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MATHEMATICAL PHYSICS

Weak Limits of Probability Distributions in Systems with Nonstationary Perturbations

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Consider a system of differential equations

$$\dot{x} = \omega, \quad \dot{\omega} = f(t), \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n \bmod 2\pi)$ are angular coordinates on an n-dimensional torus, $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n$, and f is a given vector function of t. Assume that f is twice (Riemann) integrable with respect to time t. Equations (1) describe the motion of a mechanical system with configuration space $\mathbb{T}^n = \{x\}$ and kinetic energy $T = \frac{(\omega, \omega)}{2}$ under the action of an external force f.

If $f = 0$, then (1) is a completely integrable Hamiltonian system, with the coordinates x and ω being action–angle variables. The same form is possessed by perturbations of completely integrable Hamiltonian systems in the general nondegenerate case.

Following Gibbs, we define a probability measure $\rho(x, \omega) d^n x d^n \omega$ with a summable density ρ in the phase space $\Gamma = \mathbb{T}^n \times \mathbb{R}^n$. The flow of system (1) transports this measure, so that the density $\rho_t(x, \omega)$ becomes a function of time. Since the divergence of the right-hand side of system (1) is zero, the probability density satisfies the Liouville equation

$$\frac{\partial \rho_t}{\partial t} + \left(\frac{\partial \rho_t}{\partial x}, \omega\right) + \left(\frac{\partial \rho_t}{\partial \omega}, f\right) = 0 \tag{2}$$

with initial condition $\rho_0 = \rho$.

Let $\varphi: \mathbb{T}^n \to \mathbb{R}$ be a measurable bounded function. Since $\rho_t \in L_1(\Gamma)$ for all t, the integral

$$K(t) = \int_\Gamma \rho_t(x, \omega) \varphi(x)\, d^n x\, d^n \omega$$

is a well-defined function of time. If φ is the characteristic function of a measurable domain $D \subset \mathbb{T}^n$, then $K(t)$ is the fraction of Hamiltonian systems in the Gibbs ensemble that occupy D at time t.

According to the ergodic theorem, the limit

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \rho(x - \omega t, \omega)\, dt \tag{3}$$

exists for almost all x and ω, coincides almost everywhere with an integrable function $\bar{\rho}(\omega) \geq 0$, and

$$\int_\Gamma \bar{\rho}\, d^n x\, d^n \omega = (2\pi)^n \int_{\mathbb{R}^n} \bar{\rho}(\omega) d^n \omega = 1.$$

Thus, the function $\bar{\rho}$ can be treated as the density of the limit probability measure (in a weak sense) that corresponds to a statistical equilibrium of the system under consideration.

THE MAIN RESULT

Theorem 1. *Under the assumptions made above*,

$$\lim_{t \to \pm\infty} K(t) = \int_\Gamma \bar{\rho}(\omega) \varphi(x)\, d^n x\, d^n \omega$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \varphi(x) d^n x. \tag{4}$$

Corollary. *Let φ be the characteristic function of a measurable domain D. Then*

$$\lim_{t \to \pm\infty} K(t) = \frac{\operatorname{mes} D}{\operatorname{mes} \mathbb{T}^n}.$$

Thus, as time increases indefinitely, the systems in the Gibbs ensemble become uniformly distributed on the n-dimensional configuration torus $\mathbb{T}^n$. For $f = 0$, this result was established in [1].

Theorem 1 is proved by the method described in [1]. The basic point lies in the analysis of the case where $\varphi(x) = \exp i(m, x)$, $m \in \mathbb{Z}^n$. It is necessary to show that, for $m \neq 0$,

$$\int_\Gamma \rho_t(x, \omega) e^{i(m, x)}\, d^n x\, d^n \omega \to 0 \tag{5}$$

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as $t \to \pm\infty$. For this purpose, we first solve the Liouville equation (2):

$$\rho_t(x, \omega) = \rho(x - \omega t + h(t), \omega - g(t)), \tag{6}$$

where ρ is a Cauchy datum, $\dot{g}(t) = f(t)$, $g(0) = 0$, $\dot{h}(t) = tf(t)$, and $h(0) = 0$. Formula (6) is verified by direct calculations.

Thus,

$$K(t) = \int_\Gamma \rho(x - \omega t + h, \omega - g)\varphi(x)\, d^n x\, d^n \omega$$

$$= \int_\Gamma \rho(x, \omega)\varphi(x + \omega t + \lambda(t))\, d^n x\, d^n \omega,$$

where $\dot{\lambda}(t) = g(t)$ and $\lambda(0) = 0$. It is easy to verify that $\lambda = -h$.

Now setting $\varphi = \exp i(m, x)$, we derive an explicit formula for the integral in (5):

$$e^{i(m,\lambda)} \int_{\mathbb{R}^n} \int_{\mathbb{T}^n} \rho(x, \omega) e^{i(m,x)} e^{i(m,\omega)t}\, d^n x\, d^n \omega$$

$$= e^{i(m,\lambda)} \int_{\mathbb{R}^n} \rho_m(\omega) e^{i(m,\omega)t}\, d^n \omega, \tag{7}$$

where

$$\rho_m(\omega) = \int_{\mathbb{T}^n} \rho(x, \omega) e^{i(m,x)}\, d^n x.$$

Since ρ_m is an integrable function, we conclude that, for $m \neq 0$, integral (7) approaches zero as $t \to \pm\infty$ (according to the theory of the Fourier transform), which was to be proved.

Remark. In the presence of a force f, an additional bounded oscillating factor $\exp i(m, \lambda(t))$ appears in (7).

Theorem 1 can be extended in different directions. For example, suppose that the initial density ρ belongs to $L_2(\Gamma)$ (hence, $\rho_t \in L_2$ for all t) and φ is a function from $L_2(\Gamma)$. Then

$$K(t) = \int_\Gamma \rho_t \varphi\, d^n x\, d^n \omega \tag{8}$$

is a well-defined function of time. It happens that

$$\lim_{t \to \pm\infty} K(t) = \int_\Gamma \bar{\rho}\varphi\, d^n x\, d^n \omega, \tag{9}$$

where $\bar{\rho}$ is defined by limit (3). Thus, $\bar{\rho}$ is a weak limit of ρ_t as time increases indefinitely. The state of the system with probability density $\bar{\rho}$ can be called a statistical (thermal) equilibrium. It should be emphasized that the presence of a nonstationary perturbing force $f(t)$ does not influence the approach of the system to thermal equilibrium.

Let

$$S_t = -\int_\Gamma \rho_t \ln \rho_t\, d^n x\, d^n \omega$$

be the entropy of the system at time t. It is easy to show that $S_t \equiv \text{const}$. This is a generalization of Poincaré's observation that the fine-grained entropy of autonomous dynamic systems is constant (see [2]). It is possible to introduce the entropy of a system at statistical equilibrium:

$$S_\infty = -\int_\Gamma \bar{\rho} \ln \bar{\rho}\, d^n x\, d^n \omega.$$

We have the simple inequality

$$S_t \leq S_\infty, \tag{10}$$

which corresponds to the second law of thermodynamics for irreversible processes. The formula for the entropy increment $S_\infty - S_0$ can be derived in accordance with phenomenological thermodynamics (a discussion can be found in [1]). However, in the general case, inequality (10) is valid only for adiabatic processes, without any heat inflow. For the system considered, $\dot{T} = (\omega, f) \neq 0$.

Note that the integral in (8) is also defined when $\rho \in L_p(\Gamma)$ and $\varphi \in L_q(\Gamma)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The limit relation (9) is also true in this case. In Theorem 1, $p = 1$ and $q = \infty$ (recall that L_∞ is the class of essentially bounded measurable functions).

SINGULAR LIMIT DISTRIBUTIONS

Consider the simple problem of oscillations of a unit-mass ball between two walls $0 \leq z \leq a$. Suppose that a force $f(t)$ acts on the ball. For example, we may assume that a charged ball is placed in a variable electric field. At first glance, this is a system of type (1)—an external perturbation of an integrable system. However, this is not the case, and the problem is reduced to the analysis of parametric perturbations.

Consider a two-sheeted cover of the line segment by the circle $\mathbb{T}^1 = \{x \bmod 2\pi\}$, introducing an angular variable according to the following rule: $x = \frac{\pi z}{a}$ when z increases from zero to a, and $x = 2\pi - \frac{\pi z}{a}$ when z decreases from a to zero. The equation of motion of the ball takes the form

$$\ddot{x} = -f(t)V'_x, \tag{11}$$

where $V(x) = -\frac{\pi x}{a}$ for $0 < x < \pi$ and $V(x) = \frac{\pi x}{a} - \frac{2\pi^2}{a}$ for $\pi < x < 2\pi$. The evolution of probabilities of the measure of Eq. (11) is a more complicated problem [compared to the analysis of system (1)], and it can be solved only under some additional conditions.

For example, let $f(t)$ = const. Then Eq. (11) can be explicitly integrated, and it is easy to show that the weak limit of the probability density of the measure is a function of the total energy $\frac{\dot{x}^2}{2} + fV(x)$. Integration with respect to velocity yields a probability density in the configuration space, which is generally not constant (see [1]).

Assume that $f(t)$ increases monotonically as $t \to +\infty$ and

$$\ddot{f} f \leq \frac{3}{2} \dot{f}^2. \quad (12)$$

Applying the method of [3], we can show that all solutions $x(t)$ to Eq. (11) tend to the minimum point of the potential $V(x)$ as $t \to +\infty$. Consequently, under these assumptions, the limit probability density of the ball's positions on the line segment coincides with the delta function $\delta(z - a)$.

These observations can be generalized. Suppose that $M^n = \{x\}$ is the compact configuration space of a mechanical system with n degrees of freedom, T is the kinetic energy [a positive definite quadratic form in the momenta $y = (y_1, y_2, \ldots, y_n)$], V: $M \to \mathbb{R}$ is a smooth function, and $f(t)V$ is the potential energy. The phase space Γ is the cotangent bundle of M, and the Hamiltonian is $H = T + f(t)V$. Let ρ_t be the probability density in Γ transported by the flow of the Hamiltonian system, and let $\rho_0 = \rho$ be a Cauchy datum.

Theorem 2. *Suppose that the measure* $\rho d^n x d^n \omega$ *is absolutely continuous with respect to the Liouville measure on* Γ, *the function* V *has only nondegenerate critical points on* M, *the function* $t \mapsto f(t)$ *increases monotonically with* t, *and* (12) *is fulfilled. If* φ: $M \to \mathbb{R}$ *is the characteristic function of a measurable domain on* M *not containing local minimum points of* V, *then*

$$\int_{\Gamma} \rho_t(x, y)\varphi(x)\, d^n x\, d^n \omega \to 0$$

as $t \to +\infty$.

CONCLUSIONS

Thus, the limit distribution of the Gibbs ensemble on the configuration space M is singular: this measure is concentrated on a finite set of points that are local minima of V. Theorem 2 is deduced from the result of [3]: under the conditions specified, almost all solutions to the Hamilton equations with the Hamiltonian $H = T + fV$ are such that $x(t)$ tends to a local minimum of V as time increases indefinitely. Moreover, the momenta $y(t)$ are unbounded (by the Liouville theorem on the conservation of the phase volume of Hamiltonian systems). Therefore, the frequencies of small-amplitude oscillations increase indefinitely as the system approaches a stable equilibrium.

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