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WEAK CONVERGENCE OF MEASURES IN CONSERVATIVE SYSTEMS

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Families of probability measures on the phase space of a dynamical system are considered. These measures are obtained as shifts of a given measure by the phase flow. Sufficient conditions for the existence of the weak convergence of the measures as the rate of the shift tends to infinity are suggested. The existence of such a limit leads to a new interpretation of the second law of thermodynamics. Bibliography: 5 *titles.*

1. Introduction

Let Γ be the phase space of the dynamical system generated by the differential equation

$$\dot{x} = u(x), \qquad x \in \Gamma. \tag{1.1}$$

We assume that any solution of this system can be continued to the whole time axis. Then the corresponding phase flow g^t is a well-defined one-parameter group of diffeomorphisms of the space Γ.

Following Gibbs [1], we consider, at the initial time moment $t = 0$, a probability measure μ in Γ ($\mu(\Gamma) = 1$). The flow g^t shifts the measure μ: we set $\mu_t = g^{t*}(\mu)$, where

$$g^{t*}(\mu)(D) = \mu(g^{-t}(D)) \quad \text{for any } \mu\text{-measurable set} \quad D \subset \Gamma.$$

Gibbs tried to show that as $t \to \infty$, the measures μ_t tend (in some sense) to a stationary measure, which corresponds to a heat equilibrium state. This was a motivation for introducing in Hamiltonian systems the microcanonical probability distribution, whose density depends only on the energy.

We study the problem of the *weak convergence* of the measures μ_t. This is quite natural from the viewpoint of the justification of thermodynamics – the transition to the macroscopic description of the evolution of a dynamical system.

We assume that the system (1.1) preserves a measure ν on Γ and the measure μ has a density $\rho \in L_2(\Gamma, \nu)$, i.e., $d\mu = \rho d\nu$. This allows us to determine the mean values for functions from the space $L_2(\Gamma, \nu)$.

Since the measure ν is invariant, the density ρ_t of the measure μ_t has the form

$$\rho_t(x) = \rho \circ g^{-t}(x).$$

Given a measure $\overline{\mu}$ with density $\overline{\rho} \in L_2(\Gamma, \nu)$, the measures μ_t are said to *converge weakly* to $\overline{\mu}$ (as $t \to \infty$) if for any $\varphi \in L_2(\Gamma, \nu)$, the function

$$K(t) = (\rho \circ g^{-t}, \varphi) = \int_\Gamma \varphi(x) \rho \circ g^{-t}(x)\, d\nu(x)$$

converges to $\int_\Gamma \varphi \overline{\rho}\, d\nu$.

Below we present sufficient conditions for weak convergence and describe the limit measures.

The paper is organized as follows. In Sec. 2, assuming that the weak limit exists, we present a formula for the limit measure. In Sec. 3, we introduce the class of systems (systems (3.1)) we are going to deal with. We present basic motivations and examples and discuss the main technical tool used in the proofs – a generalization of the von Neumann ergodic theorem where the time average is replaced by the average with respect to some probability measure. In Sec. 4, we specify the class of systems (so-called foliated flows) for which we prove the weak convergence of the measures μ_t. Typical systems from this class are geodesic flows and other Hamiltonian quasi-homogeneous systems. Proofs of the main results are given in Sec. 5. Finally, in Sec. 6 we prove that the entropy of the limit measure $\overline{\mu}$ is not less than the entropy of the original measure μ.

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2. Weak limit

Theorem 1. *Assume that for a function $\varphi \in L_2(\Gamma, \nu)$, there exists the limit*

$$\lim_{t\to\infty} K(t) = K_\infty.$$

Then $K_\infty = (\overline{\rho}, \varphi)$, where

$$\overline{\rho}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \rho \circ g^t(x)\, dt. \tag{2.1}$$

Relation (2.1) holds almost everywhere on Γ. Since $\rho \in L_1$, it follows from the Birkhoff ergodic theorem that $\overline{\rho}$ is defined almost everywhere, nonnegative, and g^t-invariant, and (if Γ is compact)

$$\int_\Gamma \overline{\rho}\, d\nu = 1.$$

Hence $\overline{\rho}$ is the density of a stationary probability measure.

If $\lim K(t)$ exists for any $\varphi \in L_2(\Gamma, \nu)$, then the function $\overline{\rho}$ satisfying the equation $K_\infty = (\overline{\rho}, \varphi)$ is unique. This implies the following corollary.

Corollary 2.1. *If the measures μ_t converge weakly to $\overline{\mu}$, then the corresponding density $\overline{\rho}$ satisfies* (2.1).

Proof of Theorem 1. Let $(\rho_t, \varphi) \to K_\infty$ as $t \to \infty$. Then, by the Cauchy theorem,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\rho_t, \varphi)\, dt = K_\infty.$$

Note that the integral in the left-hand side exists for all T if $\rho, \varphi \in L_2(\Gamma, \nu)$ (see, for example, [4]). By the Fubini theorem,

$$\frac{1}{T} \int_0^T (\rho_t, \varphi)\, dt = \int_\Gamma \widetilde{\rho}_T(x)\varphi(x)\, d\nu(x),$$

where

$$\widetilde{\rho}_T = \frac{1}{T} \int_0^T \rho \circ g^t\, dt.$$

By the von Neumann theorem, $\int_\Gamma (\widetilde{\rho} - \overline{\rho})\, d\nu \to 0$ as $T \to \infty$. This implies that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\rho_t, \varphi)\, dt = (\overline{\rho}, \varphi).$$

Indeed,

$$\Big[\int_\Gamma (\widetilde{\rho}_T - \overline{\rho})\varphi\, d\nu\Big]^2 \le \int_\Gamma (\widetilde{\rho}_T - \overline{\rho})^2\, d\nu \int_\Gamma \varphi^2\, d\nu \to 0$$

as $T \to \infty$. □

3. Generalization of the statistical ergodic theorem

We consider the problem of existence of the weak limit of the densities ρ_t for dynamical systems of the following form:

$$\dot{z} = v(z, \omega), \qquad \dot{\omega} = 0. \tag{3.1}$$

The phase space Γ is the cross product $\Lambda \times D$, where $\Lambda = \{z_1, \dots, z_n\}$ is a smooth n-dimensional manifold and D is a domain in $\mathbb{R}^m = \{\omega_1, \dots, \omega_m\}$. The coordinates ω are constants of motion. We assume that for a fixed ω, the system on Λ has an invariant measure $d\nu = \lambda(z, \omega)\, d^{2n}z$.

As a particular case, we obtain a Hamiltonian system. Here $m = 1$, Λ is an energy level, and ω is the value of the energy. The phase space Γ of the Hamiltonian system breaks into the cells $h_1 \le H \le h_2$, where the function H has no critical values in the interval (h_1, h_2).

First consider the particular case of (3.1) where the field $v(z, \omega)$ is the product $\omega v(z)$, with $v(z)$ being a smooth vector field on Λ. The phase flow of this system is the family of transformations $\{g^{\omega t}\}$, where $\{g^t\}$ is the flow of the dynamical system

$$\dot{z} = v(z), \qquad z \in \Lambda.$$

Its invariant measure ν does not depend on the parameter ω.

Theorem 2. *Let f_1 and f_2 be functions from $L_2(\Lambda, \nu)$, where the measure ν is absolutely continuous with respect to the measure determined on Λ by some Riemannian metric and $\nu(\Lambda) < \infty$; and let $h \in L_1(I, d\omega)$ be an integrable function on a measurable set $I \subset \mathbb{R} = \{\omega\}$.*

Then

$$\lim_{t\to\infty} \int_I h(\omega)(f_1 \circ g^t, f_2)\, d\omega = (\overline{f}_1, f_2) \int_I h(\omega)\, d\omega. \tag{3.2}$$

Theorem 2 contains, as a particular case, the classical von Neumann theorem (see [4]). Indeed, let h be the indicator of the interval $[0,1]$, i.e., $h(\omega) = 1$ for $\omega \in [0,1]$ and $h(\omega) = 0$ for $\omega \notin [0,1]$. Then for any integrable function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\int_{-\infty}^{+\infty} h(\omega)\varphi(\omega t)\, d\omega = \frac{1}{t}\int_0^t \varphi(s)\, ds.$$

Formula (3.2) looks simple in the case of an ergodic flow. Let h be the density of a probability measure on $\mathbb{R} = \{\omega\}$. Then

$$\lim_{t\to\infty} \int_{-\infty}^{+\infty} \int_\Lambda h(\omega) f_1(g^{\omega t}(x)) f_2(x)\, d\nu\, d\omega = \nu(\Lambda) \int_\Lambda f_1\, d\nu \int_\Lambda f_2\, d\nu. \tag{3.3}$$

Hence, in average, the functions

$$f_1(g^{\omega t}(x)) \quad \text{and} \quad f_2(x)$$

are statistically independent for large t: the integral of the product equals the product of the integrals. Particular cases of (3.3) are presented in [3].

The proof of Theorem 2 is based on the von Neumann ergodic theorem. For any $\varepsilon > 0$, there exists a piecewise constant function $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that

(1) $h_\varepsilon(\omega) = c_k = \text{const}$ on the intervals (ω_k, ω_{k+1}), $k = 1, \dots, N$ (the cases $\omega_1 = -\infty$ and $\omega_{N+1} = +\infty$ are allowed);

(2) $I \subset (\omega_1, \omega_{N+1})$;

(3) $\int_I |h - h_\varepsilon|\, d\omega < \varepsilon$.

Then, using the g^t-invariance of the measure ν, we obtain

$$\Big| \int_I h(\omega)(f_1 \circ g^t, f_2)\, d\omega - \int_I h_\varepsilon(\omega)(f_1 \circ g^t, f_2)\, d\omega \Big| \\ \le \int_I |h - h_\varepsilon|\, d\omega\, ||f_1||\, ||f_2|| \le \varepsilon\, ||f_1||\, ||f_2||,$$

where $||\cdot||$ is the L_2-norm. Hence it is sufficient to establish the convergence of the integrals

$$J_k(t) = \int_{\omega_k}^{\omega_{k+1}} h_\varepsilon(\omega)(f_1 \circ g^t, f_2)\, d\omega.$$

By the von Neumann ergodic theorem,

$$J_k(t) = \frac{c_k}{t} \int_{\omega_k t}^{\omega_{k+1} t} (f_1 \circ g^t, f_2)\, ds \to c_k(\omega_{k+1} - \omega_k)(\overline{f}_1, f_2)$$

as $t \to \infty$. It remains to observe that

$$\sum_1^N c_k(\omega_{k+1} - \omega_k) = \int_I h_\varepsilon(\omega)\, d\omega = \int_I h(\omega)\, d\omega + \delta,$$

with $|\delta| \le \varepsilon$. $\square$

4. Limit measures of foliated flows

In this section, we apply the method of Sec. 3 to dynamical systems with *foliated* phase flows. These systems are particular cases of (3.1), where ω is one-dimensional ($m = 1$). Their basic property is as follows: flows on Λ are conjugated for different ω after a proper change of time.

Let $I \in \mathbb{R}$ be an interval (maybe infinite), and let Λ be a smooth manifold. In the phase space $\Gamma = \Lambda \times I$, consider the dynamical system (3.1), where $z \in \Lambda$ and $\omega \in I$. We put

$$P_\gamma = \{(z,\, \omega) \in \Gamma : \omega = \gamma\}.$$

These n-dimensional manifolds are invariant. The map $\psi_\omega : (z,\, \omega) \to z$ determines a natural diffeomorphism of P_ω and Λ. The vector field v of the system (3.1) is tangent to P_ω at points $(z,\, \omega) \in \Gamma$. Let v_ω denote the restriction of v to P_ω.

Let $\{g^t\}$ be the phase flow on Γ generated by the system (3.1), and let $\{g^t_\omega\}$ be its restriction to P_ω. Since all the manifolds P_ω are diffeomorphic to Λ, we can regard $\{g^t_\omega\}$ as a one-parameter group of transformations of Λ.

Definition 4.1. *We say that the flow g^t is foliated if there exist a smooth function $\alpha : I \to (0, \infty)$ and a flow $g^s_* : \Lambda \to \Lambda$ such that the diagram*

$$\begin{array}{ccc} P_\omega & \xrightarrow{\;g^t_\omega\;} & P_\omega \\ {\scriptstyle \psi_\omega}\big\downarrow & & \big\downarrow{\scriptstyle \psi_\omega} \\ \Lambda & \xrightarrow{\;g_*^{\alpha(\omega)t}\;} & \Lambda \end{array} \tag{4.1}$$

is commutative for all $\omega \in I$ and all $t \in \mathbb{R}$. We say that a foliated flow is nondegenerate if the function $\alpha(\omega)$ has only isolated critical points.

Identifying P_ω and Λ by the diffeomorphism ψ_ω, we can present the definition of a foliated flow as follows:

$$g^t_\omega = g_*^{\alpha(\omega)t}. \tag{4.2}$$

We have the following obvious result.

Proposition 4.1. *Assume that the flow g^t_* preserves a measure ν_* on Λ, and σ is a measure on the interval I. Then the flow g^t on $\Lambda \times I$ preserves the measure $\mu = \nu_* \times \sigma$.*

Our main result is as follows.

Theorem 3. *Assume that g^t is a nondegenerate foliated flow on $\Gamma = \Lambda \times I$, the measure ν_* is absolutely continuous with respect to the measure determined on Λ by some Riemannian metric, the measure σ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, and $\nu_*(\Lambda) = 1$. Then for any $f', f'' \in L_2(\Gamma, \mu)$, there exists the limit $\lim_{t\to\infty}(f' \circ g^t, f'')$.*

Theorems 1 and 3 imply the following corollary.

Corollary 4.1. *For dynamical systems (3.1) with nondegenerate foliated phase flows, the density of the probability distribution ρ_t has weak limits as $t \to +\infty$ and $t \to -\infty$. These limits coincide and satisfy (2.1).*

Remark. Theorem 3 remains valid if $f' \in L_p(\Gamma, \mu)$ and $f'' \in L_q(\Gamma, \mu)$, where $1/p + 1/q = 1$, $1 \le p \le q \le \infty$.

We prove Theorem 3 in the next section, and now we present some examples of systems with foliated flows.

Let P_ω, $\omega > 0$, be a one-parameter family of smooth manifolds, and let $\varphi_\omega : P_1 \to P_\omega$ be a family of diffeomorphisms. The union

$$\Gamma = \cup_{\omega>0} P_\omega$$

has the structure of the cross product $\Lambda \times I$, where $\Lambda = P_1$ and $I = (0, +\infty)$. We endow Γ with the smooth structure of the cross product $\Lambda \times I$.

Definition 4.2. *A vector field v on Γ is called φ_ω-homogeneous of degree k if for some vector field v_1 on P_1,*

$$v \circ \varphi_\omega = \omega^k (D\varphi_\omega) v_1 \tag{4.3}$$

for all $\omega > 0$. (Here D stands for the differential.)

Since v_1 is tangent to P_1 and $\varphi_\omega(P_1) = P_\omega$, the field v is tangent to the leaves P_ω, $\omega > 0$.

Example. Let $(M, \langle\, ,\rangle)$ be a smooth Riemannian manifold, and let $\langle\, ,\rangle^*$ be the metric conjugated to the metric $\langle\, ,\rangle$. Let p be an element of the conjugated space T_q^*M (a momentum of the corresponding mechanical system). We put $||p||^2 = \langle p,\, p\rangle^*$ and

$$P_s = \{p \in T_q^*M : q \in M,\ ||p|| = s\}.$$

It is clear that $\Gamma = T^*M \setminus P_0$.

Let v be a Hamiltonian vector field on Γ determined by the standard symplectic structure $\sum dp_i \wedge dq_i$ and the Hamiltonian $H = ||p||^2/2$. This field generates a dynamical system on Γ, the geodesic flow. The corresponding phase flow g^t is well defined for all $t \in \mathbb{R}$ provided that $(M, \langle\, ,\rangle)$ is a complete Riemannian manifold (the length of any geodesic is infinite).

Let (q, p) be a point of Γ. We put

$$\varphi_\omega(q, p) = (q, \omega p), \qquad \omega > 0.$$

Then the vector field v determined by the Hamiltonian equations

$$\dot{q} = \partial H/\partial p, \qquad \dot{p} = -\partial H/\partial q \tag{4.4}$$

is φ_ω-homogeneous of degree $k = 1$. Indeed, the structure of equations (4.4) is such that $\dot{p}$ is quadratic in p and $\dot{q}$ is linear in p.

Example. Consider a Hamiltonian system (4.4) in $\mathbb{R}^{2n} = \{q_1, \ldots, q_n,\, p_1, \ldots, p_n\}$ with Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n}(p_i^2 + \alpha_i^2 q_i^2), \qquad \alpha_i > 0$$

(a harmonic oscillator with frequencies $\alpha_1, \ldots, \alpha_n$). We put again

$$P_\omega = \{(p, q) \in \mathbb{R}^{2n} : H(p, q) = \omega\}, \qquad \omega > 0,$$

and $\varphi_\omega : (q, p) \to (\omega q, \omega p)$. The corresponding Hamiltonian vector field on $\Gamma = \cup_{\omega>0} P_\omega$ is φ_ω-homogeneous of zero degree ($k = 0$).

Proposition 4.2. *Let v be a φ_ω-homogeneous vector field of degree k on Γ. Then v generates on Γ a foliated flow g^t, where $\alpha(\omega) = \omega^k$.*

Corollary 4.2. *For $k \neq 0$, the flow g^t is nondegenerate.*

Proof of Proposition 4.2. We put $\Lambda = P_1$. Note that any point $z \in \Gamma$ can be presented in the form

$$z = \varphi_\omega(z_1), \qquad \omega > 0, \quad z_1 \in P_1. \tag{4.5}$$

Since the map $\Lambda \times (0, +\infty) \to \Gamma$ determined by (4.5) is a diffeomorphism, it is sufficient to check that

$$g^t \circ \varphi_\omega = \varphi_\omega \circ g_1^{\omega^k t},$$

where g_1^t is the restriction of the flow g^t to P_1.

This equation obviously holds for $t = 0$. Differentiating with respect to t at $t = 0$ yields

$$\frac{d}{dt}\Big|_{t=0} g^t \circ \varphi_\omega = v \circ \varphi_\omega,$$
$$\frac{d}{dt}\Big|_{t=0} \varphi_\omega \circ g_1^{\omega^k t} = \omega^k (D\varphi_v) v_1,$$

where v_1 is the restriction of v to P_1. This equation follows from (4.3). The same equation for an arbitrary t follows from the fact that g^t is a group with respect to t. $\square$

5. Foliated flows and evolution of measures

In this section, we prove Theorem 3. Let f' and f'' be functions from $L_2(\Gamma,\mu)$, $\Gamma = \Lambda \times I$. We put

$$f'_\omega(\cdot) = f'(\cdot,\omega), \qquad f''_\omega(\cdot) = f''(\cdot,\omega).$$

By the Fubini theorem,

$$(f' \circ g^t, f'') = \int_I (f'_\omega \circ g^t_\omega, f''_\omega)\, d\sigma = \int_I (f'_\omega \circ g_*^{\alpha(\omega)t}, f''_\omega)\, d\sigma. \tag{5.1}$$

Let $A', A'' \subset \Lambda$ be measurable sets, I', I'' intervals in I, and $\chi', \chi'' : \Gamma \to \mathbb{R}$ the indicators of the sets $A' \times I'$ and $A'' \times I''$, respectively. We put

$$J(t) = (\chi' \circ g^t, \chi'').$$

Main lemma. *There exist the limits* $\lim_{t\to+\infty} J(t)$ *and* $\lim_{t\to-\infty} J(t)$.

The main lemma implies Theorem 3. Indeed, recall that ν_* is absolutely continuous with respect to the measure on Λ determined by some Riemannian metric and σ is absolutely continuous with respect to the Lebesgue measure. Therefore the space $C^0_0(\Gamma)$ of continuous functions with compact supports is dense in $L_2(\Gamma,\,\mu)$. On the other hand, the space $\mathbf{I}(\Gamma)$ of functions that are finite linear combinations of the indicators of μ-measurable sets is dense in $C^0_0(\Gamma)$ (even in the C^0-norm).

Now let f' and f'' be two functions from $L_2(\Gamma,\,\mu)$. To prove the existence of the limit

$$\lim_{t\to+\infty} (f' \circ g^t, f''),$$

it is sufficient to show that given $\varepsilon > 0$, the difference

$$(f' \circ g^{t_1}, f'') - (f' \circ g^{t_2}, f'')$$

is less than ε for all $t_1, t_2 > T(\varepsilon)$. This follows from the main lemma and the fact that the functions f' and f'' can be approximated with arbitrary precision by functions from $\mathbf{I}(\Gamma)$.

Now let us prove the main lemma. We put $A_0 = A' \cap A''$ and $I_0 = I' \cap I''$. Let $\widetilde{\chi}_0 : \Lambda \to \mathbb{R}$ be the indicator of A_0. It is clear that

$$J(t) = \int_{I_0} (\widetilde{\chi}_0 \circ g_*^{\alpha(\omega)t}, \widetilde{\chi}_0)\, d\sigma.$$

Let $D_\gamma = \{\omega \in I_0 : |\alpha'(\omega)| > \gamma\}$, $\alpha' = d\alpha/d\omega$. Since the critical points of the function $\alpha(\omega)$ are isolated, the σ-measure of the set $I \setminus D_\gamma$ tends to zero as $\gamma \to 0$. Furthermore, we may regard D_γ as the union of a finite set of intervals. Let $(\omega_1,\,\omega_2)$ be one of these intervals. Then α can be regarded as a coordinate on (ω_1, ω_2). Indeed, there exists a smooth function $\omega(\alpha)$ inverse to $\alpha : (\omega_1, \omega_2) \to \mathbb{R}$. We put $d\sigma(\omega) = h(\omega) d\omega$. According to the conditions of Theorem 3, the function $h(\omega)$ is integrable: $h \in L_1(I, d\omega)$. Hence

$$\int_{\omega_1}^{\omega_2} (\widetilde{\chi}_0 \circ g_*^{\alpha(\omega)t}, \widetilde{\chi}_0)\, h(\omega)\, d\omega = \int_{\alpha(\omega_1)}^{\alpha(\omega_2)} (\widetilde{\chi}_0 \circ g_*^{\alpha t}, \widetilde{\chi}_0)\, h(\omega(\alpha))\, \omega'(\alpha)\, d\alpha. \tag{5.2}$$

Since $h(\omega(\alpha))\omega'(\alpha) \in L_1((\alpha(\omega_2), \alpha(\omega_1)), d\alpha)$, we see that, by Theorem 2, the integral (5.2) has a limit as $t \to \infty$. $\square$

6. Increase of the entropy

The entropy is defined in statistical mechanics as the integral

$$S_t = -\int_\Gamma \rho_t \log \rho_t\, d\mu.$$

Since $\rho_t(z) = \rho \circ g^{-t}(z)$ and the flow g^t preserves the measure μ, we have $S_t = \text{const}$. This remark is a particular case of the Poincaré result (see [2]).

On the other hand, ρ_t weakly converges to $\overline{\rho}$ as $t \to \pm\infty$. Following Gibbs and Poincaré, we can say that the stationary density of the probability distribution $\overline{\rho}$ corresponds to a *heat equilibrium* of the system in question. Thus it is natural to introduce the entropy of the equilibrium state:

$$S_\infty = -\int_\Gamma \overline{\rho} \log \overline{\rho}\, d\mu.$$

Theorem 4.

$$S_t \leq S_\infty. \tag{6.1}$$

The proof is based on the fact that the function $h(x) = -x \log x$ is concave for positive x. Since $S_t = \text{const}$, by the Fubini theorem,

$$S_t = \frac{1}{T} \int_0^T \int_\Gamma h(\rho_t)\, d\mu\, dt = \int_\Gamma \Big[\frac{1}{T} \int_0^T h(\rho_t)\, dt\Big] d\mu.$$

By the Jensen inequality,

$$\frac{1}{T} \int_0^T h(\rho_t)\, dt \leq h\Big(\frac{1}{T} \int_0^T \rho_t\, dt\Big), \qquad T > 0.$$

Therefore

$$S_t \leq \int_\Gamma h\Big(\frac{1}{T} \int_0^T \rho_t\, dt\Big) d\mu.$$

It remains to consider the $T \to \infty$ limit and use Theorem 1.

Remark. For a medium without collisions in a right parallelepiped, inequality (6.1) is established in [5]. Some cases of this inequality are presented in [2, item 6].

Since ρ_t weakly converges to $\overline{\rho}$ as $t \to +\infty$ and $t \to -\infty$, the assertion of Theorem 4 concerning the entropy increase is invariant with respect to the time reversal. Recall that the kinetic Boltzmann equation implies *monotone* entropy growth as the time increases.

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