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ON NEW FORMS OF THE ERGODIC THEOREM

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Abstract. We present generalizations of the classical Birkhoff and von Neumann ergodic theorems, where the time average is replaced by a more general average, including some density.

1. Introduction

Suppose that the differential equation

$$\dot{x} = v(x), \qquad x \in M \tag{1.1}$$

on the smooth manifold M defines a dynamical system with an invariant measure μ. We will assume that μ is absolutely continuous with respect to a measure, generated on M by some Riemann metric and $\mu(M) < \infty$. Let g^t be the phase flow of system (1.1).

To any function f from $L_1(M, \mu)$ we put into correspondence its *Birkhoff average*

$$\overline{f}(x) = \lim_{t\to\infty} \frac{1}{T} \int_0^T f(g^t(x))\, dt.$$

It is well known that $\overline{f} \in L_1(M, \mu)$ and integrals of f and $\overline{f}$ over M coincide.

Let $\omega \mapsto h(\omega)$ be the density of some probability measure on $\mathbb{R} = \{\omega\}$. This is a nonnegative function from $L_1(\mathbb{R}, d\omega)$, where

$$\int_{-\infty}^{+\infty} h(\omega)\, d\omega = 1.$$

Theorem 1. *Let f be a measurable bounded function. Then for almost all x*

$$\lim_{t\to\infty} \int_{-\infty}^{+\infty} h(\omega) f(g^{\omega t})(x))\, d\omega = \overline{f}(x). \tag{1.2}$$

Theorem 2. *If $f \in L_1(M, \mu)$ is arbitrary then convergence* (1.2) *holds only in mean* (*i.e., with respect to the $L_1(M, \mu)$-norm*):

$$\lim_{t\to\infty} \int_M \left| \int_{-\infty}^{+\infty} h(\omega)[f(g^{\omega t}(x)) - \overline{f}(x)]\, d\omega \right| d\mu = 0.$$

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Theorem 3. *If $f_1, f_2 \in L_2(M, \mu)$, then*

$$\lim_{t\to\infty} \int_{-\infty}^{+\infty} h(\omega) \int_M f_1(g^{\omega t}(x)) f_2(x)\, d\mu\, d\omega = \int_M \overline{f}_1 f_2\, d\mu. \tag{1.3}$$

Remark 1. *Theorems* 1–3 *include as particular cases classical theorems of ergodic theory. Indeed, let h be the density of the uniform distribution on the unit interval $0 \le \omega \le 1$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be an arbitrary integrable function. Then, obviously,*

$$\int_{-\infty}^{+\infty} h(\omega)\varphi(\omega t)\, dt = \frac{1}{t}\int_0^t \varphi(s)\, ds.$$

Consider the case where the flow of system (1.1) is ergodic. Then equality (1.3) turns into

$$\int_{-\infty}^{+\infty} h(\omega)\Big[\int_M f_1(g^{\omega t}(x)) f_2(x)\, d\mu\Big]\, d\omega \to \mu(M)\int_M f_1\, d\mu \int_M f_2\, d\mu. \tag{1.4}$$

In particular, let h be the density of normal distribution with dispersion σ. Then, for $\sigma \to \infty$,

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\sigma^2}} \int_M f_1(g^t(x)) f_2(x)\, d\mu\, dt \to \mu(M)\int_M f_1\, d\mu \int_M f_2\, d\mu. \tag{1.5}$$

This equation implies that when σ increases infinitely, the functions $f_1(g^t(x))$ and $f_2(x)$ become statistically independent in mean: integral of the product equals product of integrals. Particular cases of (1.4) are presented in [1].

Theorems 1–3 differ from the classical ergodic theorems in the following aspect. Time averaging is replaced here by the averaging with respect to the parameter ω with some density. The result of the averaging in the limit does not depend on the density.

Theorems 1–3 are closely related with the *summation method* S_h: we say that

$$f(t) \to \overline{f}\ (S_h)$$

as $t \to \infty$ if

$$\lim_{t\to\infty}\int_{-\infty}^{+\infty} h(\omega) f(\omega t)\, d\omega = \overline{f}.$$

Since the integral in the left-hand side equals

$$\int_{-\infty}^{+\infty} \frac{1}{t} h\Big(\frac{\alpha}{t}\Big) f(\alpha)\, d\alpha,$$

S_h is a particular form of a linear summation method with the kernel $h(\alpha/t)/t$, where t is a parameter. Theory of such methods can be found in [2]. In particular, all such methods are regular: if $f(t) \to \overline{f}$ in the usual sense, then $f(t) \to \overline{f}(S_h)$. If h is the limit of a uniformly converging sequence of piecewice-constant functions, then the method S_h includes the Cesàro method.

Some ergodic theorems, where the Cesàro average is replaced by another linear (weighted) average, are presented in [5]. Weighted average contains as a particular case the average used in (1.2). However, according to [5], other conditions should be imposed on the density h. For example, to apply [5, Theorem 2.1, Sec. 8.2] to our case, h should be monotone.

2. Proof of Theorems 1–3

We call a function $\varphi : \mathbb{R} \to \mathbb{R}$ piecewise-constant if for some finite system of intervals

$$(\omega_j, \omega_{j+1}), \qquad j = 1, \dots, N, \qquad \omega_1 = -\infty, \quad \omega_{N+1} = +\infty$$

we have $\varphi|_{(\omega_j, \omega_{j+1})} = c_j = \text{const}$. Values of φ at the points $\omega_2, \dots, \omega_N$ are arbitrary.

According to Remark 1, Theorems 1–3 hold if $h(\omega)$ is piecewise-constant. Hence for the proof, it is sufficient to approximate h by such functions. More precisely, consider a piecewise-constant function h_ε such that

$$\int_{-\infty}^{+\infty} |h(\omega) - h_\varepsilon(\omega)| \, d\omega < \varepsilon. \tag{2.1}$$

The function h_ε obviously has a compact support.

1. Theorem 1 follows from the estimate

$$\Big| \int_{-\infty}^{+\infty} h(\omega) f(g^{\omega t}(x)) \, d\omega - \int_{-\infty}^{+\infty} h_\varepsilon(\omega) f(g^{\omega t}(x)) \, d\omega \Big| < \varepsilon \sup |f|.$$

2. Consider the case of an arbitrary $f \in L_1(M, \mu)$. As we have mentioned above, the integral

$$a(x,t) = \int_{-\infty}^{+\infty} h_\varepsilon(\omega) f(g^{\omega t}(x)) \, d\omega$$

converges as $t \to \infty$ almost everywhere to

$$a(x, \infty) = \int_{-\infty}^{+\infty} h_\varepsilon(\omega) \, d\omega \, \overline{f}(x).$$

We show that such convergence takes place in mean as well. Indeed, the convergence

$$a(x,t) \to a(x, \infty) \tag{2.2}$$

almost everywhere implies the convergence in measure ([4]). Therefore, for any $\varepsilon_1, \varepsilon_2 > 0$ there exists $T > 0$ such that for $t > T$ the measure of the set

$$M_{\varepsilon_1} = \{x \in M : |a(x,t) - a(x,\infty)| > \varepsilon_1\}$$

does not exceed ε_2:

$$\mu(M_{\varepsilon_1}) \le \varepsilon_2.$$

Suppose that (2.2) does not converge in mean. Then

$$\int_M |a(x,t) - a(x,\infty)|\, d\mu(x) = \Big(\int_{M\setminus M_{\varepsilon_1}} + \int_{M_{\varepsilon_1}}\Big)|a(x,t) - a(x,\infty)|\, d\mu(x)$$

does not converge. Since the first integral in the right-hand side does not exceed $\varepsilon_2\mu(M)$, there is no convergence of

$$\int_{M_{\varepsilon_1}} |a(x,t) - a(x,\infty)|\, d\mu(x) \quad \text{as} \quad \varepsilon_1 \to 0,\ t \to \infty.$$

Therefore (since $a(x,\infty) \in L_1(M,\mu)$), there is no convergence of

$$B = \int_{M_{\varepsilon_1}} a(x,t)\, d\mu(x) \quad \text{as } \varepsilon_1 \to 0,\ t \to \infty. \tag{2.3}$$

We have

$$\begin{aligned} 0 &= \int_M a(x,t)\, d\mu(x) - \int_M a(x,\infty)\, d\mu(x) = B + B_1 + B_2,\\ B_1 &= -\int_{M_{\varepsilon_1}} a(x,\infty)\, d\mu(x),\\ B_2 &= -\int_{M\setminus M_{\varepsilon_1}} (a(x,t) - a(x,\infty))\, d\mu(x). \end{aligned}$$

We obtain a contradiction since B_1 and B_2 converge (to zero) as $\varepsilon_1 \to 0$, $t \to \infty$. Theorem 2 is proved.

3. The proof of Theorem 3 is contained in [3]. For completeness, we quote it here.

Let $(\cdot,\cdot)$ be an inner product in $L_2(M,\mu)$ and $\|\cdot\|$ be the corresponding norm. The family of operators $U^s : L_2(M,\mu) \to L_2(M,\mu)$, $U^s f = f \circ g^s$ is isometric. Therefore,

$$\begin{aligned} \Big|\int_{-\infty}^{+\infty} h(\omega)(U^{\omega t} f_1, f_2)\, d\omega - \int_{-\infty}^{+\infty} h_\varepsilon(\omega)(U^{\omega t} f_1, f_2)\, d\omega\Big| \\ \le \int_{-\infty}^{+\infty} |h - h_\varepsilon|\, d\omega\, \|f_1\|\, \|f_2\| \le \varepsilon\, \|f_1\|\, \|f_2\|, \end{aligned}$$

Hence, it is sufficient to verify the convergence of the integrals

$$J_k(t) = \int_{\omega_k}^{\omega_{k+1}} h_\varepsilon(\omega)(U^{\omega t} f_1, f_2)\, d\omega,$$

where (ω_k, ω_{k+1}) is a constancy interval for h_ε.

According to the von Neumann ergodic theorem [6],

$$J_k(t) = \frac{c_k}{t} \int_{\omega_k t}^{\omega_{k+1} t} (U^s f_1, f_2)\, ds \to c_k(\omega_{k+1} - \omega_k)(\overline{f}_1, f_2)$$

as $t \to \infty$. It remains to note that

$$\sum_1^N c_k(\omega_{k+1} - \omega_k) = \int_{-\infty}^{+\infty} h_\varepsilon(\omega)\, d\omega = \int_{-\infty}^{+\infty} h(\omega)\, d\omega + \delta,$$

where $|\delta| \le \varepsilon$.

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