V. V. KOZLOV
Steklov Institute of Mathematics
Russian Academy of Sciences
117966 Moscow, Russia
E-mail: kozlov@pran.ru

NOTES ON DIFFUSION IN COLLISIONLESS MEDIUM

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A collisionless continuous medium in Euclidean space is discussed, i. e. a continuum of free particles moving inertially, without interacting with each other. It is shown that the distribution density of such medium is weakly converging to zero as time increases indefinitely. In the case of Maxwell's velocity distribution of particles, this density satisfies the well-known diffusion equation, the diffusion coefficient increasing linearly with time.

1. Dynamics of collisionless medium in the Euclidean space

We are going to consider a very simple object — *a collisionless continuous medium*, i. e. a continuum of free particles moving inertially, without interacting with each other. The configurational space of a particle is the n-dimensional Euclidean space $\mathbb{R}^n$ with orthogonal coordinates $x_1, \ldots, x_n$; let $\mathbb{R}^n = \{\omega_1, \ldots, \omega_n\}$ be the velocity space. The direct product $\mathbb{R}^n_x \times \mathbb{R}^n_\omega = \Gamma$ is the phase space of a free particle.

Let $\rho(\omega, x)$ be the particle distribution density at the initial time $t = 0$. We can assume that the density is normalized to the total mass of the collection of particles (the total mass is supposed to be finite). In other words, ρ is the density of some probability measure: $\rho \geqslant 0$ and

$$\int\limits_\Gamma \rho d^n\omega d^n x = 1.$$

From the very beginning, it is possible to use the probabilistic approach and treat a collisionless medium as a *Gibbs ensemble* of identical systems, where each system is a free particle in the Euclidean space $\mathbb{R}^n$.

According to the elementary principles of statistical mechanics, density ρ_t at time t is given by

$$\rho_t(\omega, x) = \rho(\omega, x - \omega t).$$

It is clear that $\rho_0 = \rho$. It is also clear that

$$u(x,t) = \int\limits_{\mathbb{R}^n} \rho(\omega, x - \omega t) d^n\omega \tag{1.1}$$

is the density of the collisionless medium at point x at time t. Our objective is to study the concentration u, its evolution and limit behavior as $t \to \pm\infty$.

In Ref. [1], the problem of evolution of a collisionless medium inside a box with mirror walls was discussed. Upon a simple regularization, this problem can be reduced to the problem with periodic boundary conditions: the configurational space $\mathbb{R}^n = \{x\}$ is factorized using the lattice $(2\pi\mathbb{Z})^n$. As a result, the phase space Γ is the direct product of torus $\mathbb{T}^n = \{x_1, \ldots, x_n mod 2\pi\}$ and $\mathbb{R} = \{\omega\}$.

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Theorem 1. *Let $\rho \in L_1(\Gamma)$ and φ be the characteristic function of a bounded measurable region $D \subset \mathbb{R}^n = \{x\}$. Then*

$$\int\limits_{\mathbb{R}^n} u(x,t)\varphi(x)d^n x = \int\limits_{D} u(x,t)d^n x \to 0 \tag{1.2}$$

as $t \to \pm\infty$.

This theorem is intuitively obvious: the particles scatter to infinity, each with its own velocity, and, therefore, their concentration in any finite region of $\mathbb{R}^n = \{x\}$ is decreasing indefinitely.

In fact, (1.2) holds for any essentially bounded measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$. Equation (1.2) has the following meaning: weak density limit ρ_t is zero as $t \to \pm\infty$. This assertion was proven in Ref. [1] for a compact configurational space. In any case,

$$\lim_{t\to\pm\infty} \int\limits_{\Gamma} \rho(\omega, x - \omega t)\varphi(x)d^n x d^n\omega = \int_{\Gamma} \bar{\rho}\varphi d^n x d^n\omega, \tag{1.3}$$

where $\bar{\rho}$ is the Birkhoff average of ρ:

$$\bar{\rho}(\omega, x) = \lim_{\tau\to\infty} \frac{1}{2\tau} \int\limits_{-\tau}^{\tau} \rho(\omega, x - \omega t)dt.$$

It is easy to calculate:

$$\bar{\rho} = \lim_{\tau\to\infty} \frac{1}{2\omega\tau} \int\limits_{x-\omega\tau}^{x+\omega\tau} \rho(\omega, \xi)d\xi = 0$$

for almost all ω, because (according to Fubini's theorem) the integral of $\rho(\omega, \xi)$ over the variable $\xi \in \mathbb{R}$ exists (and is finite) for almost all ω. Specifically, the integral (1.3) is also zero.

The question whether $u(x,t)$ itself tends to zero as $t \to \pm\infty$ is somewhat more interesting. We discuss it for the case where ρ is the product of two summable functions $h(\omega)$ and $\varphi(x)$. Then, the integral (1.1) takes the form

$$u(x,t) = \int\limits_{\mathbb{R}^n} h(\omega)\varphi(x - \omega t)d^n\omega. \tag{1.4}$$

Let us present φ in terms of Fourier transformation

$$\varphi(z) = \frac{1}{(\sqrt{2\pi})^n} \int\limits_{\mathbb{R}^n} \Phi(\xi)e^{i(z,\xi)}d^n\xi,$$

and put

$$H(z) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} h(\omega)e^{-i(z,\omega)}d^n\omega.$$

Theorem 2. *Suppose that Φ is a bounded summable function and $H \in L_1(\mathbb{R}^n)$. Then, for all $x \in \mathbb{R}^n$, function* (1.4) *tends to zero as $t \to \pm\infty$.*

Indeed, according to Fubini's theorem,

$$u = \frac{1}{(\sqrt{2\pi})^n} \int\limits_{\mathbb{R}^n} \Phi(\xi)e^{i(x,\xi)} \left[\int\limits_{\mathbb{R}^n} h(\omega)e^{-it(\omega,\xi)}d^n\omega \right] d^n\xi = \int\limits_{\mathbb{R}^n} \Phi(\xi)H(t\xi)e^{i(x,\xi)}d^n\xi.$$

When $t > 0$, this integral is equal to

$$\frac{1}{t} \int\limits_{\mathbb{R}^n} H(z)\Phi(\tfrac{z}{t})e^{i(x,z)/t}d^n z,$$

which is of order $O(1/t)$ as $t \to \infty$ (by the assumption of the theorem).

2. Heat conduction equation

The equation is

$$u_\tau = \sigma^2 \triangle u, \tag{2.1}$$

where τ is a time variable, $\triangle$ is the Laplace operator, while $\sigma = \text{const}$. Equation (2.1) is a special form of the diffusion equation; σ^2 is the diffusion coefficient.

The solution of (2.1) is well-known

$$u(x,\tau) = \frac{1}{(2\sigma\sqrt{\pi\tau})^n} \int\limits_{\mathbb{R}^n} e^{\frac{|x-\xi|^2}{4\sigma^2\tau}} \varphi(\xi) d^n\xi,\ \tau > 0, \tag{2.2}$$

where φ is the initial temperature distribution, and $|q| = q_1^2 + \ldots + q_n^2$. The function φ is customarily supposed to be *continuous* and *bounded*. The latter condition ensures convergence of integral (2.2), while the continuity property allows proving that

$$\lim_{\tau\to 0} u(x,z) = \varphi(x).$$

The point is that the exponential term in (2.2) (together with the term outside the integral) tends to the delta-function $\delta(x-\xi)$ as $\tau \to 0$.

However, the integral (2.2) also converges on the assumption of summability of φ (i. e. when $\varphi \in L_1$). It turns out that (2.2) can be presented in the form of (1.4), and this, among other things, implies that $u(x,0) = \varphi(x)$ if φ is summable.

Let $\tau = t^2/2$ and $\xi = x - \omega\tau$, where $\omega = (\omega_1, \ldots, \omega_n)$. Then, $d\xi_j = -t d\omega_j$ and

$$\begin{aligned} &\frac{1}{(\sigma\sqrt{2\pi}t)^n} \int\limits_{\mathbb{R}^n} e^{-\frac{|x-\xi|}{2\sigma^2 t}} \varphi(\xi) d^n\xi = \\ &= \frac{(-1)^n}{(\sigma\sqrt{2\pi})^n} \int\limits_{\infty}^{-\infty} \ldots \int\limits_{\infty}^{-\infty} e^{-\frac{|\omega|^2}{2\sigma^2}} \varphi(x-\omega t) d^n\omega = \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \int\limits_{\mathbb{R}^n} e^{-\frac{|\omega|^2}{2\sigma^2}} \varphi(x-\omega t) d^n\omega. \end{aligned} \tag{2.3}$$

In particular, $u(x,0) = \varphi(x)$.

Thus, if we adopt the normal law of velocity distribution

$$h(\omega) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{|\omega|^2}{2\sigma^2}}, \tag{2.4}$$

then the density $u(x,t)$ of the collisionless medium, given by the integral (1.4), satisfies the diffusion equation

$$u_t = t\sigma^2 \triangle u. \tag{2.5}$$

The diffusion coefficient $t\sigma^2$ increases indefinitely with time. As distinct from the heat conduction equation, this equation is invariant under time reversion $t \mapsto -t$. This reflects the property of reversibility of the equations of motion for a free particle. Specifically, concentration of particles at any point $x \in \mathbb{R}^n$ decreases indefinitely both as $t \to +\infty$ and as $t \to -\infty$.

From the statistical mechanics point of view, it would be more appropriate to treat the distribution (2.4) as a Maxwell distribution, dispersion σ^2 being proportional to the absolute temperature of the gas. This distribution does not vary with time (because the medium is collisionless), and the temperature field is proportional to the density of the collisionless medium (after identifying $t^2/2$ with τ).

The simple equation (2.3) is also useful for the analysis of heat propagation in $\mathbb{R}^n$. For example, let $\varphi > 0$ inside an open bounded region $D \subset \mathbb{R}^n$ and $\varphi = 0$ outside this region. Then $u(x, t) > 0$ at any point $x \in \mathbb{R}$ for arbitrarily small $t > 0$. This property immediately follows from the law of motion of a collisionless medium : however distant a point $x \in \mathbb{R}^n$ may be, it will be reached in an arbitrarily small time by very fast particles, located initially in D.

3. An example of a nonstandard diffusion equation

It would be a mistake to think that functions of the form (1.4) satisfy the diffusion equation in its well-known form. Let us put, for example, $n = 1$ and

$$h(\omega) = e^{\omega} \text{ when } \omega \leqslant 0 \quad \text{and} \quad h(\omega) = 0 \text{ when } \omega > 0.$$

Then (1.4) becomes

$$u(x,t) = \int\limits_{-\infty}^{0} e^{\omega} \varphi(x - \omega t)\, d\omega. \tag{3.1}$$

Integrating by parts yields the equation

$$u = t u_x + \varphi(x), \tag{3.2}$$

which does not contain the derivative of u_t at all. Putting $t = 0$ in (3.2), we find that $u(x, 0) = \varphi(x)$.

However, Theorem 2 cannot be applied straightforwardly because

$$H(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{0} e^{\omega} e^{-iz\omega}\, d\omega = \frac{1}{\sqrt{2\pi}(1 - iz)}$$

does not belong to $L_1(\mathbb{R})$. However, if, instead of the boundedness of $|\Phi(z)|$, we require that this function vanishes at infinity as $O(|z|^{-\alpha}), \alpha > 0$, then again we can say that the integral (3.1) tends to zero as $t \to \pm\infty$, for each $x \in \mathbb{R}$.

The function φ can be eliminated from (3.2) if we replace (3.2) by the equation

$$u_t = (t u_x)_t \tag{3.3}$$

and the Cauchy condition $u|_{t=0} = \varphi(x)$. Thus, (3.3) should also be considered a diffusion equation.

Equation (3.3) is not invariant under time reversal. This fact can be easily explained: all the particles of a collisionless medium move to the left. To have symmetry between the past and the future, one should assume that there is symmetry between "left" and "right" in the distribution of velocities. We come to a simple

Proposition. *If $h(-\omega) = h(\omega)$, then $u(x, -t) = u(x, t)$.*

Indeed,

$$u(x, -t) = \int\limits_{-\infty}^{\infty} h(\omega)\varphi(x + \omega t)\, d\omega = -\int\limits_{\infty}^{-\infty} h(-\omega')\varphi(x - \omega' t)\, d\omega' =$$

$$= \int\limits_{-\infty}^{\infty} h(\omega)\varphi(x - \omega t)\, dt = u(x, t).$$

Let, for example,

$$h(\omega) = \frac{1}{2} e^{-|\omega|}.$$

Integrating by parts, we obtain

$$\int\limits_{-\infty}^{0} e^{\omega} \varphi(x - \omega t)\, d\omega = \varphi(x) + t\varphi'(x) + t^2 \int\limits_{-\infty}^{0} e^{\omega} \varphi''(x - \omega t)\, d\omega,$$

$$\int\limits_{0}^{\infty} e^{-\omega} \varphi(x - \omega t)\, d\omega = \varphi(x) - t\varphi'(x) + t^2 \int\limits_{0}^{\infty} e^{-\omega} \varphi''(x - \omega t)\, d\omega.$$

This leads to

$$u = \varphi(x) + \frac{t^2}{2} u_{xx},$$

with $u|_{t=0} = \varphi(x)$. This equation is equivalent to the equation of evolution

$$u_t = \left(\frac{t^2}{2} u_{xx} \right)_t,$$

with the initial condition $u|_{t=0} = \varphi(x)$. Unlike (3.3), this equation is invariant under time reversal.

4. Diffusion in the compact case

Now, let the configurational space be an n-dimensional torus $\mathbb{T}^n = \{x_1, \ldots, x_n \mod 2\pi\}$. In this case, the density of a collisionless medium is also given by (1.1). As it was shown in [1], the weak limit of $u(x,t)$ as $t \to \pm\infty$ is equal to

$$\overline{u} = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{T}^n} \int\limits_{\mathbb{R}^n} \rho(\omega, x)\, d^n\omega\, d^n x. \tag{4.1}$$

Under certain additional conditions, we can state that $u(x,t) \to \overline{u}$ as $t \to \pm\infty$ for all $x \in \mathbb{R}^n$. To this end, suppose that the density $u(x,t)$ is given by the integral (1.4). Let

$$\sum \varphi_m e^{i(m,x)}, \quad m \in \mathbb{Z}^n \tag{4.2}$$

be the Fourier series of a bounded measurable function $\varphi\colon\ \mathbb{T}^n \to \mathbb{R}$.

Theorem 3. *Suppose that the series*

$$\sum |\varphi_m| \tag{4.3}$$

converges; then $u(x,t) \to \overline{u}$ as $t \to \pm\infty$ for all $x \in \mathbb{T}^n$.

Inserting (4.2) into (1.4), we obtain:

$$u(x,t) = \overline{u} + \sum_{m \neq 0} \varphi_m e^{i(m,x)} \int\limits_{\mathbb{R}^n} h(\omega) e^{-i(m,\omega)t}\, d^n\omega, \tag{4.4}$$

where

$$\overline{u} = \varphi_0 \int\limits_{\mathbb{R}^n} h(\omega)\, d^n\omega$$

coincides with (4.1). Because of the convergence of the series (4.3) the summation over m and integration over ω can be interchanged.

Let ε be an arbitrary small positive number. Since the series (4.3) converges, there exists a number N, depending on ε, such that

$$\left|\sum_{|m|>N} \varphi_m e^{i(m,x)} \int\limits_{\mathbb{R}^n} h(\omega) e^{-i(m,\omega)t}\, d^n\omega\right| \leqslant$$
$$\leqslant \left(\sum_{|m|>N} |\varphi_m|\right) \int\limits_{\mathbb{R}^n} h(\omega)\, d^n\omega$$

is less than $\varepsilon/2$. Then, the terms with indices subject to $|m| \leqslant N$ and $m \neq 0$ tend to zero as $t \to \infty$, because (by the Riemann–Lebesgue theorem) so does the integral

$$\int\limits_{\mathbb{R}^n} h(\omega) e^{-i(m,\omega)t}\, d^n\omega$$

Thus, there exists $T(\varepsilon)$ such that for $t > T$ the sum of this finite number of terms is less than $\varepsilon/2$, which is the required result.

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References

[1] *V. V. Kozlov.* Kinetics of collisionless continuous medium. Reg. & Chaot. Dyn. 2001. V. 6. №3. P. 235–251.