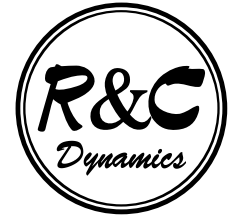


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## ON JUSTIFICATION OF GIBBS DISTRIBUTION

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The paper develop a new approach to the justification of Gibbs canonical distribution for Hamiltonian systems with finite number of degrees of freedom. It uses the condition of nonintegrability of the ensemble of weak interacting Hamiltonian systems.

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**Gibbs distribution.** We consider the probability distribution in the phase space of Hamiltonian system with the density

$$\rho = ce^{-\frac{H}{k\tau}}, \quad (1)$$

where  $H$  is a Hamiltonian,  $\tau$  is an absolute temperature,  $k$  is the Boltzmann constant,  $c$  is a normalized factor. It plays the key role in the equilibrium statistical mechanics. Gibbs show in [1] that the averaging with respect to probability measure with density (1.1) give rise to the fundamental relations of equilibrium thermodynamics.

To deduce the canonical Gibbs distribution one usually consider the ensemble of Hamiltonian systems with Hamiltonian function of the following form

$$\mathcal{H} = \mathcal{H}_0(P, Q) + \varepsilon \mathcal{H}_1(P, Q), \quad (2)$$

where

$$\mathcal{H}_0 = \sum_{s=1}^n H_0(p^{(s)}, q^{(s)}),$$
$$p^{(s)} = (p_1^{(s)}, \dots, p_m^{(s)}), \quad q^{(s)} = (q_1^{(s)}, \dots, q_m^{(s)}). \quad (3)$$

Thus at  $\varepsilon = 0$  the system with Hamiltonian (1.2) is decomposed on  $n$  identical systems with  $m$  degrees of freedom and Hamiltonian  $H_0$ . The canonical variables  $P, Q$  are the momenta  $p^{(1)}, \dots, p^{(n)}$  and coordinates  $q^{(1)}, \dots, q^{(n)}$  of separate subsystems. The perturbing function  $\mathcal{H}_1$  is the energy of interaction of  $n$  subsystems; it usually depends on their coordinates  $Q$ . Small parameter  $\varepsilon$  is the characteristic of intensity of subsystems' interaction.

We consider the case, when the Hamiltonian  $\mathcal{H}$  is sufficiently smooth with respect to variables  $P, Q$ . However, in application we often see cases with singular interaction. The classical example is the Boltzmann-Gibbs gas, the assembly of rigid balls in cube that elastically collide with each other (see [1, 2, 3]).

The traditional approach to the deduction of Gibbs distribution suggested by Fowler and Darwin ([4], the rigorous exposition see in [5, 6]) essentially uses the ergodic hypothesis: for all small  $\varepsilon > 0$  the Hamiltonian system with Hamiltonian (1.2) is ergodic on fixed energy manifolds  $\mathcal{H} = \text{const}$ . With some additional conditions some systems with Hamiltonian  $H_0$  are distributed in accordance with formula (1.1) as  $\varepsilon \rightarrow 0$  and  $n \rightarrow \infty$ .

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But the proof of ergodic hypothesis for specific Hamiltonian systems is usually pretty difficult problem. Moreover the ergodic hypothesis often contradicts with results of KAM theory. In particular if the Hamiltonian system with Hamiltonian  $H_0$  is completely integrable and the energy surfaces  $H_0 = \text{const}$  are compact, then the ergodic property is not present with certainty.

In view of this remark we can set the following interesting problem: prove that in analytical (or even in infinitely differentiable) case if the energy surfaces  $H_0 = \text{const}$  are compact, then the Hamiltonian system with Hamiltonian (1.2) never satisfy ergodic hypothesis.

We can try to modify the Fowler–Darwin method assuming that  $\varepsilon \neq 0$  and  $n$  depend on  $\varepsilon$  in such way that  $n(\varepsilon) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ . We can assume that for some functions  $\varepsilon \mapsto n(\varepsilon)$  the system with Hamiltonian (1.2) is ergodic on the surfaces  $\mathcal{H} = \text{const}$  as a result of huge number of its degrees of freedom. This somewhat weakened version of ergodic hypothesis is closely related to the unsolved problem of estimation of small parameter in KAM theory, when the “last” Kolmogorov torus disappears. The weaker conjecture on transitivity of system with Hamiltonian (1.2) on energy surfaces  $\mathcal{H} = \text{const}$  for large values of  $n$  and small  $\varepsilon$  is not proved yet. If this problem has the positive answer, then we can assert the presence of diffusion in Hamiltonian systems with many degrees of freedom (see [7–9]).

**Probability density as an integral of Hamiltonian equations.** A different approach to the deduction of canonical Gibbs distribution was proposed in the paper [10]. This approach is based on the fact that the stationary density of probability distribution is the integral of Hamiltonian differential equations uniquely defined in the whole phase space. In this case the number of interacting subsystem  $n \geq 2$  is fixed.

More exactly in [10] we consider the case, when the subsystems have one degree of freedom:  $m = 1$ . Under some natural condition we can proceed to the angle–action variables in each subsystem, and using the well-known Poincaré method we can obtain the constructive condition of nonexistence of new integrals (see [11, 12]). The results of paper [10] can be easily converted to the more general case, when  $H_0$  is a Hamiltonian of completely integrable system. We are going to find out the sufficient conditions (constructive if possible) of nonintegrability of systems (1.2)–(1.3).

We study the conditions of existence of an integral  $\mathcal{F}(P, Q, \varepsilon)$  of the canonical differential equations

$$\dot{P} = -\frac{\partial \mathcal{H}}{Q}, \quad \dot{Q} = \frac{\partial \mathcal{H}}{P} \quad (4)$$

with a Hamiltonian  $\mathcal{H}$  of the form (1.2)–(1.3). We emphasize that the integral  $\mathcal{F}$  depends on the parameter  $\varepsilon$ . Poincaré considered the analytical case; in particular we can construct the integral  $\mathcal{F}$  as a power series with respect to  $\varepsilon$ . We suppose that  $\mathcal{F}$  is a function of class  $C^2$  with respect to all the variables  $P, Q$  and  $\varepsilon$ . Hence we can suppose

$$\mathcal{F} = \mathcal{F}_0(P, Q) + \varepsilon \mathcal{F}_1(P, Q) + o(\varepsilon), \quad (5)$$

where  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are functions of class  $C^2$  and  $C^1$  with respect to  $P$  and  $Q$  correspondingly. Probably we can weaken the requirements to the class of smoothness of integral  $\mathcal{F}$ , and the following arguments still will be correct. But this is a separate problem and we are not going to discuss it here.

**Non-perturbed problem.** Suppose  $\varepsilon = 0$ . Then we have a system of  $n$  independent subsystems. It is strongly nonergodic: at  $\varepsilon = 0$  system of differential equations (2.1) has  $n$  independent first integrals

$$H_s = H_0(p^{(s)}, q^{(s)}), \quad 1 \leq s \leq n. \quad (6)$$

It is clear that the function  $\mathcal{F}_0$  from expansion (2.2) is a first integral of this unjointed system. Let’s show that under the specific conditions the function  $\mathcal{F}_0$  depends only on  $H_1, \dots, H_n$ . In particular these conditions will imply that any separate subsystem with  $m$  degrees of freedom does not have

first integrals independent on the integral of energy. The ideas of our arguments follows Poincaré method [11].

Let  $M$  be a phase space of Hamiltonian system with Hamiltonian function (3.1). Certainly for all  $s$  these spaces are identical. A phase space of new system is the direct product

$$\mathcal{M} = M \times \dots \times M, \quad \dim \mathcal{M} = 2nm.$$

Let  $h_s$  be a value of total energy of system with number  $s$ , and

$$\sum(h_s) = \{p^{(s)}, q^{(s)} : H_s(p^{(s)}, q^{(s)}) = h_s\}$$

is the a energy surface. If the value of  $h_s$  is uncritical, then  $\sum$  is a smooth  $2m - 1$ -dimensional manifold. At fixed values of  $h = (h_1, \dots, h_n)$  and  $\varepsilon = 0$  unjoined Hamiltonian system (2.1) is reduced to the direct product of dynamical systems defined on

$$S(h) = \sum(h_1) \times \dots \times \sum(h_n) \subset \mathcal{M}.$$

The ergodic property of system with Hamiltonian  $H_s$  on  $\sum(h_s)$  does not necessarily imply the constancy of integral  $\mathcal{F}_0$  on the invariant set  $S(h)$ . Consider the following simple example

EXAMPLE. Suppose the following dynamical system

$$\dot{x}_i = \omega_i, \quad \dot{y}_j = \omega_j; \quad i, j = 1, \dots, k \tag{7}$$

with constant incommensurable systems  $\omega = (\omega_1, \dots, \omega_k)$  is defined on direct product of  $k$ -dimensional tori  $\mathbb{T}^k\{x \bmod 2\pi\} \times \mathbb{T}^k\{y \bmod 2\pi\}$ . According to the Weyl theorem, each separate subsystem is ergodic on  $\mathbb{T}^k$ . But equations (3.2) have single-valued nonconstant integrals  $\sin(x_i - y_i)$  ( $1 \leq i \leq k$ ).

REMARK. However, if Hamiltonian systems are weakly mixing (mixing) systems on  $\sum(h_s)$ , then their direct product also has weakly mixing (mixing) property on  $S(h)$ . In particular in these cases the function  $\mathcal{F}_0$  is constant on any connected component of manifold  $S(h)$ .

Let  $\mathbb{T}^1$  be a nondegenerate periodic trajectory of system with number  $s$  with energy  $h_s$ ,  $T_s$  its period, and  $\omega_s = 2\pi/T_s$  its frequency. According to Floquet–Lyapunov theorem, in a neighborhood of this trajectory on  $\sum(h_s)$  we can express the coordinates  $\varphi_s \bmod 2\pi, z_1^{(s)}, \dots, z_{2m-2}^{(s)}$ , so that in the new variables the motion equation obtain the following form:

$$\dot{\varphi}_s = \omega_s + f_s(\varphi_s, z^{(s)}), \quad \dot{z}^{(s)} = \Omega_s z^{(s)} + g_s(\varphi_s, z^{(s)}). \tag{8}$$

Here  $f_s = O(|z^{(s)}|), g_s = o(|z^{(s)}|)$ , and constant square matrix  $\Omega_s$  of order  $2m - 2$  is nondegenerate. Assuming in (3.3)  $z^{(s)} = 0$  we obtain the equation of periodic trajectory:

$$\dot{\varphi}_s = \omega_s \quad (1 \leq s \leq n). \tag{9}$$

According to the assumption on non degeneracy of periodic trajectory  $\mathbb{T}^1$ , nondegenerate periodic trajectories with similar period are situated on close energy surfaces  $\sum(h_s)$ ; periods and frequencies continuously depend on energy  $h_s$ .

It is clear that the direct product  $\mathbb{T}^1 \times \dots \times \mathbb{T}^1 = \mathbb{T}^n$  is  $n$ -dimensional invariant torus of canonical system of differential equations (2.1) at  $\varepsilon = 0$  situated on  $S(h)$ . In the neighborhood of this torus the equation of motion have form (3.3). Hence, such torus is reducible and nondegenerate (see, for example, [12]). On the torus the equation are reduced to a conditionally-periodic form (3.4). As usually, we call an invariant torus nonresonance if the frequencies  $\omega_1, \dots, \omega_n$  are independent on the ring of integers. In the further analysis the following condition is essential

- A) For almost all admissible values of  $h \in \mathbb{R}^n$  nonresonance tori are everywhere dense on the manifold  $S(h)$ .

EXAMPLE. Let separate subsystems describe the inertial motion on the manifold  $\mathcal{N}$  of negative curvature. Energy  $h$  is non-negative. All periodic trajectories with positive energy are hyperbolic; hence they are not degenerate. Periodic trajectories are the motions on closed geodesics on  $\mathcal{N}$  with different speed. If  $l$  is a length of closed geodesic, then the period is equal to  $\frac{l}{\sqrt{2h}}$ . Hence the frequency  $\omega$  is defined by the formula

$$2\pi \frac{\sqrt{2h}}{l}.$$

Since the lengths of  $n$  geodesics  $l_1, \dots, l_n$  are fixed, then for almost all positive values of energy  $h_1, \dots, h_n$  the frequencies  $\omega_1, \dots, \omega_n$  are incommensurable. It is possible to show (and this is a separate problem) that in a considered situation condition A is fulfilled.

**Proposition 1.** *If condition A is fulfilled, then for all  $h \in \mathbb{R}^n$  the function  $\mathcal{F}_0$  is constant on any connected component of  $S(h)$ .*

*Proof.* Since  $\mathcal{F}_0$  is an integral of system of equations (2.1) at  $\varepsilon = 0$ , then (according to the Kronecker theorem)  $\mathcal{F}_0$  is constant on any nonresonance torus  $\mathbb{T}^n$ . Since this torus is reducible and nondegenerate, then  $d\mathcal{F}_0 = 0$  in points  $\mathbb{T}$  (see [12], ch. IV). According to condition A, for almost all values of  $h \in \mathbb{R}^n$  nonresonance tori are everywhere dense on  $S(h)$ . Hence,  $d\mathcal{F}_0 = 0$  on such manifolds  $S(h)$ . Therefore  $\mathcal{F}_0$  is constant on their connected components. For other values of  $h$  the conclusion of proposition 1 follows by continuity.

REMARK. The proof shows that in condition A instead of “for almost all admissible values of  $h \in \mathbb{R}^n$ ” we can say “for everywhere dense set of values of  $h \in \mathbb{R}^n$ ”. However, such weakening of condition A practically does not give anything new.

In further analysis we will use the proposition on everywhere density of the set of maximum resonance tori (when all frequencies are rationally expressed through one frequency). This condition together with condition A produces “an alternation” of resonance and nonresonance invariant tori and replaces the condition of nondegeneracy of non-perturbed completely integrable system in the Poincaré theory.

**Energy surfaces.** Let us consider a case, when function the  $H_0 : M \rightarrow \mathbb{R}$  has a finite number of critical values  $a_1 < a_2 < \dots < a_r$ , and  $a_1 = \min H_0$ . Such situation is common in applications. When the total energy  $h_0$  passes through the critical value, the continuous dependence of energy surface  $\Sigma(h_0)$  on  $h_0$  is lost. In that moment its topology generally changes.

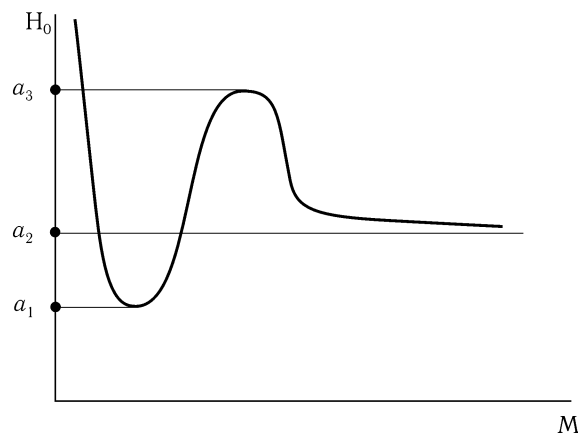


Fig. 1

In fig. 1 we present the plot of Hamiltonian  $H_0$  with three critical values. The points  $a_1$  and  $a_3$  are stationary values of  $H_0$ , and the critical point  $a_2$  is not a stationary value. The presence of

the nonstationary critical points is the characteristic property of potentials describing gravitational or Coulomb interaction.

Let's denote as  $K_{i_1 i_2 \dots i_n}$  an open parallelepiped in  $\mathbb{R}^n = \{h_1, \dots, h_n\}$ . This parallelepiped is a direct product of the intervals

$$a_{i_1} < h_1 < a_{i_1+1}, \dots, a_{i_n} < h_n < a_{i_n+1}. \tag{10}$$

If number  $i_s + 1$  is larger than  $r$ , then we replace  $a_{i_s+1}$  with a symbol  $\infty$ . In fig. 2 these domains are shown for  $n = 2$  and  $r = 3$ . Each point  $h \in K_i, i = (i_1, \dots, i_n)$  corresponds to a smooth regular manifold, which may consist of several connected parts. The quantity of connected components of  $S(h)$  does not depend on a point  $h \in K_i$ ; we denote this number by symbol  $\varkappa_i$ .

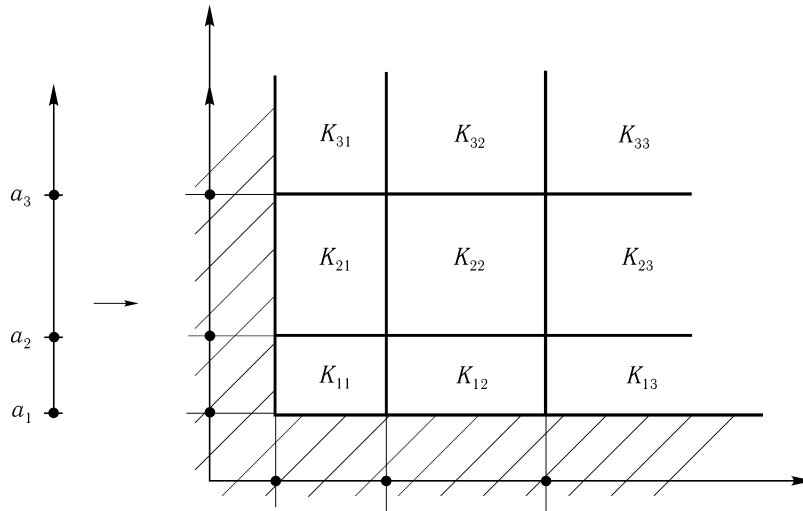


Fig. 2

Let us introduce in the phase space  $\mathcal{M}$  open areas  $\mathcal{K}_i$  defined by the inequalities similar to (4.1):

$$a_{i_1} < H_1(p^{(1)}, q^{(1)}) < a_{i_1+1}, \dots, a_{i_n} < H_n(p^{(n)}, q^{(n)}) < a_{i_n+1}.$$

It is clear that the closure of these domains in the whole covers all  $\mathcal{M}$ . Also each  $\mathcal{K}_i$  has exactly  $\varkappa_i$  connected components.

**Proposition 2.** *For any connected component of domain  $\mathcal{K}_i$  there exists the continuously differentiable function*

$$f_i : K_i \rightarrow \mathbb{R},$$

*such that the following equality is fulfilled in this domain*

$$\mathcal{F}_0 = f_i(H_1, H_2, \dots, H_n). \tag{11}$$

REMARK. Actually function  $f_i$  belongs to the class of smoothness  $C^2$ . However it is not essential for the further analysis.

*Proof.* It is clear that the domain  $\mathcal{K}_i$  is foliated to the regular surfaces  $S$ . The function  $\mathcal{F}_0$  is constant on these surfaces (more exactly on their connected components) (the proposition 1). Hence, on any connected component  $\mathcal{K}_i$  the function  $\mathcal{F}_0$  has natural representation (4.2). By definition in any point  $\mathcal{K}_i$  functions  $H_1, \dots, H_n$  are independent. Therefore we can introduce locally new coordinates so that  $H_1, \dots, H_n$  will appear as  $n$  of new variables. The transition to such coordinates is carried out with certainty with the help of continuously differentiable reversible transformation. In new variables the function  $\mathcal{F}_0$  is continuously differentiable and depends only on  $n$  variables  $H_1, \dots, H_n$ . The proposition is proved.

**Resonances.** Let us consider again an invariant torus  $\mathbb{T}$  of non-perturbed system. The equations of motion on the torus are reduced to form (3.4). This torus we call completely resonance if there exist  $n - 1$  linearly independent integer vectors

$$u = (u_1, \dots, u_n), \quad v = (v_1, \dots, v_n), \dots, \quad w = (w_1, \dots, w_n),$$

such that

$$(u, \omega) = u_1\omega_1 + \dots + u_n\omega_n = 0, \quad (u, \omega) = 0, \dots, \quad (w, \omega) = 0. \tag{12}$$

In other words all frequencies  $\omega_s$  are rationally expressed through one of them. This is equivalent to the proposition that all solutions of differential equations (3.4) on the torus  $\mathbb{T}^n$  are periodic with the same period.

Let  $\Phi$  be a restriction of perturbing function  $\mathcal{H} : M \rightarrow \mathbb{R}$  on the invariant torus  $\mathbb{T}^n$ . It is clear that  $\Phi$  is  $2\pi$ -periodic function of  $(\varphi_1, \dots, \varphi_n) = \varphi$ . We associate it with multiple Fourier series

$$\Phi = \sum_{k \in \mathbb{Z}^n} \varphi_k \exp i(k, \varphi). \tag{13}$$

A nonresonance torus we call the Poincaré torus if the factors of Fourier decomposition (5.2) with numbers  $u, v, \dots, w$  are nonzero. Since the Poincaré tori consist of the separate closed trajectories, they have no “rigidity” property and collapse after the addition of perturbation. We can not exclude the possibility that the family of degenerate periodic trajectories, components of the Poincaré torus, at perturbation give rise to the finite number of nondegenerate periodic trajectories with close transition.

Let us introduce finally the Poincaré set  $\mathbb{P} \subset \mathbb{R}^n$ . It is a set of points from  $\mathbb{R}^n = \{h_1, \dots, h_n\}$ , which are the images of the Poincaré tori under the “energy” mapping  $\mathcal{M} \rightarrow \mathbb{R}^n$ : a point with coordinates

$$(P, Q) = (p^{(1)}, \dots, p^{(n)}, q^{(1)}, \dots, q^{(n)})$$

passes to a point with coordinates

$$h_1 = H_1(p^{(1)}, q^{(1)}), \dots, \quad h_n = H_n(p^{(n)}, q^{(n)}).$$

REMARK. In the Poincaré theory [11, 12] the usually supposition is that the non-perturbed system with the Hamiltonian  $\mathcal{H}_0$  is completely integrable and nondegenerate. Therefore the set of its first integrals is a set of action variables “numerating” the invariant tori. The Poincaré set (introduced in [12]) is defined here as a set of points in space of action variables corresponding to the completely resonance tori collapsing after the addition of perturbation. In our case functions  $H_1, \dots, H_n$  are the set of integrals of the non-perturbed problem and the Poincaré set is a set of points in space of values of these integrals.

**Proposition 3.** *In points of the Poincaré set  $\mathbb{P}$  functions*

$$\mathcal{H}_0 = \sum_{s=1}^n \quad \text{and} \quad \mathcal{F}_0 = f(H_1, \dots, H_n)$$

are dependent.

*Proof.* Let  $\{ \}$  be a Poisson bracket connected with symplectic structure on  $\mathcal{M}$ . Since function  $\mathcal{F}$  is the first integral of initial system (2.1), then  $\{ \mathcal{H}, \mathcal{F} \} = 0$  for all values of  $\varepsilon$ . Since  $\{ \mathcal{H}_0, \mathcal{F}_0 \} = 0$ , then

$$\lim_{\varepsilon \rightarrow 0} \frac{\{ \mathcal{H}, \mathcal{F} \}}{\varepsilon} = 0. \tag{14}$$

Using decomposition (1.2) and (2.2) we receive from (5.3) the equality

$$\{ \mathcal{H}_0, \mathcal{F}_1 \} = \{ \mathcal{F}_0, \mathcal{H}_1 \}. \tag{15}$$

Well known that  $\{\mathcal{F}_0, \mathcal{H}_1\}$  is a derivative of  $\mathcal{F}_1$  by virtue of the initial system of differential equations with Hamiltonian  $\mathcal{H}_0$ . The Poisson bracket  $\{\mathcal{F}_0, \mathcal{H}_1\}$  has similar sense. Besides by formula (4.2) we obtain

$$\{\mathcal{F}_0, \mathcal{H}_1\} = \sum_{s=1}^n \frac{\partial f}{\partial H_s} \{H_s, \mathcal{H}_1\}. \tag{16}$$

Now let us restrict equality (5.4) on the invariant torus  $\mathbb{T}^n$ . It is clear that  $\{\mathcal{H}_0, \mathcal{F}_1\}$  is equal to the derivative of restriction of function  $\mathcal{F}_1$  on  $\mathbb{T}^n$  by virtue of system of differential equations (3.4). Let  $\Psi : \mathbb{T}^n \rightarrow \mathbb{R}$  be a restriction of  $\mathcal{F}_1$  on  $\mathbb{T}^n$  and

$$\Psi = \sum_{k \in \mathbb{Z}^n} \psi_k \exp i(k, \varphi) \tag{17}$$

its Fourier series. In points  $\varphi \in \mathbb{Z}^n$  the left part of (5.4) becomes

$$\sum_{k \in \mathbb{Z}^n} i(k, \omega) \psi_k \exp i(k, \varphi). \tag{18}$$

In view of formulas (5.2) and (5.5) the right part of relation (5.4) in points of the invariant torus is equal to

$$\sum_{k \in \mathbb{Z}^n} i \left( \frac{\partial f}{\partial H_1} k_1 \omega_1 + \dots + \frac{\partial f}{\partial H_n} k_n \omega_n \right) \varphi_k \exp i(k, \varphi). \tag{19}$$

Comparing (5.7) and (5.8) we obtain a chain of equalities

$$(k, \omega) \psi_k = (\xi k_1 \omega_1 + \dots + \xi_n k_n \omega_n) \psi_k, \quad k \in \mathbb{Z}^n. \tag{20}$$

Now let  $\mathbb{T}^n$  be the Poincaré torus. Setting  $k$  equal to  $u, v, \dots, w$  we obtain that  $\omega$  as a vector of  $n$ -dimensional space is orthogonal to the hyperplane  $\Pi$  generated by linearly independent vectors  $u, v, \dots, w$ . Since  $(u, \omega) = \dots = (w, \omega) = 0$ , and  $\varphi_u \neq 0, \dots, \varphi_w \neq 0$ , then we obtain from (5.9)  $n - 1$  linear relations:

$$\begin{aligned} u_1(\xi_1 \omega_1) + \dots + u_n(\xi_n \omega_n) &= 0, \\ \dots\dots\dots & \\ w_1(\xi_1 \omega_1) + \dots + w_n(\xi_n \omega_n) &= 0. \end{aligned}$$

Hence the vector with components  $\xi_1 \omega_1, \dots, \xi_n \omega_n$  is orthogonal to the hyperplane  $\Pi$  and consequently it is collinear to the vector  $\omega = (\omega_1, \dots, \omega_n)$ . Hence  $\xi_1 = \dots = \xi_n$ .

Thus in points of the Poincaré set the derivatives  $\frac{\partial f}{\partial H_1}, \dots, \frac{\partial f}{\partial H_n}$  are equal to each other. It means obviously the dependence of functions  $\mathcal{H}_0$  and  $\mathcal{F}_0$ .

REMARK. Since functions  $\Phi$  and  $\Psi$  are assumed to be only once continuously differentiable, then the Fourier series (5.2), (5.6)-(5.8) may diverge. In this case relations (5.9) can be deduced in another way. For this purpose we multiply the derivative with respect to time

$$\dot{\Phi} = \frac{\partial \phi}{\partial \varphi_1} + \dots + \frac{\partial \phi}{\partial \varphi_n} \omega_n$$

on  $\exp i(k, \varphi)$ , apply the operation of averaging over  $\mathbb{T}^n$

$$\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} (\cdot) \partial \varphi_1 \dots d\varphi_n,$$

and integrate by parts. In result we obtain the left part of relation (5.9) up to the factor  $-i$ . The right part of (5.9) is obtained similarly.

Let us introduce the additional condition

B) The Poincaré set  $\mathbb{P}$  is everywhere dense in any parallelepiped  $K_i$ .

The proposition 3 imply the following corollary

**Corollary.** *If conditions A and B are fulfilled, then the functions  $\mathcal{F}_0$  and  $\mathcal{F}_0$  are everywhere dependent.*

Hence we immediately obtain the following representation on any parallelepiped  $K_i$ :

$$\mathcal{F}_0 = F_i(\mathcal{H}_0), \tag{21}$$

where  $F_i$  is some continuously differentiable function.

Indeed, introducing the new variables  $H_1, \dots, H_{n-1}, \mathcal{H}_0 = \sum H_s$  instead of  $H_1, \dots, H_n$  we can write down formula (4.2) in the another form:

$$\mathcal{F}_0 = f_i(H_1, \dots, H_{n-1}, \mathcal{H}_0 - H_1 - \dots - H_{n-1}).$$

Since functions  $\mathcal{F}_0$  and  $\mathcal{H}_0$  are dependent, then the right part of this equality actually does not depend on  $H_1, \dots, H_{n-1}$ .

**Deduction of Gibbs distribution.** Now let  $\varepsilon$  tend to zero. At the limit we obtain the  $n$  unjoined subsystems moving independently: the change of the initial data  $p^{(l)}, q^{(l)}$  for  $l \neq s$  does not affect the dynamics of subsystem with number  $s$ . In order to be consistent we shall also assume at  $\varepsilon = 0$  that the subsystem with number  $s$  being in some fixed state  $(p^{(s)}, q^{(s)}) \in M$  is a random event. The following natural condition plays the main role in the process of deduction of Gibbs distribution

C) These random events are independent.

If  $\rho_s(p^{(s)}, q^{(s)}), 1 \leq s \leq n$  is a density of probability distribution of the subsystem with number  $s$ , and

$$\rho(P, Q, \varepsilon) = \rho_0(P, Q) + O(\varepsilon)$$

is a density of probability distribution in the initial system with Hamiltonian (1.2), then using condition C and the rules of multiplication of probabilities of independent events we obtain as  $\varepsilon \rightarrow 0$  the following equality:

$$\rho_0 = \rho_1 \dots \rho_n. \tag{22}$$

Equality (6.1) is also called the Gibbs hypothesis on the preservation of thermodynamic equilibrium of subsystems at vanishing interaction ([1], see also [13]). The sense of this term will be explained below.

Our main result is the following theorem

**Theorem.** *Suppose conditions A, B and C are fulfilled. Then*

$$\rho_s = c_s e^{-\frac{H_s}{k\tau}}, \quad c_s = \text{const} > 0 \tag{23}$$

for all  $1 \leq s \leq n$ .

In particular, according to (6.1),

$$\rho_0 = c_0 e^{-\frac{H_s}{k\tau}}, \quad c_0 = c_1 \dots c_n.$$

From (6.1) we see that all separate subsystems have the same distribution. We compute the factors  $c_s$  using the normalizing condition

$$\int_M \rho_s d^n p d^n q = 1.$$

*Proof of theorem.* First note that  $\rho_s$  is a function of Hamiltonian  $H_s$  only, and it is continuously differentiable in all open intervals

$$(a_1, a_2), \quad (a_2, a_3), \dots, \quad (a_r, \infty). \tag{24}$$



More exactly the number of such functions is equal to the number of the connected components of level surface  $\sum(h_s)$ , when the energy  $h_s$  changes in each of intervals (6.3). Some of these functions may coincide.

Indeed, for almost all  $h_s \geq a_1$  the Hamiltonian system with Hamiltonian  $H_s$  has everywhere dense set of nondegenerate periodic trajectories on energy surfaces  $\sum(h_s)$ . Otherwise condition A is not fulfilled because of the identity of separate subsystems. Then, according to Poincaré [11], the points of nondegenerate periodic trajectories are stationary for the restriction of any first integral on  $\sum(h_s)$ . The continuity imply that the first integrals of Hamiltonian system with Hamiltonian function  $H_s$  are constant on the connected components of  $\sum(h_s)$ . At last we should note that  $\rho_s$  is the first integral and use the (simplified) arguments of section 4.

Now let us consider again equation (6.1) true on any parallelepiped  $K_i$ :

$$\rho_0(H_1 + \dots + H_n) = \rho_1(H_1) \dots \rho_n(H_n). \tag{25}$$

This functional equation is easily solved. We differentiate (6.4) sequentially with respect to  $H_1, \dots, H_n$  and divide the result on the product  $\rho_1 \dots \rho_n$ . In result we obtain the following chain of equations

$$\frac{\rho'_1}{\rho_1} = \dots = \frac{\rho'_n}{\rho_n} = -\beta.$$

Here  $\beta$  is some constant independent on the number  $s$ . Hence

$$\rho_s = c_s e^{-\beta H_s}, \quad c_s = \text{const}. \tag{26}$$

The dimension of constant  $\beta$  is equal to the inverse energy dimension. Usually one suppose that  $\beta = (k\tau)^{-1}$ , where  $\tau$  is the absolute temperature, and  $k$  is the Boltzmann constant.

We should note that formula (6.5) may depend on the choice of multiindex  $i = (i_1, \dots, i_n)$ . More precisely, any connected component of the set  $\mathcal{K}_i \subset \mathcal{M}$  has its own set of the factors  $\beta$  and  $c_s$  in (6.5).

However, we can easily show that the constant  $\beta$  has an universal character. Indeed, suppose the constants  $\beta$  in formula (6.5) are equal to the values  $\beta_1, \dots, \beta_n$  on some connected component of domain  $\mathcal{K}_i$  with some index  $i$ . Then, according to (6.1), in this domain

$$\rho_0 = c_0 e^{\sum \beta_s H_s}. \tag{27}$$

If some  $\beta_s$  are not equal to each other, then functions (6.6) and  $\mathcal{H}_0 = \sum(H_s)$  are independent. But this statement contradicts to the corollary of proposition 3.

This argument has the evident physical meaning: at thermodynamic equilibrium all components of system have identical temperature.

The last remaining possibility is that constants  $c_s$  in formula (6.5) are different on the different intervals of values of energy (6.3). But in reality this possibility is not realized because of the continuity property of functions  $\rho_s : M \rightarrow \mathbb{R}$ .

The theorem is completely proved.

In conclusion of the work we shall make one important remark. The Gibbs theory presented in [1] does not imply that the densities of probability distributions  $\rho_1, \dots, \rho_n$  should be continuous functions on  $M$ . We can consider more general situation and assume, for example, that functions  $\rho_s$  are continuously differentiable only on those domains of phase space  $M$ , in which energy is contained between its neighboring critical values

$$a_r < H_s < a_{r+1} \quad (r = 1, \dots, p; \quad a_{p+1} = \infty). \tag{28}$$

Naturalness of such assumption is already evident if we consider the example of mathematical pendulum: the separatrices on phase cylinder separate the domains with essentially different type of motion (fluctuations and rotations).

Applying the developed above method we again obtain formula (6.2), but the constants  $c_s$  have different values in different domains (6.7). Moreover, these constants may be different for different connected components of domains (6.7). It is quite possible that such generalized discontinuous Gibbs distribution could be useful for the study of concrete thermodynamic systems.

Let us assume, for example, that the phase space  $M$  has only two domains  $M_+$  and  $M_-$  of form (6.7). In domains  $M_{\pm}$  we have the following densities of distributions

$$c_{\pm} e^{-\frac{H}{k\tau}}.$$

We calculate the constants  $c_+$  and  $c_-$  using the normalizing condition

$$c_+ \int_{M_+} e^{-\frac{H}{k\tau}} d^n p d^n q + c_- \int_{M_-} e^{-\frac{H}{k\tau}} d^n p d^n q = 1.$$

If the difference  $\Delta = c_+ - c_-$  is given, then factors  $c_{\pm}$  are uniquely defined by this equality:

$$\begin{aligned} c_+ \int_M e^{-\frac{H}{k\tau}} d^n p d^n q &= 1 + \Delta \int_{M_-} e^{-\frac{H}{k\tau}} d^n p d^n q, \\ c_- \int_M e^{-\frac{H}{k\tau}} d^n p d^n q &= 1 - \Delta \int_{M_+} e^{-\frac{H}{k\tau}} d^n p d^n q. \end{aligned}$$

Since  $c_{\pm} > 0$ , then the right parts of these equalities are positive. It happens with certainty if the jump  $\Delta$  satisfies the following inequality

$$|\Delta| < \left[ \int_M e^{-\frac{H}{k\tau}} d^n p d^n q \right]^{-1}.$$

Could we obtain the probability distribution with piecewise smooth function of distribution using the Fowler–Darwin method?

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