Doklady Mathematics, Vol. 64, No. 3, 2001, pp. 390–392. Translated from Doklady Akademii Nauk, Vol. 381, No. 5, 2001, pp. 596–598.
Original Russian Text Copyright © 2001 by Kozlov.
English Translation Copyright © 2001 by MAIK "Nauka/Interperiodica" (Russia).

MATHEMATICS

Diffusion in Systems with Integral Invariants on the Torus

Academician V. V. Kozlov

Received August 27, 2001

1. It is well known that differential equations on the two-dimensional torus with an integral invariant and without equilibrium positions reduce to the form

$$\dot{x}_1 = \frac{\omega_1}{f}, \quad \dot{x}_2 = \frac{\omega_2}{f}, \tag{1}$$

where $\omega_1, \omega_2 = \text{const}$ ($\omega_1^2 + \omega_2^2 \neq 0$) and f is a smooth positive function on $\mathbb{T}^2 = \{x_1, x_2 \bmod 2\pi\}$ (the density of the integral invariant [1]). Equations (1) on $\mathbb{T}^2$ were considered as early as almost a century ago by Poincaré [2].

Kolmogorov proved [1] that, for almost all rotation numbers $\gamma = \frac{\omega_1}{\omega_2}$ ("poorly" approximated by rationals), there exists an invertible change of variables reducing Eqs. (1) to the conditionally periodic form

$$\dot{x}_1 = \frac{\omega_1}{\Lambda}, \quad \dot{x}_2 = \frac{\omega_2}{\Lambda}, \tag{2}$$

where

$$\Lambda = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1, x_2)dx_1\,dx_2.$$

This statement of Kolmogorov's result is given in [3].

On the contrary, if an irrational γ is approximated by rationals anomalously fast, then such a reduction is impossible [1] (see also [4, 5]). A sufficient irreducibility condition is that the integral

$$\int_0^t [f(\gamma s, s) - \Lambda]ds \tag{3}$$

of a conditionally periodic function is unbounded.

Suppose that γ is rational. Then the torus $\mathbb{T}^2$ is fibered into a family of closed trajectories Γ. If they have different periods, then the reduction of Eqs. (1) to (2) is also impossible. This is equivalent to the condition that the mean value of the density f over the trajectories Γ is nonconstant.

Moscow State University, Vorob'evy gory, Moscow, 119899 Russia

2. Suppose that g^t is the phase flow of system (1) and F, G are two square-summable functions on $\mathbb{T}^2$. Let us introduce the function of time

$$K(t) = \int_{\mathbb{T}^2} F(g^{-t}(x))G(x)f(x)dx_1\,dx_2. \tag{4}$$

For brevity, we put

$$\langle H\rangle = \int_{\mathbb{T}^2} H(x)f(x)dx_1\,dx_2.$$

If, as $t \to \infty$, the function $K(t)$ tends to

$$\frac{\langle F\rangle\langle G\rangle}{\Lambda}, \tag{5}$$

then (1) is a mixing system. In [1], the problem on the possibility of mixing in the case where (1) does not reduce to (2) is stated. Actually, this problem was considered by Poincaré [2], who conjectures that mixing occurs if integral (3) is unbounded.

A negative answer to the question about mixing is given in [6, 7]. In [3], the following stronger assertion about uniform returning is obtained: there exists an unbounded sequence of moments of time t_n such that

$$\|g^{t_n}(x) - x\| \to 0$$

as $n \to \infty$ uniformly with respect to x. This, certainly, immediately implies the absence of not only mixing but also diffusion.

Definition. System (1) is called a diffusion system if function (4) has a limit as $t \to \pm\infty$.

To better understand the sense of this definition, consider the case where F is the density of the probability measure on $\mathbb{T}^2$ (in the sense of Gibbs), in particular, $\langle F\rangle = 1$. Let G be the characteristic function of a measurable domain D. Then $K(t)$ has the meaning of the probability that the system is in the domain D at time t. For mixing systems, this probability tends toward the share $\frac{\text{mes}\,D}{\text{mes}\,\mathbb{T}^2}$ of the domain D as $t \to \infty$. For diffusion systems, the probability tends to a certain limit. This limit may be zero for some domains. However, if $D = \mathbb{T}^2$, then $K(t) \equiv 1$. The presence of diffusion witnesses the noninvertibility of the behavior of the dynamical system.

If γ is rational, then system (1) is nonergodic, and mixing is out of the question. However, since the periods of different trajectories do not generally coincide, neither is this system uniformly returning; therefore, diffusion may occur.

3. Thus, suppose that $\gamma = \frac{p}{q}$, where the integers p and q are coprime integers. There exist integers r and s such that $ps - qr = 1$. Consider the linear automorphism of the torus

$$z_1 = sx_1 + rx_2, \quad z_2 = -qx_1 + px_2.$$

In the new angular variables z_1, $z_2 \bmod 2\pi$, differential equations (1) take the form

$$\dot{z}_1 = \frac{\Omega}{f}, \quad \dot{z}_2 = 0. \tag{6}$$

Here, $\Omega = s\omega_1 - r\omega_2$, and f is the density of the invariant measure written in the variables z_1, z_2.

Equations (6) can be further simplified by passing to the new angular coordinates

$$v_1 = \frac{1}{\lambda}\int_0^{z_1} f(s, z_2)ds, \quad v_2 = z_2, \tag{7}$$

where

$$\lambda(v_2) = \frac{1}{2\pi}\int_0^{2\pi} f(s, v_2)ds.$$

The Jacobian of this change of variables is

$$\frac{\partial(v_1, v_2)}{\partial(z_1, z_2)} = \frac{f}{\lambda}. \tag{8}$$

In the variables v_1, v_2, Eqs. (6) take the form

$$\dot{v}_1 = \frac{\Omega}{\lambda(v_2)}, \quad \dot{v}_2 = 0. \tag{9}$$

4. Let us examine these equations in more detail. First, we rewrite them in the general form

$$\dot{x}_1 = \omega(y), \quad \dot{y} = 0; \tag{10}$$

x and y are the angular coordinates on the torus, and ω is a nowhere vanishing smooth 2π-periodic function. The general solution to (10) has the form

$$x = \omega(y)t + x_0, \quad y = y_0; \quad x_0, y_0 = \text{const}.$$

Let A and B be square-summable functions on $\mathbb{T}^2 = \{x, y \bmod 2\pi\}$. Put

$$a(y) = \frac{1}{2\pi}\int_0^{2\pi} A(x, y)dx, \quad b(y) = \frac{1}{2\pi}\int_0^{2\pi} B(x, y)dx.$$

Theorem 1 (on diffusion). *If all critical points of a function $y \mapsto \omega(y)$ are nondegenerate, then*

$$\lim_{t\to\infty}\int_{\mathbb{T}^2} A(x - \omega(y)t, y)B(x, y)dx\,dy = 2\pi\int_0^{2\pi} abdy. \tag{11}$$

Remark. If the function $\omega(\cdot)$ is of class C^∞, then it suffices to require that its critical points be finite-to-one. This condition is obviously fulfilled if $\omega(\cdot)$ is a nonconstant analytic function.

5. Let us apply Theorem 1 to Eqs. (9). In this case, $\omega = \frac{\Omega}{\lambda(y)}$. Therefore, the critical points (and their nondegeneracy properties) of the functions with respect to ω and λ coincide.

Let us show that averaging with respect to the coordinate v_1 is equivalent to averaging in initial system (1) with respect to time along periodic trajectories. Suppose that $H(v_1, v_2)$ is a function on the torus and $H'(z_1, z_2)$ is the same function represented in other variables. Using transition formulas (7), we can write the mean

$$h(v_2) = \frac{1}{2\pi}\int_0^{2\pi} H(v_1, v_2)dv_1$$

in the form

$$\frac{1}{2\pi}\int_0^{2\pi} H'(z_1, z_2)\frac{f(z_1, z_2)}{\lambda(z_2)}dz_1.$$

According to (6), this equals

$$\frac{1}{T}\int_0^T \mu\, dt.$$

Here, μ is the function H with z replaced by solutions to (6), and T is the period of the closed trajectory (depending on z_2).

Theorem 2 (on complete diffusion). *Suppose that* γ *is rational and all critical points of the periodic function* $\lambda(\cdot)$ *are nondegenerate. Suppose also that F and G are the characteristic functions of measurable domains X and Y of positive measure on the torus* $\mathbb{T}^2 = \{x_1, x_2 \bmod 2\pi\}$, *and almost all trajectories of system* (1) *intersect X. Then*

$$\lim_{t\to\infty} K(t) > 0.$$

6. Let us fix functions F and G from L_2 and consider a sequence of rational rotation numbers $\{\gamma_n\}_1^\infty$ tending toward an irrational limit γ. Then the function $K(t)$ defined by (4) depends on n; we denote it by $K_n(t)$. Each γ_n is assigned its own periodic function $\lambda_n(\cdot)$ obtained by averaging the density of the invariant measure over the closed trajectories of the nth dynamical

system. If λ_n are Morse functions, then (by Theorem 1) $K_n(t) \to \kappa_n$ as $t \to \infty$.

Theorem 3 (on limit mixing). *If* λ_n *are Morse functions at all n, then*

$$\lim_{t \to \infty} \kappa_n = \frac{\langle F \rangle \langle G \rangle}{\Lambda}.$$

Compare this formula with (5). In computer modeling (when an irrational γ is replaced by an irreducible fraction $\frac{p}{q}$ with large p and q), system (1) is virtually indistinguishable from a mixing system.

Note that, for a typical density f, the means $\lambda(\cdot)$ calculated for all rational γ's are Morse functions.

ACKNOWLEDGMENTS

This work was financially supported by the Russian Foundation for Basic Research (project no. 99-01-01096) and by the program "Leading Scientific Schools" (project no. 00-15-96146).

REFERENCES

1. Kolmogorov, A.N., *Dokl. Akad. Nauk SSSR*, 1953, vol. 93, no. 5, pp. 763–766.
2. Poincaré, H., *Teoriya veroyatnostei* (Probability Theory), Izhevsk, 1999.
3. Kozlov, V.V., *Vestn. Mosk. Univ., Ser. 1: Mat. Mekh.,* 1978, no. 1, pp. 106–115.
4. Shklover, M.D., *Izv. Vyssh. Uchebn. Zaved. Mat.,* 1967, no. 10, pp. 113–124.
5. Anosov, D.V., *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1973, vol. 37, no. 6, pp. 1259–1274.
6. Katok, A.B., *Funkts. Anal. Ego Prilozh.*, 1967, vol. 1, no. 4, pp. 75–85.
7. Kochergin, A.V., *Dokl. Akad. Nauk SSSR*, 1972, vol. 205, no. 3, pp. 515–518.