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BRIEF COMMUNICATIONS

Poincaré Rotation Numbers and the Riesz and Voronoi Means

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1. Let T be an orientation-preserving homeomorphism of the circle $\{x \bmod 2\pi\}$. Clearly,

$$Tx = x + f(x),$$

where f is a continuous 2π-periodic function. Poincaré proved [1] that for all x,

$$f(T^n x) \to \lambda \quad (C) \qquad \text{as} \quad n \to \infty. \tag{1}$$

The symbol C denotes the Cesaro convergence. The number $\mu = \lambda/(2\pi)$ is called the *rotation number* of the homeomorphism T.

Let us supplement this statement with the following two remarks to be used below:

(1) relation (1) also remains valid as $n \to -\infty$;
(2) the convergence in (1) is uniform in x.

The first statement is obvious and the second one readily follows from the proof of the Poincaré theorem (see [1, 2]).

2. Suppose that

$$p_0 > 0, \qquad p_n \ge 0, \qquad \sum p_n = \infty.$$

The *Riesz means* of a sequence $\{s_n\}_0^\infty$ are the ratios

$$t_n = \frac{p_0 s_0 + \dots + p_n s_n}{p_0 + \dots + p_n}.$$

If $p_n = p_0$, then this definition yields the Cesaro means (arithmetic means). The sequence p_n specifies the *Riesz summation method*: if $t_n \to s$ as $n \to \infty$, then (by the definition) $s_n \to s$ (R, p_n). This method is regular: if $s_n \to s$, then $t_n \to s$. Let $p_{n+1} \ge p_n$. Then the Cesaro method includes the Riesz method: the convergence $s_n \to s \quad (R, p_n)$ implies the convergence $s_n \to s \quad (C)$. If p_n grows exponentially with n, then the Riesz method loses its significance and becomes equivalent to ordinary convergence.

In addition to Riesz convergence, we will use *Voronoi convergence*. Again, suppose that $q_0 > 0$ and $q_n \ge 0$. Set

$$u_n = \frac{q_n s_0 + \dots + q_0 s_n}{q_0 + \dots + q_n}.$$

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If $u_n \to s$, then (by the definition) $s_n \to s \quad (W, q_n)$.[1] The criterion for the regularity of the W-method is as follows:

$$\frac{q_n}{q_0 + \dots + q_n} \to 0.$$

The theory of Riesz and Voronoi summation is developed in detail in [3].

3. The Poincaré theorem can be strengthened by replacing the Cesaro method by the weaker Riesz or Voronoi methods.

Theorem 1. *If $q_{n+1} \le q_n$ and $\sum q_n = \infty$, then*

$$f(T^n x) \to \lambda \quad (W, q_n).$$

Theorem 2. *If $p_{n+1} \ge p_n$ and $p_n / \big(\sum_0^n p_s\big) \to 0$, then*

$$f(T^n x) \to \lambda \quad (R, p_n). \tag{2}$$

These two statements are in fact equivalent. Let us prove, for instance, Theorem 2. By setting $x = T^{-n} z$ (z depends on n), we obtain

$$\frac{p_0 f(x) + \dots + p_n f(T^n x)}{p_0 + \dots + p_n} = \frac{p_n f(z) + \dots + p_0 f(T^{-n} z)}{p_0 + \dots + p_n}.$$

Now expressions that appear on the right-hand side of this equation are the Voronoi means. The second condition of Theorem 2 is the regularity criterion for the (W, p_n)-method. Since we additionally have $p_{n+1} \ge p_n$, by Hardy's theorem [3, Theorem 23], the (W, p_n)-method includes the Cesaro method. Consequently, by the Poincaré theorem with due account of supplements 1 and 2 from Sec. 1, the Riesz means (2) tend to λ as $n \to \infty$ for all x, which completes the proof.

Theorems 1 and 2 do strengthen the Poincaré theorem, because the Riesz and Voronoi methods that they involve are included in the Cesaro method. For instance, the sequence

$$p_n = \exp\left(\frac{n}{\ln^\alpha n}\right), \qquad \alpha > 0,$$

satisfies conditions of Theorem 2 and the (R, p_n)-method is considerably weaker than the Cesaro method. Furthermore, for small values of α, this method is practically equivalent to ordinary convergence.

4. In connection with Theorems 1 and 2, an interesting problem arises: to describe all the matrix summation methods S for which $f(T^n x) \to \lambda(S)$.

For the Riesz method $S = (R, p_n)$, a weaker condition suffices for (2) to be true.

Theorem 3. *If $p_n / \big(\sum_0^n p_s\big) \to 0$ and*

$$p_0(n+1) + |p_1 - p_0| n + |p_2 - p_1|(n-1) + \dots + |p_n - p_{n-1}| \le H(p_0 + \dots + p_n) \tag{3}$$

for a certain $H > 0$, then

$$f(T^n x) \to \lambda(R, p_n).$$

Proof. Indeed, if the (W, p_n) method is regular and condition (3) holds, then (W, p_n) includes the Cesaro method (this follows from [3, Theorem 19]). After this remark it only remains to apply the idea of the proof of Theorem 2. □

Theorem 2 follows from Theorem 3, because the condition $p_{k+1} \ge p_k$ implies inequality (3) with the constant $H = 1$.

[1] *Editor's note.* We have kept the standard notation W used by the authors and based on a different transliteration (Woronoj) of Voronoi.

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