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# WEAK CONVERGENCE OF MEASURES IN CONSERVATIVE SYSTEMS

Abstract. Families of probability measures on the phase space of a dynamical system are considered. These measures are obtained as shifts of a given measure by the phase flow. Sufficient conditions for the existence of the weak convergence of the measures as the rate of the shift tends to infinity are proposed. Existence of such a limit leads to a new interpretation of the second law of thermodynamics.

## 1. Introduction

Let $\Gamma$ be the phase space of the dynamical system, generated by the differential equation

$$\dot{x} = u(x), \qquad x \in \Gamma. \tag{1.1}$$

We assume that any solutions of this system can be continued to the whole axis of time. Then the corresponding phase flow $g^t$ is a well-defined one-parametric group of diffeomorphisms of $\Gamma$.

Following Gibbs [1], we consider at the initial time moment $t = 0$ a probability measure $\mu$ in $\Gamma$ ($\mu(\Gamma) = 1$). The flow $g^t$ shifts $\mu$: $\mu_t = g^{t*}(\mu)$, where

$$g^{t*}(\mu)(D) = \mu(g^{-t}(D)) \quad \text{for any } \mu\text{-measurable} \quad D \subset \Gamma.$$

Gibbs tried to show that for $t \to \infty$ the measures $\mu_t$ tend (in some sense) to a stationary measure which corresponds to a heat equilibrium state. That was a motivation for the introduction in Hamiltonian systems the micro-canonical probability distribution, where the density depends only on the energy.

We study the problem of a *weak convergence* of the measures $\mu_t$. This is quite natural from the point of view of the justification of thermodynamics – the transition to the macroscopic description of the evolution of a dynamical system.

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We assume that the system (1.1) preserves the measure $\nu$ on $\Gamma$ and $\mu$ has a density $\rho \in L_2(\Gamma, \nu)$ i.e., $d\mu = \rho d\nu$. This gives us a possibility to determine mean values for functions from the space $L_2(\Gamma, \nu)$.

Since the measure $\nu$ is invariant, the density $\rho_t$ of the measure $\mu_t$ has the form

$$\rho_t(x) = \rho \circ g^{-t}(x).$$

Given a measure $\overline{\mu}$ with the density $\overline{\rho} \in L_2(\Gamma, \nu)$. The measures $\mu_t$ are said to converge weakly to $\overline{\mu}$ (as $t \to \infty$) if for any $\varphi \in L_2(\Gamma, \nu)$ the function

$$K(t) = (\rho \circ g^{-t}, \varphi) = \int\limits_{\Gamma} \varphi(x)\rho \circ g^{-t}(x)\, d\nu(x)$$

converges to $\int_\Gamma \varphi\overline{\rho}\, d\nu$.

Below we present sufficient conditions for weak convergence and describe the limit measures.

The plan of the paper is as follows. In §2, assuming that the weak limit exists, we present a formula for it. In §3 we introduce the class of systems (systems (3.1)) we are going to deal with. We present basic motivations and examples, and discuss the main technical tool for the proofs – a generalization of the von Neumann ergodic theorem, where the time average is replaced by the average with respect to some probability measure. In §4 we specify the class of systems (the so called foliated flows), for which we prove weak convergence of the measures $\mu_t$. Typical systems from this class are geodesic flows and other Hamiltonian quasi-homogeneous systems. We present proofs of main results in §5. Finally, in §6 we prove that entropy of the limit measure $\overline{\mu}$ is not less than entropy of the original measure $\mu$.

## 2. Weak limit

**Theorem 1.** *Suppose that for some function $\varphi \in L_2(\Gamma, \nu)$ there exists a limit*

$$\lim_{t\to\infty} K(t) = K_\infty.$$

*Then $K_\infty = (\overline{\rho}, \varphi)$, where*

$$\overline{\rho}(x) = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho \circ g^t(x)\, dt. \qquad (2.1)$$

The equation (2.1) holds almost everywhere on $\Gamma$. Since $\rho \in L_1$, by the Birkhoff ergodic theorem $\overline{\rho}$ is defined almost everywhere, non-negative, $g^t$-invariant, and (if $\Gamma$ is compact)

$$\int_\Gamma \overline{\rho}\, d\nu = 1.$$

Hence, $\overline{\rho}$ is the density of a stationary probability measure.

If $\lim K(t)$ exists for any $\varphi \in L_2(\Gamma, \nu)$, then $\overline{\rho}$, satisfying the equation $K_\infty = (\overline{\rho}, \varphi)$, is unique. This implies

**Corollary 2.1.** *If $\mu_t$ converges weakly to $\overline{\mu}$ then the corresponding density $\overline{\rho}$ satisfies* (2.1).

**Proof of Theorem** 1. Let $(\rho_t, \varphi) \to K_\infty$ as $t \to \infty$. Then by the Cauchy theorem

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\rho_t, \varphi)\, dt = K_\infty.$$

Note that the integral in the left-hand side exists for all $T$ if $\rho, \varphi \in L_2(\Gamma, \nu)$ (see for example, [4]). By the Fubini theorem

$$\frac{1}{T} \int_0^T (\rho_t, \varphi)\, dt = \int_\Gamma \widetilde{\rho}_T(x) \varphi(x)\, d\nu(x),$$

where

$$\widetilde{\rho}_T = \frac{1}{T} \int_0^T \rho \circ g^t\, dt.$$

By the von Neumann theorem $\int_\Gamma (\widetilde{\rho} - \overline{\rho})\, d\nu \to 0$ as $T \to \infty$. This implies that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\rho_t, \varphi)\, dt = (\overline{\rho}, \varphi).$$

Indeed,

$$\Big[ \int_\Gamma (\widetilde{\rho}_T - \overline{\rho}) \varphi\, d\nu \Big]^2 \leqslant \int_\Gamma (\widetilde{\rho}_T - \overline{\rho})^2\, d\nu \int_\Gamma \varphi^2\, d\nu \to 0,$$

as $T \to \infty$. $\bullet$

## 3. Generalization of the statistical ergodic theorem

We consider the problem of the existence of weak limit for the density $\rho_t$ for dynamical systems of the following form:

$$\dot{z} = v(z, \omega), \qquad \dot{\omega} = 0. \tag{3.1}$$

The phase space $\Gamma$ is the cross product $\Lambda \times D$, where $\Lambda = \{z_1, \dots, z_n\}$ is a smooth $n$-dimensional manifold, and $D$ is a domain in $\mathbb{R}^m = \{\omega_1, \dots, \omega_m\}$. The coordinates $\omega$ are constants of motion. We assume that for fixed $\omega$ the system on $\Lambda$ has an invariant measure $d\nu = \lambda(z, \omega)\, d^{2n} z$.

As a particular case we have a Hamiltonian system. Here $m = 1$, $\Lambda$ is an energy level, and $\omega$ is the value of the energy. The phase space of $\Gamma$ of the Hamiltonian system breaks into cells $h_1 \leqslant H \leqslant h_2$, where in the interval $(h_1,\, h_2)$ the function $H$ has no critical values.

First, consider the particular case of equations (3.1), when the field $v(z, \omega)$ is the product $\omega v(z)$, where $v(z)$ is a smooth vector field on $\Lambda$. Phase flow of this system is the family of transformations $\{g^{\omega t}\}$, where $\{g^t\}$ is the flow of dynamical system

$$\dot{z} = v(z), \qquad z \in \Lambda.$$

Its invariant measure $\nu$ does not depend on the parameter $\omega$.

**Theorem 2.** *Let $f_1, f_2$ be functions from $L_2(\Lambda, \nu)$, where the measure $\nu$ is absolutely continuous with respect to the measure, determined on $\Lambda$ by some Riemannian metric, $\nu(\Lambda) < \infty$, and $h \in L_1(I, d\omega)$ is an integrable function on the measurable set $I \subset \mathbb{R} = \{\omega\}$.*

*Then*

$$\lim_{t\to\infty} \int\limits_I h(\omega)(f_1 \circ g^t, f_2)\, d\omega = (\overline{f}_1, f_2) \int\limits_I h(\omega)\, d\omega. \tag{3.2}$$

Theorem 2 contains as a particular case the classical von Neumann theorem [4]. Indeed, let $h$ be the indicator of the interval $[0, 1]$ i.e., $h(\omega) = 1$ for $\omega \in [0, 1]$ and $h(\omega) = 0$ for $\omega \notin [0, 1]$. Then for any integrable function $\varphi : \mathbb{R} \to \mathbb{R}$

$$\int\limits_{-\infty}^{+\infty} h(\omega)\varphi(\omega t)\, d\omega = \frac{1}{t} \int\limits_0^t \varphi(s)\, ds.$$

Formula (3.2) looks simple in the case of ergodic flow. Let $h$ be the density of a probability measure on $\mathbb{R} = \{\omega\}$. Then

$$\lim_{t\to\infty} \int_{-\infty}^{+\infty} \int_{\Lambda} h(\omega) f_1(g^{\omega t}(x)) f_2(x)\, d\nu\, d\omega = \nu(\Lambda) \int_{\Lambda} f_1\, d\nu \int_{\Lambda} f_2\, d\nu. \tag{3.3}$$

Hence, in average, for large $t$ the functions

$$f_1(g^{\omega t}(x)) \quad \text{and} \quad f_2(x)$$

are statistically independent: integral of the product equals product of the integrals. Particular cases of (3.3) are presented in [3].

Proof of Theorem 2 is based on the von Neumann ergodic theorem. For any $\varepsilon > 0$ there exists a piecewise-constant function $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that

1) $h_\varepsilon(\omega) = c_k = \mathrm{const}$ on the intervals $(\omega_k, \omega_{k+1})$, $k = 1, \dots, N$ (it is possible that $\omega_1 = -\infty$ and $\omega_{N+1} = +\infty$),

2) $I \subset (\omega_1, \omega_{N+1})$,

3) $\int_I |h - h_\varepsilon|\, d\omega < \varepsilon$.

Then, using $g^t$-invariance of the measure $\nu$, we get:

$$\Big| \int_I h(\omega)(f_1 \circ g^t, f_2)\, d\omega - \int_I h_\varepsilon(\omega)(f_1 \circ g^t, f_2)\, d\omega \Big| \leqslant \int_I |h - h_\varepsilon|\, d\omega\, \|f_1\|\, \|f_2\| \leqslant \varepsilon\, \|f_1\|\, \|f_2\|,$$

where $\|\cdot\|$ is the $L_2$ norm. Hence, it is sufficient to establish convergence of the integrals

$$J_k(t) = \int_{\omega_k}^{\omega_{k+1}} h_\varepsilon(\omega)(f_1 \circ g^t, f_2)\, d\omega.$$

By the von Neumann ergodic theorem

$$J_k(t) = \frac{c_k}{t} \int_{\omega_k t}^{\omega_{k+1} t} (f_1 \circ g^t, f_2)\, ds \to c_k(\omega_{k+1} - \omega_k)(\overline{f}_1, f_2)$$

as $t \to \infty$. It remains to note that

$$\sum_{1}^{N} c_k(\omega_{k+1} - \omega_k) = \int\limits_I h_\varepsilon(\omega)\, d\omega = \int\limits_I h(\omega)\, d\omega + \delta,$$

with $|\delta| \leqslant \varepsilon$. •

## 4. Limit measures of foliated flows

In this section we apply the method of §3 to dynamical systems with *foliated* phase flows. These systems are particular cases of (3.1), when $\omega$ is one-dimensional ($m = 1$). Their basic property is as follows: flows on $\Lambda$ are conjugated for different $\omega$ after a proper change of time.

Let $I \in \mathbb{R}$ be an interval (may be, infinite) and $\Lambda$ a smooth manifold. Consider in the phase space $\Gamma = \Lambda \times I$ a dynamical system (3.1), where $z \in \Lambda$, $\omega \in I$. We put

$$P_\gamma = \{(z, \omega) \in \Gamma : \omega = \gamma\}.$$

These $n$-dimensional manifolds are invariant. The map $\psi_\omega : (z, \omega) \to z$ determines a natural diffeomorphism of $P_\omega$ and $\Lambda$. The vector field $v$ system (3.1) is tangent to $P_\omega$ at points $(z, \omega) \in \Gamma$. Let $v_\omega$ denote the restriction of $v$ to $P_\omega$.

Let $\{g^t\}$ be the phase flow on $\Gamma$, generated by the system (3.1), and $\{g^t_\omega\}$ its restriction to $P_\omega$. Since all the manifolds $P_\omega$ are diffeomorphic to $\Lambda$, it is possible to regard $\{g^t_\omega\}$ as a one-parametric group of transformations of $\Lambda$.

**Definition 4.1.** *We call the flow $g^t$ foliated, if there exists a smooth function $\alpha : I \to (0, \infty)$ and a flow $g^s_* : \Lambda \to \Lambda$ such that the diagram*

$$\begin{tikzcd} P_\omega \arrow[r, "g^t_\omega"] \arrow[d, "\psi_\omega"'] & P_\omega \arrow[d, "\psi_\omega"] \\ \Lambda \arrow[r, "g_*^{\alpha(\omega)t}"] & \Lambda \end{tikzcd} \qquad (4.1)$$

*is commutative for all $\omega \in I$ and all $t \in \mathbb{R}$. We call a foliated flow non-degenerate if the function $\alpha(\omega)$ has only isolated critical points.*

Identifying $P_\omega$ and $\Lambda$ by the diffeomorphism $\psi_\omega$, we can present the definition of a foliated flow as follows:

$$g^t_\omega = g_*^{\alpha(\omega)t}. \qquad (4.2)$$

We have the following obvious

**Proposition 4.1.** *Suppose that the flow $g_*^t$ preserves the measure $\nu_*$ on $\Lambda$, and $\sigma$ is any measure on the interval $I$. Then the flow $g^t$ on $\Lambda \times I$ preserves the measure $\mu = \nu_* \times \sigma$.*

Our main result is as follows.

**Theorem 3.** *Let $g^t$ be a non-degenerate foliated flow on $\Gamma = \Lambda \times I$, the measure $\nu_*$ is absolutely continuous with respect to a measure, determined on $\Lambda$ by some Riemannian metrics, the measure $\sigma$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, and $\nu_*(\Lambda) = 1$. Then for any $f', f'' \in L_2(\Gamma, \mu)$ there exists a limit $\lim_{t\to\infty}(f' \circ g^t, f'')$.*

Theorems 1 and 3 imply

**Corollary 4.1.** *For dynamical systems* (3.1) *with non-degenerate foliated phase flows the density of probability distribution $\rho_t$ has weak limits for $t \to +\infty$ and $t \to -\infty$. These limits coincide and satisfy* (2.1).

**Remark.** Theorem 3 remains true if $f' \in L_p(\Gamma, \mu)$ and $f'' \in L_q(\Gamma, \mu)$, where $1/p + 1/q = 1$, $1 \leqslant p \leqslant q \leqslant \infty$.

We prove Theorem 3 in the next section, and now we present some examples of systems whose flows are foliated.

Let $P_\omega$, $\omega > 0$ be a one-parametric family of smooth manifolds and $\varphi_\omega : P_1 \to P_\omega$ a family of diffeomorphisms. The union

$$\Gamma = \cup_{\omega>0} P_\omega$$

has the structure of the the cross product $\Lambda \times I$, where $\Lambda = P_1$, and $I = (0, +\infty)$. We endow $\Gamma$ with the smooth structure of the cross product $\Lambda \times I$.

**Definition 4.2.** *We call the vector field $v$ on $\Gamma$ $\varphi_\omega$-homogeneous of degree $k$, if for some vector field $v_1$ on $P_1$*

$$v \circ \varphi_\omega = \omega^k (D\varphi_\omega) v_1 \tag{4.3}$$

*for all $\omega > 0$. (Here $D$ is the notation for the differential.)*

Since $v_1$ is tangent to $P_1$ and $\varphi_\omega(P_1) = P_\omega$, the field $v$ is tangent to the leaves $P_\omega$, $\omega > 0$.

**Example.** Let $(M, \langle\,,\rangle)$ be a smooth Riemannian manifold and let $\langle\,,\rangle^*$ be the metric conjugated to the metric $\langle\,,\rangle$. Let $p$ be an element of the conjugated space $T_q^*M$ (a momentum of the corresponding mechanical system). We put $||p||^2 = \langle p, p\rangle^*$ and

$$P_s = \{p \in T_q^*M : q \in M,\ ||p|| = s\}.$$

It is clear that $\Gamma = T^*M \setminus P_0$.

Let $v$ be a Hamiltonian vector field on $\Gamma$ determined by the standard symplectic structure $\sum dp_i \wedge dq_i$ and by the Hamiltonian $H = ||p||^2/2$. This field generates a dynamical system on $\Gamma$, a geodesic flow. The corresponding phase flow $g^t$ is well defined for all $t \in \mathbb{R}$ provided $(M, \langle\,,\rangle)$ is a complete Riemannian manifold (length of any geodesic is infinite).

Let $(q, p)$ be a point of $\Gamma$. We put

$$\varphi_\omega(q, p) = (q, \omega p), \qquad \omega > 0.$$

Then the vector field $v$, determined by the Hamiltonian equations

$$\dot{q} = \partial H/\partial p, \qquad \dot{p} = -\partial H/\partial q \tag{4.4}$$

is $\varphi_\omega$-homogeneous of degree $k = 1$. Indeed, the structure of the equations (4.4) is such that $\dot{p}$ is quadratic in $p$ and $\dot{q}$ is linear in $p$.

**Example.** Consider in $\mathbb{R}^{2n} = \{q_1, \dots, q_n, p_1, \dots, p_n\}$ a Hamiltonian system (4.4) with Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n}(p_i^2 + \alpha_i^2 q_i^2), \qquad \alpha_i > 0$$

(a harmonic oscillator with frequencies $\alpha_1, \dots, \alpha_n$). We put again

$$P_\omega = \{(p, q) \in \mathbb{R}^{2n} : H(p, q) = \omega\}, \qquad \omega > 0,$$

and $\varphi_\omega : (q, p) \to (\omega q, \omega p)$. The corresponding Hamiltonian vector field on $\Gamma = \cup_{\omega > 0} P_\omega$ is $\varphi_\omega$-homogeneous of zero degree ($k = 0$).

**Proposition 4.2.** *Let $v$ be a $\varphi_\omega$-homogeneous vector field on $\Gamma$ of degree $k$. Then $v$ generates on $\Gamma$ a foliated flow $g^t$, where $\alpha(\omega) = \omega^k$.*

**Corollary 4.2.** *For $k \neq 0$ the flow $g^t$ is non-degenerate.*

Proof of Proposition 4.2. We put $\Lambda = P_1$. Note that any point $z \in \Gamma$ can be presented in the form

$$z = \varphi_\omega(z_1), \qquad \omega > 0, \quad z_1 \in P_1. \tag{4.5}$$

Since the map $\Lambda \times (0, +\infty) \to \Gamma$, determined by the equation (4.5), is a diffeomorphism, it is sufficient to check that

$$g^t \circ \varphi_\omega = \varphi_\omega \circ g_1^{\omega^k t},$$

where $g_1^t$ is the restriction of the flow $g^t$ to $P_1$.

This equation obviously holds for $t = 0$. Differentiating with respect to $t$ at $t = 0$, we get:

$$\frac{d}{dt}\Big|_{t=0} g^t \circ \varphi_\omega = v \circ \varphi_\omega,$$
$$\frac{d}{dt}\Big|_{t=0} \varphi_\omega \circ g_1^{\omega^k t} = \omega^k (D\varphi_v) v_1,$$

where $v_1$ is the restriction of $v$ to $P_1$. This equation follows from (4.3). The same equation for arbitrary $t$ follows from the fact that $g^t$ is a group with respect to $t$. •

## 5. Foliated flows and evolution of measures

In this section we prove Theorem 3. Let $f'$ and $f''$ be functions from $L_2(\Gamma, \mu)$, $\Gamma = \Lambda \times I$. We put

$$f'_\omega(\cdot) = f'(\cdot, \omega), \qquad f''_\omega(\cdot) = f''(\cdot, \omega).$$

By the Fubini theorem

$$(f' \circ g^t, f'') = \int_I (f'_\omega \circ g^t_\omega, f''_\omega)\, d\sigma = \int_I (f'_\omega \circ g_*^{\alpha(\omega)t}, f''_\omega)\, d\sigma. \tag{5.1}$$

Let $A', A'' \subset \Lambda$ be measurable sets, $I', I''$ intervals in $I$, and let $\chi', \chi'' : \Gamma \to \mathbb{R}$ be indicators of the sets $A' \times I'$, $A'' \times I''$ respectively. We put

$$J(t) = (\chi' \circ g^t, \chi'').$$

**Main Lemma.** *There exist limits* $\lim_{t\to+\infty} J(t)$ *and* $\lim_{t\to-\infty} J(t)$.

Main Lemma implies Theorem 3. Indeed, recall that $\nu_*$ is absolutely continuous with respect to a measure on $\Lambda$ determined by some Riemannian metric, and $\sigma$ is absolutely continuous with respect to the Lebesgue measure. Therefore the space $C^0_0(\Gamma)$ of continuous functions with compact supports is dense in $L_2(\Gamma, \mu)$. On the other hand, the space $\mathbf{I}(\Gamma)$ of functions which are finite linear combinations of indicators of $\mu$-measurable sets is dense in $C^0_0(\Gamma)$ (even in $C^0$-norm).

Now let $f'$ and $f''$ be two functions from $L_2(\Gamma, \mu)$. To prove the existence of the limit

$$\lim_{t\to+\infty} (f' \circ g^t, f''),$$

it is sufficient to show that given $\varepsilon > 0$, the difference

$$(f' \circ g^{t_1}, f'') - (f' \circ g^{t_2}, f'')$$

is less than $\varepsilon$ for all $t_1, t_2 > T(\varepsilon)$. This follows from Main Lemma and the fact that the functions $f', f''$ can be approximated with arbitrary precision by functions from $\mathbf{I}(\Gamma)$.

Now let us prove Main Lemma. We put $A_0 = A' \cap A''$, $I_0 = I' \cap I''$. Let $\widetilde{\chi}_0 : \Lambda \to \mathbb{R}$ be the indicator of $A_0$. It is clear that

$$J(t) = \int\limits_{I_0} (\widetilde{\chi}_0 \circ g_*^{\alpha(\omega)t}, \widetilde{\chi}_0)\, d\sigma.$$

Let $D_\gamma = \{\omega \in I_0 : |\alpha'(\omega)| > \gamma\}$, $\alpha' = d\alpha/d\omega$. Since critical points of the function $\alpha(\omega)$ are isolated, $\sigma$-measure of the set $I \setminus D_\gamma$ tends to zero as $\gamma \to 0$. Furthermore, it possible to regard $D_\gamma$ as a union of a finite set of intervals. Let $(\omega_1, \omega_2)$ be one of these intervals. Then $\alpha$ can be regarded as a coordinate on $(\omega_1, \omega_2)$. Indeed, there exists a smooth function $\omega(\alpha)$ inverse to $\alpha : (\omega_1, \omega_2) \to \mathbb{R}$. We put $d\sigma(\omega) = h(\omega) d\omega$. According to conditions of Theorem 3 the function $h(\omega)$ is integrable: $h \in L_1(I, d\omega)$. Hence,

$$\int\limits_{\omega_1}^{\omega_2} (\widetilde{\chi}_0 \circ g_*^{\alpha(\omega)t}, \widetilde{\chi}_0)\, h(\omega)\, d\omega = \int\limits_{\alpha(\omega_1)}^{\alpha(\omega_2)} (\widetilde{\chi}_0 \circ g_*^{\alpha t}, \widetilde{\chi}_0)\, h(\omega(\alpha))\, \omega'(\alpha)\, d\alpha. \quad (5.2)$$

Since $h(\omega(\alpha))\omega'(\alpha) \in L_1((\alpha(\omega_2), \alpha(\omega_1)), d\alpha)$, we see that by Theorem 2 the integral (5.2) has a limit as $t \to \infty$. $\bullet$

## 6. Increase of the entropy

Entropy is defined in statistical mechanics as the integral

$$S_t = -\int\limits_{\Gamma} \rho_t \ln \rho_t \, d\mu.$$

Since $\rho_t(z) = \rho \circ g^{-t}(z)$ and the flow $g^t$ preserves the measure $\mu$, we have: $S_t = \text{const}$. This remark is a particular case of the Poincaré result [2].

On the other hand, $\rho_t$ weakly converges to $\overline{\rho}$ as $t \to \pm\infty$. Following Gibbs and Poincaré, we can say that the stationary density of probability distribution $\overline{\rho}$ corresponds to a *heat equilibrium* of the system in question. Therefore it is natural to introduce entropy of the equilibrium state

$$S_\infty = -\int\limits_\Gamma \overline{\rho} \ln \overline{\rho}\, d\mu.$$

**Theorem 4.**

$$S_t \leqslant S_\infty. \tag{6.1}$$

The proof is based on the fact that the function $h(x) = -x \ln x$ is concave for positive $x$. Since $S_t = \text{const}$, by the Fubini theorem

$$S_t = \frac{1}{T} \int\limits_0^T \int\limits_\Gamma h(\rho_t)\, d\mu\, dt = \int\limits_\Gamma \Big[\frac{1}{T} \int\limits_0^T h(\rho_t)\, dt\Big]\, d\mu.$$

By the Jensen inequality,

$$\frac{1}{T} \int\limits_0^T h(\rho_t)\, dt \leqslant h\Big(\frac{1}{T} \int\limits_0^T \rho_t\, dt\Big), \qquad T > 0.$$

Therefore,

$$S_t \leqslant \int\limits_\Gamma h\Big(\frac{1}{T} \int\limits_0^T \rho_t\, dt\Big)\, d\mu.$$

It remains to consider the limit $T \to \infty$ and to use Theorem 1.

**Remark.** For a media without collisions in a right parallelepiped the inequality (6.1) is established in [5]. In some cases it is presented in [2] (item 6).

Since $\rho_t$ weakly converges to $\overline{\rho}$ as $t \to +\infty$ and as $t \to -\infty$, the statement of Theorem 4 concerning the entropy increase is invariant with respect to the reverse of time. Recall that the kinetic Boltsman equation implies *monotone* entropy growth as time increases.

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