equality (2.4) implies the conclusion of Theorem 1. Equality (2.4) also implies the useful inequality

$$|i^+ - i^-| \leq 2s \tag{2.5}$$

Example. If the conditions for the stability exchange principle to be valid are satisfied, a simple pair of real eigenvalues appears, the remaining eigenvalues remaining pure imaginary. Indeed, here $i^- = 1$, $i^+ = n - 1$. Consequently, by inequality (2.5), $s \geqslant k - 1$, where $k = n/2$. Thus $s = k - 1$.

It would be useful to extend these observations to the case of multiple roots with non-trivial Jordan cells.

3. STRONG STABILITY

An equilibrium $x = 0$ of system (1.1) is said to be *strongly stable* if the eigenvalues of the operator A are pure imaginary and different. The property of strong stability is preserved under small perturbations of system (1.1). Clearly, a strongly stable equilibrium will be stable in Lyapunov's sense. The converse is, of course, not true. However, the conditions for the pure imaginary eigenvalues of the operator A to be identical define the boundary of the stability domain.

We will now investigate the case in which the pseudo-Euclidean space $(\mathbb{R}^n, f)$ is an Artin space $(i^- = i^+)$. The collection of all $k = n/2$-dimensional planes in $\mathbb{R}^n$ that pass through the point $x = 0$ is a smooth Grassman manifold G of dimension k^2. The set of all k-dimensional Lagrangian (singular) planes is a smooth submanifold $L(S)$ in G of dimension $k(k + 1)/2$ ($k(k - 1)/2$, respectively). Since $\dim G = \dim L + \dim S$, it is natural to seek conditions under which L and S intersect.

Theorem 2. If an equilibrium of system (1.1) is strongly stable, then L and S do not intersect.

The result was proved in [1] for $n = 4$. Theorem 1 becomes false if strong stability is replaced by stability in Lyapunov's sense (for examples, see [1]).

Corollary. If a singular Lagrangian $n/2$-dimensional plane exists, the equilibrium $x = 0$ is not strongly stable.

Proof. Since the eigenvalues of the operator A are pure imaginary and distinct, and system (1.1) is Hamiltonian, canonically conjugate coordinates $p_1, \ldots, p_k, q_1, \ldots, q_k$ $(2k = n)$ exist, in which the Hamiltonian has the form

$$f = \lambda_1(p_1^2 + q_1^2)/2 + \ldots + \lambda_k(p_k^2 + q_k^2)/2 \tag{3.1}$$

where $|\lambda_j|$ is the frequency of small oscillations, with $\lambda_i^2 \neq \lambda_j^2$ (see, e.g., [5]). The Hamiltonian (3.1) is the quadratic form (1.2) expressed in the new variables. In particular, the indices of inertia of the form (1.2) and (3.1) are the same. Since the linear space $\mathbb{R}^{2k}$ with pseudo-Euclidean metric (3.1) is an Artin space, the numbers of positive and negative coefficients λ_j in (3.1) are equal. In particular, k is even, and therefore the dimensionality of the phase space must be divisible by 4.

Remark. At first glance, this last conclusion seems to contradict the example of a mechanical system with gyroscopic forces and an odd number of degrees of freedom (see Section 1). However, if $P < 0$ (only in that case does the total energy generate the structure of an Artin space), then the equilibrium of system (1.3) will be unstable by the classical Thomson theorem (since the Poincaré degree of instability is odd).

Let Λ be a Lagrangian plane. We will first consider the case in which the equation of Λ may be written in a form that is solvable for the momentum:

$$\dot{p} = Mq \tag{3.2}$$

where $M = ||m_{ij}||$ is a symmetric $k \times k$ matrix. Let us assume that the plane Λ is singular. Substituting relation (3.2) into expression (3.1) for the Hamiltonian, we arrive at the equation

$$(Jq, q) + (JMq, Mq) = 0 \tag{3.3}$$

where $J = \mathrm{diag}(\lambda_1, \dots, \lambda_k)$. Since the form (3.3) must vanish for all $q \in \mathbb{R}^k$, it follows that

$$MJM = -J \tag{3.4}$$

Note that if Eq. (3.4) is replaced by the more general one $M^TJM = -J$, it is always solvable in the larger class of non-singular $k \times k$ matrices (since the index of inertia of the form (Jq, q) is $k/2$). We must show that this equation has no symmetric solutions.

Since $\lambda_j \neq 0$, it follows that $|J| \neq 0$. Consequently, $|M| \neq 0$, and in particular M has an inverse. Multiplying Eq. (3.4) on the left and right by M^{-1}, we obtain

$$M^{-1}JM^{-1} = -J \tag{3.5}$$

Multiplying together the left- and right-hand sides of Eqs (3.4) and (3.5), respectively, we arrive at the relation

$$MJ^2M^{-1} = J^2$$

that is to say, the matrices M and J^2 commute, $MJ^2 = J^2M$.

We will show that the matrix M is also diagonal. Indeed, let

$$J^2 = \mathrm{diag}(\mu_1, \dots, \mu_k), \quad \mu_j = \lambda_j^2$$

Then

$$MJ^2 = \|\mu_j m_{ij}\|, \quad J^2M = \|\mu_i m_{ij}\|$$

Consequently, $\mu_i m_{ij} = \mu_j m_{ij}$ for all i, j. Since $\mu_i \neq \mu_j$ $(i \neq j)$, it follows that $m_{ij} = 0$ for all $i \neq j$.

Thus, $M = \mathrm{diag}(m_1, \dots, m_k)$. But then the matrix equation (3.4) reduces to the contradictory relations $\lambda_j m_j^2 = -\lambda_j$. Hence Eq. (3.4) cannot have any symmetric solutions.

We will now consider the case when the equation of the Lagrangian plane cannot be written in the form (3.2), solvable for the momenta. In the most general case, one can always choose a set of canonical coordinates

$$p_{i_1}, \dots, p_{i_m}, \quad q_{j_1}, \dots, q_{j_{k-m}} \tag{3.6}$$

is such a way that

(1) $(i_1, \dots, i_m)$ $(j_1, \dots, j_{k-m})$ is a partition of the set $(1, \dots, n)$ into two disjoint parts,

(2) the equation of the Lagrangian k-dimensional plane Λ has the form

$$\begin{aligned} p_{i_1} &= m_{1,1}q_{i_1} + \dots + m_{1,m}q_{i_m} + m_{1,m+1}p_{j_1} + \dots + m_{1,k}p_{k-m} \\ &\dots\dots\dots \\ -q_{j_{k-m}} &= m_{n,1}q_{i_1} + \dots + m_{k,m}q_{i_m} + m_{k,m+1}p_{j_1} + \dots + m_{k,k}p_{k-m} \end{aligned} \tag{3.7}$$

where $\|m_{r,s}\|$ is a symmetric $k \times k$ matrix. Note that one of the two non-intersecting parts of the set of indices $1, \dots, n$ may be empty.

Assuming that the plane Λ is singular, we substitute expressions (3.7) into formula (3.1) for the Hamiltonian and equate the resulting quadratic form to zero. As a result we again obtain Eq. (3.4) for the symmetric matrix M, which equation (as shown previously) has no symmetric solutions if no two of the numbers $\lambda_1^2, \dots, \lambda_k^2$ are equal. This completes the proof of Theorem 2.

4. SOME GENERALIZATIONS

Let us consider the more general case in which all the eigenvalues of the non-degenerate operator A are pure imaginary and (in the case of multiple eigenvalues) without Jordan cells. This case corresponds to the property of Lyapunov stability of the equilibrium position of system (1.1). In certain suitable canonical coordinates $p_1, \dots, p_k, q_1, \dots, q_k$, the Hamiltonian function again has the form (3.1). We may assume without loss of generality that

$$|\lambda_1| \le |\lambda_2| \le \dots \le |\lambda_k| \tag{4.1}$$

As before, we will assume that the index of inertia of the non-degenerate quadratic form (3.1) is k. We will now describe all cases in which there is a Lagrangian singular k-dimensional plane.

Let

$$\mu_1 = |\lambda_1| = \dots = |\lambda_{k_1}|, \quad \mu_2 = |\lambda_{k_1+1}| = \dots = |\lambda_{k_2}|$$
$$\mu_m = |\lambda_{k_{m-1}+1}| = \dots = |\lambda_k| \tag{4.2}$$

The linear space $\mathbb{R}^{2k}$ decomposes as a direct sum of m subspaces $\Pi_1, \dots, \Pi_m$ of dimensions $k_1, k_2 - k_1, \dots, k - k_{m-1}$, respectively; the space Π_j is defined by the linear relations

$$p_1 = q_1 = \dots = p_{k_j-1} = q_{k_j-1} = 0, \quad p_{k_j+1} = q_{k_j+1} = \dots = p_k = q_k = 0.$$

Clearly, all these subspaces are invariant with respect to the phase flow of the linear Hamiltonian system with Hamiltonian (3.1) (and consequently also of the original system (1.1)).

Theorem 3. System (1.1) with the integral (1.2) admits of a singular Lagrangian plane if and only if the index of inertia of the restriction of the quadratic form f to each subspace Π_j is $\dim\Pi_j/2$.

In particular, the dimensions of the subspaces $\Pi_1, \dots, \Pi_k$ must be multiples of four. If the inequalities in the chain (4.1) are strict, Theorem 3 implies Theorem 2.

Proof. We will first verify the sufficiency of the conditions. To that end, it will suffice to consider the case in which $n = 4$ and the Hamiltonian is

$$a(p_1^2 + q_1^2)/2 - a(p_2^2 + q_2^2)/2, \quad a > 0 \tag{4.3}$$

In the general case, as pointed out previously, k is a multiple of 4 and the matrix M of Eq. (3.4) may be found as a partitioned matrix with symmetric 4×4 matrices along the diagonal, defining the equations of a Lagrangian singular plane for the system with Hamiltonian (4.3).

We will describe all two-dimensional singular Lagrangian planes for the Hamiltonian system with Hamiltonian (4.3):

$$\Lambda_\alpha^\pm : p_1 = \operatorname{sh}\alpha q_1 \pm \operatorname{ch}\alpha q_2, \qquad p_2 = \pm\operatorname{ch}\alpha q_1 + \operatorname{sh}\alpha q_2$$

$$\Lambda_\infty^\pm : p_1 = \pm p_2 \quad q_1 = \mp q_2$$

where α is a real parameter. As $\alpha \to \pm\infty$. the plane $\Lambda_\alpha^\pm$ obviously tends to the Lagrangian singular plane $\Lambda_\infty^\pm$. In fact, the union of the two continuous families of planes $\Lambda_\infty^\pm$, $\alpha \in \mathbb{R}$, and the two singular planes $\Lambda_\infty^\pm$ in the four-dimensional Grassman manifold G is a topological circle $\mathbb{T}^1$ (as a hyperbola in the projective plane, it is in fact an oval).

In the general case $k = 4s$, $s \in N$, and the Lagrangian singular planes form an s-dimensional manifold parameterized by the points of an s-dimensional torus $\mathbb{T}^s$.

Necessity is proved in the same way as Theorem 2. One has to solve the matrix equation (3.4), from which it follows, in particular, that the matrices M and J^2 commute. Let $J^2 = \operatorname{diag}(\lambda_1^2, \dots, \lambda_k^2)$, where the numbers λ_j satisfy conditions (4.2). Then $M = \operatorname{diag}(M_1, M_2, \dots, M_m)$, where $M_1, M_2, \dots, M_m$ are square symmetric matrices of orders $k_1, k_2 - k_1, \dots, k - k_{m-1}$, respectively. This follows at once from a comparison of the explicit forms of the matrices MJ^2 and J^2M.

Thus, the problem reduces to checking the equations

$$M_j J_j M_j = -J_j, \quad M_j^T = M_j \tag{4.4}$$

for solvability in each of the subspaces Π_j. Note that the matrix J_j in (4.4) is diagonal, each diagonal element being one of the numbers $\pm\lambda_j(\lambda_j \neq 0)$.

It remains to remark that there is the same number of positive diagonal elements and negative ones, for otherwise (by the law of inertia) it would not be possible by the linear transformation defined by M_j to transform the quadratic form $(J_j x, x)$, $x \in \Pi_j$, to the quadratic form $-(J_j x, x)$. This completes the proof of the theorem.

5. COMPLETE INSTABILITY

System (1.1) with the maximum possible degree of instability ($u = n/2$) is said to be *completely unstable*. In that case the spectrum of the operator A has no pure imaginary eigenvalues at all.

Our main result is the following theorem.

Theorem 4. If all the eigenvalues of the operator A are simple, then a Lagrangian singular plane exists if and only if system (1.1) is completely unstable.

Equality of the eigenvalues is an exceptional phenomenon. Hence if there is at least one singular Lagrangian plane, the equilibrium $x = 0$ is almost surely unstable.

Proof. Sufficiency follows from theory of normal Williamson forms [5]. If the non-degenerate system (1.1) is completely unstable, the spectrum of the operator A contains either real parts $\pm a (a > 0)$ or complex quadruplets $\pm a \pm ib$ $(a, b > 0)$. When that is the case the Hamiltonian splits into a sum of partial Hamiltonians corresponding to the pairs and quadruplets, and system (1.1) itself is a direct product of Hamiltonian subsystems whose Hamiltonians are these partial Hamiltonians. As pointed out in Section 3, in the unstable case the indices of inertia of the partial Hamiltonians equal half the dimensions of the corresponding phase spaces. It turns out that each of these subsystems has a singular Lagrangian plane. Indeed, the partial Hamiltonian of a pair of real eigenvalues $\pm a$ is

$$apq \tag{5.1}$$

and there are therefore two such planes: $p = 0$ and $q = 0$. The following partial Hamiltonian corresponds to a quadruplet of eigenvalues $\pm a \pm ib$

$$-a(p_1q_1 + p_2q_2) + b(p_1q_2 - p_2q_1) \tag{5.2}$$

Here there are again two singular Lagrangian planes: $p_1 = p_2 = 0$ and $q_1 = q_2 = 0$. The required singular Lagrangian planes of system (1.1) are the direct products of the singular Lagrangian planes of its subsystems.

We will now prove necessity. In normal canonical Williamson coordinates, Hamiltonian of a system with simple eigenvalues is

$$(KP, Q) + (Jq, p) + (Jp, q) \tag{5.3}$$

where P and Q are sets of canonical variables corresponding to the real pairs and complex quadruplets of eigenvalues of A, and the canonical variables p and q correspond to the pairs of pure imaginary eigenvalues. The matrix J is diagonal with different diagonal elements. We shall look for Lagrangian singular planes in the form

$$\begin{Vmatrix} P \\ p \end{Vmatrix} = \begin{Vmatrix} M_1 & N \\ N & M_2 \end{Vmatrix} \begin{Vmatrix} Q \\ q \end{Vmatrix}$$

where M_1 and M_2 are certain symmetric matrices. Substituting these expressions into formula (5.3) for the Hamiltonian, we obtain a quadratic form in the coordinates Q and q

$$(RQ, Q) + (SQ, q) + (Tq, q)$$

where $T = M_2JM_2 + J$. If this form is identically zero, then, in particular, $T = 0$. Hence we obtain a quadratic matrix equation for M_2

$$M_2JM_2 = -J \tag{5.4}$$

However, by Theorem 2, this equation is contradictory, since all the elements of the diagonal matrix J are different. The case in which the equation of the Lagrangian plane is not solvable for momenta is considered as in the proof of Theorem 2.

Theorem 4 can be generalized to the case of multiple real pairs and complex quadruplets of eigenvalues of A, provided the multiple pairs of pure imaginary eigenvalues do not have Jordan cells. In that case, the conditions for the existence of singular Lagrangian planes reduce to the conditions for matrix equation (5.4) to be solvable, which were described in Theorem 3. Thus, the only case still not considered is that of multiple pairs of pure imaginary eigenvalues with non-trivial Jordan cells.

Theorem 5. If system (1.1) is completely unstable and the eigenvalues of A are simple, then the number of distinct k-dimensional singular Lagrangian planes is

$$2^{(k+r)/2} \tag{5.5}$$

where r is the real degree of instability of system (1.1).

Since $k = u$ and the numbers u, r have the same parity, $(k + r)/2$ is an integer. Formula (5.5) links the number of pairs of real eigenvalues of the operator A of a completely unstable system with the number of intersections of submanifolds L and S of the Grassman manifold G.

Proof. We will first consider the case in which all the eigenvalues of A are real: $\pm\lambda_j$, $\lambda_j \neq 0$. In particular, $r = k$. Then the Hamiltonian will be a sum of partial Hamiltonians of the form (5.1). A linear canonical transformation converts this function to the form

$$\sum_{j=1}^{k} \lambda_j \frac{p_j^2 - q_j^2}{2} \tag{5.6}$$

Since $\lambda_j \neq 0$, the index of inertia of the quadratic form (5.6) is obviously equal to k. We will seek the Lagrangian singular planes in the form $p = Mq$, where M is a symmetric $k \times k$ matrix satisfying the matrix equation

$$MJM = J, \quad J = \operatorname{diag}(\lambda_1, \dots, \lambda_k) \tag{5.7}$$

This equation is similar to Eq. (3.4) and can be solved in the same way. Since no two of the numbers $\lambda_1, \dots, \lambda_k$ are equal, the matrix M will be diagonal: $M = \operatorname{diag}(m_1, \dots, m_k)$. Consequently, Eq. (5.7) splits into k independent relations $m_j^2\lambda_j = \lambda_j$, $1 \leq j \leq k$. Since $\lambda_j \neq 0$, it follows that $m_j = \pm 1$. Thus, we have 2^k different Lagrangian singular planes

$$\Lambda = \{p, q : p_j = m_j q_j, 1 \leq j \leq k\}$$

These planes differ from one another in the combinations of signs in the equations $p_j = \pm q_j$.

In the general case, when the spectrum of A contains complex quadruplets, there will be functions of the form (5.2) among the partial Hamiltonians. In that case, too, the Hamiltonian reduces to the form of (5.6), but the corresponding canonical transformation will be complex.

We first apply a canonical transformation

$$P_1 = (p_1 - ip_2)/\sqrt{2}, \quad Q_1 = (q_1 + iq_2)/\sqrt{2}$$

$$P_2 = (p_1 + ip_2)/\sqrt{2}, \quad Q_2 = (q_1 - iq_2)/\sqrt{2}$$

In the new variables P, Q, the partial Hamiltonian (5.2) becomes

$$\lambda P_1 Q_1 + \bar{\lambda} P_2 Q_2, \quad \lambda = -a - ib, \quad \bar{\lambda} = -a + ib. \tag{5.8}$$

Further, the linear canonical transformation $P, Q \to u, v$, defined by

$$P_j = (u_j + v_j)/\sqrt{2}, \quad Q_j = (-u_j + v_j)/\sqrt{2}$$

converts Hamiltonian (5.8) to the form of (5.6),

$$\lambda(v_1^2 - u_1^2)/2 + \bar{\lambda}(v_2^2 - u_2^2)/2$$

Since no two of the eigenvalues λ_j are equal, the Lagrangian singular planes are again defined by equations of the form

$$p = \pm q, \quad v_1 = m_1 u_1, \quad v_2 = m_2 u_2, \quad m_j^2 = 1$$

But the equations $\upsilon_1 = u_1$ and $\upsilon_2 = -u_2$, as well as $\upsilon_1 = -u_1$ and $\upsilon_2 = u_2$, define non-real Lagrangian planes. On the other hand, the equations $\upsilon_j = u_j$ and $\upsilon_j = -u_j$ $(j = 1, 2)$ define real Lagrangian singular planes $q_1 = q_2 = 0$ and $p_1 = p_2 = 0$, respectively, for the system with partial Hamiltonian (5.2).

Thus, the existence of a complex quadruplet in the spectrum of the Operator A halves the number if Lagrangian singular planes. Consequently, the exponent in formula (5.5) equals $k - (k - 2)/2 = (k + 2)/2$, which it was required to prove.

Corollary. If the operator A has simple eigenvalues, the manifolds S and L are either disjoint or the number of their points of intersection lies in the range $[2^{k/2}, 2^k]$.

The lower limit $2^{k/2}$ corresponds to the case in which all the eigenvalues are combined in complex quadruplets.

Remark. If a completely unstable system has equal eigenvalues, then the number of different singular Lagrangian planes may be reduced. Let us consider as an example the case in which $k = 2$ and a pair of real eigenvalues with non-zero Jordan cell exists. The classical method of [5] reduces the Hamiltonian to the form $a(p_1q_1 + p_2q_2) + p_1q_2$. It can be shown that here there are only three Lagrangian singular planes

$$p_1 = p_2 = 0, \quad q_1 = q_2 = 0, \quad p_1 = q_2 = 0$$

6. SOLUTIONS OF THE QUADRATIC MATRIX EQUATION

Let us find the conditions for Eq. (1.5) to be solvable for the symmetric matrix D. Put $P = -M^2$, were $M = \mathrm{diag}(\mu, \ldots, \mu_k)$, with all $\mu_j > 0$. We shall look for solutions in the form of power series in a parameter ε, replacing Γ by $\varepsilon\Gamma$, and then put $\varepsilon = 1$. Thus,

$$D = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots \tag{6.1}$$

where the coefficients $D_j, j \geq 1$, are found successively from the recurrent relations

$$\begin{aligned} &D_0D_1 + D_1D_0 + (\Gamma D_0 - D_0\Gamma)/2 = 0 \\ &D_0D_2 + D_2D_0 + D_1 + (\Gamma D_1 - D_1\Gamma)/2 - \Gamma^2/4 = 0 \\ &D_0D_3 + D_3D_0 + D_1D_2 + D_2D_1 + (\Gamma D_2 - D_2\Gamma)/2 = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{6.2}$$

The unperturbed matrix D_0 satisfies the simple matrix equation $D_0^2 = M^2$. It has 2^k distinct solutions: $D_0 = \mathrm{diag}(\pm\mu_1, \ldots, \pm\mu_k)$. The solutions differ in the combinations of signs of the diagonal elements. This simple observation corresponds to the conclusion of Theorem 5: when there are no gyroscopic forces, all the eigenvalues of system (1.3) are real if $P < 0$.

Thus, let $D_0 = \mathrm{diag}(d_1, \ldots, d_k)$, where $d_j = \pm\mu_j$.

Lemma 1. If $d_i + d_j \neq 0$ for all $1 \leq i, j \leq k$, then the equation $D_0X + XD_0 = Y$ is solvable for X in the class of symmetric matrices, with

$$\|X\| \leq c\|Y\|, \quad c \leq \max|d_i + d_j|^{-1} \tag{6.3}$$

where $\|\cdot\|$ is any matrix norm.

Indeed, if $Y = \|y_{ij}\|$ and $X = \|x_{ij}\|$, then

$$x_{ij} = y_{ij}/(d_i + d_j)$$

Note that the condition of the lemma is surely satisfied if no two of the numbers $\mu_1, \ldots, \mu_k$ are equal. It also holds in the case when $D_0 = M$ or $D_0 = -M$.

Lemma 1 guarantees the solvability of the sequence of relations (6.2) with respect to $D_1, D_2, \ldots$. Let D_1 be a solution of the first equation in the sequence (6.2). Put

$$\|D_1 - \Gamma/2\| = d^-, \quad \|D_1 + \Gamma/2\| = d^+, \quad 2d = d^+ + d^-$$

The other equations of the sequence may be written in the form

$$\begin{aligned} &D_0D_2 + D_2D_0 + (D_1 + \Gamma/2)(D_1 - \Gamma/2) = 0 \\ &D_0D_3 + D_3D_0 + (D_1 + \Gamma/2)D_2 + D_2(D_1 - \Gamma/2) = 0 \\ &D_0D_4 + D_4D_0 + D_2^2 + (D_1 + \Gamma/2)D_3 + D_3(D_1 - \Gamma/2) = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \tag{6.4}$$

Hence we obtain successively

$$\begin{aligned} &\|D_2\| \le cd^+d^- \le c(d^+ + d^-)^2/4 = cd^2 \\ &\|D_3\| \le c\|D_2\|(d^+ + d^-) \le 2c^2d^2 \\ &\dots\dots\dots\dots\dots\dots\dots \\ &\|D_m\| \le \kappa_m c^{m-1} d^m \end{aligned}$$

There is a recurrent rule for calculating the coefficients κ_m, $m \geqslant 1$:

$$\begin{aligned} &\kappa_2 = 1, \quad k_3 = 2, \quad \kappa_4 = \kappa_2^2 + 2\kappa_3, \quad \kappa_5 = 2\kappa_2\kappa_3 + 2\kappa_4 \\ &\kappa_6 = \kappa_2^2 + 2\kappa_2\kappa_4 + 2\kappa_5, \dots \end{aligned} \tag{6.5}$$

We introduce the function

$$g(z) = \sum_{m=1}^{\infty} \kappa_m z^m, \quad \kappa_1 = 1 \tag{6.6}$$

Lemma 2. The function f satisfies the equation $f^2 = f - z$.
The proof follows at once from formulae (6.5).
Thus,

$$g(z) = [1 - (1 - 4z)^{1/2}]/2$$

and consequently, the radius of convergence of the power series (6.6) is 1/4. This implies that when $\varepsilon = 1$ the original series (6.1) is convergence if

$$cd < 1/4 \tag{6.7}$$

In fact there are 2^k conditions (6.7) (depending on the number of solutions of the initial matrix equation $D_0^2 = -P$). Each of them is surely satisfied if the norm $||\Gamma||$ is small. Indeed, by the first equation of (6.2) and Lemma 1, the norm $||D_1||$ is small together with $||\Gamma||$. Next, $d^\pm \leqslant ||D_1|| + ||\Gamma||/2$. Thus $d = (d^+ + d^-)/2 \to 0$, if $||\Gamma|| \to 0$.

Theorem 6. Suppose no two of the numbers $\mu_1, \dots, \mu_k$ are equal and all 2^k conditions (6.7) are satisfied. Then all the eigenvalues of the linear system (1.3) are real.

Proof. We again replace Γ by $\varepsilon\Gamma$ and let the parameter ε vary in the range [0, 1]. Then the coefficients of the characteristic equation

$$|\lambda^2 E + \lambda\varepsilon\Gamma + P| = 0 \tag{6.8}$$

will be analytic functions of ε. We first observe that, for almost all $\varepsilon \in [0, 1]$ (except possibly a finite number), the roots of the characteristic equation are all simple.

Indeed, the discriminant of the characteristic polynomial (6.8) is an entire function of all its coefficients. Consequently, the discriminant will be an analytic function of the real parameter ε which is non-zero when $\varepsilon = 0$ (because, when there are no gyroscopic forces, the roots of Eq. (6.8) are different real pairs μ_j). Hence the discriminant may vanish only at finite number of points in the range [0, 1].

Now, by conditions (6.7), the 2^k solutions of the matrix equation (1.5) (with Γ replaced by $\varepsilon\Gamma$) are analytic matrix functions of ε in the range [0, 1]. These functions are pairwise distinct, since when $\varepsilon = 0$ their values are equal to the 2^k distinct solutions of the matrix equation $D_0^2 = M^2$. Consequently, for almost all $\varepsilon \in [0, 1]$, Eq. (1.5) admits of exactly 2^k distinct solutions which are symmetric $k \times k$ matrices.

Combining these arguments and applying Theorems 4 and 5, we conclude that for almost all ε all the roots of the characteristic equation (6.8) split into k different real pairs. Since these roots are continuous functions of the parameter ε, it follows that when $\varepsilon = 1$ they must still be real.

Example. It turns out that complex quadruplets of eigenvalues in a system with gyroscopic forces (1.3) already arise at $k = 2$. Put

$$\Gamma = \begin{Vmatrix} 0 & \gamma \\ -\gamma & 0 \end{Vmatrix}, \quad \Pi = \begin{Vmatrix} -a & 0 \\ 0 & -b \end{Vmatrix}, \quad a > b > 0$$

If $\gamma = 0$, then there are two real pairs $\pm\sqrt{a}$, $\pm\sqrt{b}$ of eigenvalues. All $|\gamma|$ increases, they begin to move toward each other, coming together at points $\pm(ab)^{1/4}$, when $|\gamma| = \sqrt{a} - \sqrt{b}$. Next they leave the real axis, and when $\sqrt{a} - -\sqrt{b} < |\gamma| < \sqrt{a} + \sqrt{b}$ there is a complex quadruplet of eigenvalues. When $|\gamma| = \sqrt{a} + \sqrt{b}$, the eigenvalues collide with the points $\pm i(ab)^{1/4}$ of the imaginary axis. As $|\gamma|$ continues to move, they diverge along the imaginary axis and the equilibrium becomes stable.

We will now determine the boundary beyond which the eigenvalues cease to be real, as defined by inequality (6.7). Put

$$D_0 = \mathrm{diag}(\pm\sqrt{a}, \pm\sqrt{b})$$

Then

$$D_1 = \begin{Vmatrix} 0 & \mp\dfrac{\gamma(\sqrt{a}+\sqrt{b})}{2(\sqrt{a}-\sqrt{b})} \\ \mp\dfrac{\gamma(\sqrt{a}+\sqrt{b})}{2(\sqrt{a}-\sqrt{b})} & 0 \end{Vmatrix}, \quad c = \frac{1}{\sqrt{a}-\sqrt{b}} \tag{6.9}$$

and consequently

$$d = d^{\pm} = |\gamma|\sqrt{a}/(\sqrt{a}-\sqrt{b})$$

Thus, inequality (6.7) yields the sufficient condition for the eigenvalues to be real:

$$|\gamma| < (\sqrt{a}-\sqrt{b})^2/(4\sqrt{a}) \tag{6.10}$$

It is clear that the right-hand side of this inequality does not exceed $\sqrt{a} - \sqrt{b}$, if $a \geqslant b$.

Note that if it is assumed that equality (6.9) holds, then inequality (6.7) yields the condition

$$|\gamma| < (\sqrt{a}+\sqrt{b})^2/(4\sqrt{a}) \tag{6.11}$$

which includes condition (6.10). However, by Theorem 6, the interval in which the eigenvalues are real reduces to the intersection of the intervals (6.10) and (6.11). By Theorem 1, inequality (6.10) is a sufficient condition for system (1.3) not to be strongly stable.

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