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MATHEMATICS

The Spectrum of a Linear Hamiltonian System and Symplectic Geometry of a Complex Artin Space

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1. SINGULAR LAGRANGIAN PLANES

Consider the linear Hamiltonian system

$$\dot{x} = IAx, \quad x \in \mathbb{R}^{2n}, \tag{1}$$

where I is the symplectic unit matrix ($I^2 = -E$) and A is a nondegenerate symmetric matrix inducing the Hamiltonian function

$$H = \frac{A(x, x)}{2}. \tag{2}$$

System (1) can be considered in the complex phase space $\mathbb{C}^{2n}$ with complex coordinates $x = (x_1, x_2, \ldots, x_{2n})$.

The spectrum of linear system (1) is invariant with respect to reflections in the real and imaginary axes. By virtue of the nondegeneracy of quadratic form (2), the spectrum can contain only real pairs, purely imaginary pairs, and quadruples of complex numbers. In the typical case where all eigenvalues of the operator IA are simple, the canonical linear transformation $(x_1, x_2, \ldots, x_{2n}) \to (p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n)$ (which is generally complex) reduces system (1) to a Hamiltonian system with Hamiltonian function

$$H = \sum_{1}^{n} \lambda_j p_j q_j. \tag{3}$$

The numbers $\pm\lambda_j$ ($1 \le j \le n$) form the spectrum of system (1). By virtue of the nondegeneracy assumption, quadratic form (3) over the field $\mathbb{C}$ is neutral. Therefore, the linear space $\mathbb{C}^{2n} = \{x\}$ with the metric determined by quadratic form (2) is an Artin space (see [1] for the geometry of Artin spaces).

Consider the set of all n-planes in $\mathbb{C}^{2n}$ that pass through the origin. They form a complex Grassmann manifold G of dimension n^2. The singular planes (contained entirely in the isotropic cone $\{H = 0\}$) form a smooth submanifold S of dimension $\frac{n(n+1)}{2}$. The set of all Lagrangian planes [on which the symplectic 2-form (Ix', x'') vanishes] is also a smooth submanifold L of dimension $\frac{n(n-1)}{2}$. We have

$$\dim G = \dim S + \dim L,$$

therefore, in the typical situation, S and L intersect in a finite set of points. It turns out that the number of intersection points of S and L and the structure of singular Lagrangian planes are closely related to the structure of the spectrum of system (1).

First, we mention the following simple proposition.

Proposition 1. *A singular Lagrangian plane is invariant for system* (1).

Similarly, invariant Lagrangian planes are singular planes.

Theorem 1. *If all eigenvalues of Hamiltonian system* (1) *are simple, then the manifolds S and L intersect in precisely* 2^n *different points.*

If the eigenvalues are multiple, then the intersection $S \cap L$ may be a continuum (see [2]).

2. THE MAIN RESULT

Let Λ be a singular Lagrangian plane. It is determined by n linearly independent (over $\mathbb{C}$) equations

$$(a, x) = a_1 x_1 + a_2 x_2 + \ldots + a_{2n} x_{2n} = 0, \quad a_j \in \mathbb{C}. \tag{4}$$

The vectors a from (4) with complex components form an n-dimensional linear space. The set of vectors from this space that have real components is, obviously, a linear subspace (over the field $\mathbb{R}$). We refer to the dimension of this subspace as the real codimension of the plane Λ and denote it by $\mathrm{codim}_{\mathbb{R}}\Lambda$. The number $2n - \mathrm{codim}_{\mathbb{R}}\Lambda = \dim_{\mathbb{R}}\Lambda$ is called the real dimension of the plane Λ in $\mathbb{C}^{2n}$. Clearly, $\dim_{\mathbb{R}}\Lambda \ge n$.

Suppose that the eigenvalues of the operator IA are simple. Suppose that r is the number of real pairs of eigenvalues $\pm a$, s is the number of purely imaginary pairs $\pm ib$, and u is the number of complex quadruples $\pm a \pm ib$. Since the operator IA is nondegenerate, we have

$$r + s + 2u = n. \tag{5}$$

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Proposition 2. *If* Λ *is a singular Lagrangian plane, then*

$$n + s \le \dim_{\mathbb{R}}\Lambda \le n + s + 2u = 2n - r.$$

In particular, if the spectrum of system (1) contains no complex quadruples, then all singular Lagrangian planes have the same real dimension.

Theorem 2. *The number of different singular Lagrangian planes with real dimension* $n + s + 2j$, *where* $0 \le j \le u$, *equals*

$$C_u^j \cdot 2^{r+s+u}. \tag{6}$$

By virtue of (5), we have

$$\sum_{j=0}^{u} C_u^j \cdot 2^{r+s+u} = 2^n,$$

this is what Theorem 1 asserts.

The main idea of the proof of Theorem 2 is as follows. It uses Williamson's theory of normal forms [3]. The phase space $\mathbb{R}^{2n}$ decomposes into a direct sum of skew-orthogonal (with respect to the standard symplectic structure $\mathbb{R}^{2n}$) invariant subspaces in such a way that Hamiltonian (2) is represented as the sum of quadratic forms on these subspaces. Such forms are called real partial Hamiltonians. In suitable canonical variables, a pair $\pm a$ of simple real eigenvalues corresponds to the partial Hamiltonian

$$apq, \tag{7}$$

a purely imaginary pair $\pm ib$, to the Hamiltonian

$$\pm\frac{b(p^2+q^2)}{2}, \tag{8}$$

and a quadruple $\pm a \pm ib$ of eigenvalues, to the Hamiltonian

$$-a(p_1q_1 + p_2q_2) + b(p_1q_2 - p_2q_1). \tag{9}$$

A singular Lagrangian plane for the system with Hamiltonian (7) has one of the forms

$$p = 0 \quad \text{and} \quad q = 0. \tag{10}$$

The Hamiltonian system with Hamiltonian (8) has two complex singular Lagrangian planes

$$p = \pm iq, \tag{11}$$

and the system with Hamiltonian (9) has four planes

$$p_1 = p_2 = 0, \quad q_1 = q_2 = 0;$$
$$p_1 = ip_2, \quad q_1 = iq_2; \quad p_1 = -ip_2, \quad q_1 = -iq_2. \tag{12}$$

Choosing one equation from each family of forms (10) and (11) and two pairs of equation from each family of form (12), we obtain a system of n linear equations in $\mathbb{R}^{2n}$ with complex coefficients; they determine n-dimensional singular Lagrangian planes. Obviously, the total number of such planes is 2^n. Theorem 1 asserts that there are no other singular Lagrangian planes provided that all eigenvalues of system (1) are simple. The real codimension of Λ equals the sum of the number of equations of form (10) and of the number of those equations of form (12) that do not contain the imaginary unit. Elementary combinatorial relations imply required formula (6).

Let us mention one of the consequences of the approach developed in this paper. Suppose that we know all singular Lagrangian n-planes of Hamiltonian system (1). Then we can calculate their real dimensions and, thereby, the number of real pairs, imaginary pairs, and complex quadruples of eigenvalues. Indeed, according to Proposition 2, the minimal (maximal) value of the real dimension equals $n + s$ $(2n - r)$. This gives the numbers s and r. The remaining number of complex quadruples are found from (5).

3. AN APPLICATION TO SYSTEMS WITH GYROSCOPIC FORCES

We illustrate the aforesaid by an example from the theory of linear oscillations of mechanical systems, which are described by the second-order equation

$$\ddot{x} + \Gamma\dot{x} + Px = 0, \quad x \in \mathbb{R}^n. \tag{13}$$

Here, $\Gamma^T = -\Gamma$ is the matrix of gyroscopic forces and P is the symmetric matrix inducing the potential energy $V = \frac{(Px, x)}{2}$. It can be assumed that P has the diagonal form $\operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$.

The Hamiltonian function coincides with the total energy $\frac{(\dot{x}, \dot{x})}{2} + V$, the canonical momenta are determined by the rule $y = \dot{x} + \frac{\Gamma x}{2}$, and the symplectic structure of the phase space $\mathbb{R}^{2n} = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ has the canonical form $\sum dy_j \wedge dx_j$. Let $\Lambda = \{y = Dx\}$ be the n-plane passing through the origin. The condition that Λ is singular and Lagrangian reduces to the matrix quadratic equation

$$D^2 - \frac{D\Gamma - \Gamma D}{2} + P - \frac{\Gamma^2}{4} = 0, \tag{14}$$

and $D^T = D$. Note that (14) is an invariance condition for the plane Λ.

Suppose that $\mu_1, \mu_2, \ldots, \mu_s > 0$ and $\mu_{s+1}, \mu_{s+2}, \ldots, \mu_n < 0$. The number $r = n - s$ is called the degree of instability (in the sense of Poincaré) of system (13). We assume that none of the numbers μ_j are equal.

First, we set $\Gamma = 0$. Then, matrix equation (14) has the 2^n different solutions

$$D_0 = \operatorname{diag}(\pm i\sqrt{\mu_1}, \ldots, \pm i\sqrt{\mu_s}, \pm\sqrt{-\mu_{s+1}}, \ldots, \pm\sqrt{-\mu_n}).$$

These matrices differ in the combinations of the signs + and –. Clearly, $u = 0$ and the real dimension of the plane $\{y = D_0x\}$ equals $2n - r = n + s$.

We seek solutions to quadratic equation (14) in the form of series in powers of the parameter ε; we replacing Γ by εΓ, after which we set $\varepsilon = 1$:

$$D = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots . \quad (15)$$

The coefficients D_j, where $j \geq 1$, are found successively from the recursive relations

$$D_0D_1 + D_1D_0 + \frac{\Gamma D_0 - D_0\Gamma}{2} = 0,$$

$$D_0D_2 + D_2D_0 + D_1^2 + (\Gamma D_1 - D_1\Gamma) - \frac{\Gamma^2}{4} = 0, \quad (16)$$

$$D_0D_3 + D_3D_0 + D_1D_2 + D_2D_1 + \frac{\Gamma D_2 - D_2\Gamma}{2} = 0,$$

....

So, suppose that $D_0 = \mathrm{diag}(d_1, d_2, \dots, d_n)$ and $d_j^2 = -\mu_j$.

Lemma 1. *If* $d_i + d_j \neq 0$ *for all* $1 \leq i, j \leq n$, *then the equation* $D_0X + XD_0 = Y$ *is solvable with respect to X in the class of complex symmetric matrices and*

$$\|X\| \leq c\|Y\|,$$

where $\|\cdot\|$ *is an arbitrary matrix norm and*

$$c \leq \max_{i,j} |d_i + d_j|^{-1}.$$

Note that the conditions of the lemma surely hold if none of the numbers $\mu_1, \mu_2, \dots, \mu_n$ are equal. This assertion ensures the solvability of the chain of relations (16) with respect to $D_1, D_2, \dots$. We set

$$\left\|D_1 - \frac{\Gamma}{2}\right\| = d^-, \quad \left\|D_1 + \frac{\Gamma}{2}\right\| = d^+, \quad 2d = d^- + d^+.$$

Theorem 3. *Series* (15) *converges for all* $|\varepsilon| \leq 1$ *if*

$$cd < \frac{1}{4}. \quad (17)$$

Note that, if pairs of real or purely imaginary eigenvalues of system (13) collide and transform into complex quadruples under a variation of the parameter ε, then the analyticity of the matrix function $\varepsilon \to D(\varepsilon)$ is violated.

Theorem 4. *Suppose that none of the numbers* $\mu_1, \mu_2, \dots, \mu_n$ *are equal and all the* 2^n *conditions of form* (17) *hold.*

Then, linear system (13) *has precisely r real pairs and* $s = n - r$ *purely imaginary pairs of eigenvalues.*

The proof of Theorem 4 uses Theorem 2 and the observation that, in the analyticity interval of the matrix function $D(\varepsilon)$, the real dimension of the planes $\{y = D(\varepsilon)x\}$ does not change. In addition, for almost all ε from this interval, the spectrum of system (13) is simple.

As an example, consider the case $n = 2$; let

$$\Gamma = \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix},$$

where $a > b > 0$. If $\gamma = 0$, then system (13) has two pairs, $\pm\sqrt{a}$ and $\pm\sqrt{b}$ of real eigenvalues. As $|\gamma|$ increases, these pairs move toward each other and merge at $|\gamma| = \sqrt{a} - \sqrt{b}$. Theorem 4 gives the following sufficient condition for the eigenvalues of system (13) to be real:

$$|\gamma| < \frac{(\sqrt{a} - \sqrt{b})^2}{4\sqrt{a}}.$$

Clearly, the right-hand side of this inequality does not exceed $\sqrt{a} - \sqrt{b}$ if $a \geq b$.

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