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INTEGRAL ANALOGUE OF THE GAUSS PRINCIPLE

UDC 531

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Abstract. *The famous Gauss principle states that an actual motion is the one among conceivable motions that deviates least from the released motion. Herz based his forceless dynamics on this principle [1]. Gauss called the deviations of the conceivable motions from the released one the constraint. An explicit expression of the constraint in generalized coordinates was obtained first by Lipshitz [2].*

In this paper, two new theorems are pointed out.

1. The famous Gauss principle states that an actual motion is the one among conceivable motions that deviates least from the released motion. Herz based his forceless dynamics on this principle [1].

Gauss called the deviations of the conceivable motions from the released one the constraint. An explicit expression of the constraint in generalized coordinates was obtained first by Lipshitz (see [2]).

Let $x = (x_1, \dots, x_n)$ be generalized coordinates of a mechanical system, let $T(\dot{x}, x, t)$ be the kinetic energy, and let $F = (F_1, \dots, F_n)$ be generalized forces. We suppose that the constraint

$$\Phi(\dot{x}, x, t) = 0, \quad (1)$$

is also imposed on the system. This constraint is considered being nonintegrable in the general case. Everything to be said below readily transfers to the case of several constraints that have the explicit form (1). We suppose that the constraint (1) is regular, i.e. $\partial\Phi/\partial\dot{x} \neq 0$. The virtual velocities δx are defined by the Chetaev equation

$$\frac{\partial\Phi}{\partial\dot{x}} \cdot \delta x = 0 \quad (2)$$

One can find the actual motions from the equations of motion with the Lagrange multipliers

$$[T] = F + \lambda \frac{\partial \Phi}{\partial \dot{x}}, \quad \Phi = 0. \tag{3}$$

Here

$$[T] = \left(\frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x}$$

is the variational, or the Lagrange derivative of the function T .

Let $\ddot{x}_a, \ddot{x}_r, \ddot{x}_c$ be the accelerations of the actual, released and conceivable motions respectively. One should calculate them in a fixed moment of time and a fixed state of the system.

According to Lipschitz, the constraint is determined by the following expression:

$$Z(\ddot{x}_c, \ddot{x}_r) = A^{-1} \Delta \cdot \Delta, \quad \Delta = A(\ddot{x}_c - \ddot{x}_r). \tag{4}$$

Here

$$A = \frac{\partial^2 T}{\partial \dot{x}^2}$$

is a positively defined symmetric $n \times n$ -matrix. In these notations the Gauss principle takes the form

$$Z(\ddot{x}_a, \ddot{x}_r) \leq Z(\ddot{x}_c, \ddot{x}_r). \tag{5}$$

Lipschitz has used coordinate representation, and not the matrix one. His formula can be written more concisely:

$$Z(\ddot{x}_c, \ddot{x}_r) = A(\ddot{x}_c - \ddot{x}_r) \cdot (\ddot{x}_c - \ddot{x}_r).$$

However notation (4) is "more correct" from the point of view of the generalization of the Gauss principle that is to be presented below.

2. Let $t \rightarrow x(t)$ be some smooth path being defined in the time interval $t_1 \leq t \leq t_2$. Its variations δx are smooth functions of time t , which satisfy equations (2). We should point out that we do not require the variation δx turning to zero at the ends of the interval $[t_1, t_2]$.

Let us calculate the value of the covector

$$R(t) = ([T] - F)_{x(t)}$$

on this path.

Theorem 1. *The path $x(t)$ is a motion of the system if and only if for every variation of this path $\delta x(t)$ the inequality*

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (A^{-1}(R + A\delta x)) \cdot (R + A\delta x) dt \geq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} A^{-1} R \cdot R dt \tag{6}$$

holds.

We now prove the necessity. Let $x(t)$ be a motion of the mechanic system. Then from equations (3) and (4) we have $R \cdot \delta x = 0$ for all values of t . Consequently,

$$\int_{t_1}^{t_2} R \delta x dt = 0. \quad (7)$$

But in this case the left-hand part of inequality (6) differs from the right-hand part by

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} A \delta x \cdot \delta x dt \geq 0.$$

Proof of sufficiency. If inequality (6) holds for all variations (2), then relation (7) readily follows. Since it holds for all smooth functions $\delta x(t)$ that satisfy (2), then along the path $x(t)$ the equality

$$R = \lambda \frac{\partial \Phi}{\partial \dot{x}}$$

is valid. Hence the path $x(t)$ is a motion of the system. The theorem is now proved.

Let us derive the usual Gauss principle from theorem 1. To this end we turn the interval of integrating $t_2 - t_1$ in inequality (6) to zero. Then (6) becomes equivalent to local inequality

$$A^{-1}(R + A\delta x) \cdot (R + A\delta x) \geq A^{-1}R \cdot R. \quad (8)$$

Since

$$[T] = A\ddot{x} + P(\dot{x}, x, t),$$

then

$$R = A\ddot{x}_a + P - F \quad 0 = A\ddot{x}_r + P - F.$$

Hence,

$$R = A(\ddot{x}_a - \ddot{x}_r). \quad (9)$$

Further, since the actual and the conceivable motions satisfy the same constraint equation (1), then

$$\delta x = \ddot{x}_c - \ddot{x}_a \quad (10)$$

is a virtual velocity.

Substituting (9) and (10) in inequality (8), we obtain the desired inequality (5).

3. Theorem 1 differs from the classical Gauss principle only formally. However one can apply the same idea to more interesting cases, when equations (3) are substituted by the equations of the Lagrange variational problem:

$$\delta \int_{t_1}^{t_2} L dt = 0, \quad \Phi = 0, \quad x(t_1) = const, \quad x(t_2) = const. \quad (11)$$

Here L is some function (the Lagrangian) of \dot{x} , x , t , moreover we suppose again that the matrix of the second derivatives

$$A = \frac{\partial^2 L}{\partial \dot{x}^2}$$

is positively defined.

In the Lagrange problem the varied paths also satisfy the constraint equation. Thus the equation for variations δx takes more complicated form:

$$\delta\Phi = \frac{\partial\Phi}{\partial\dot{x}}(\delta\dot{x}) + \frac{\partial\Phi}{\partial x}(\delta x) = 0. \quad (12)$$

Theorem 2. *The path $t \rightarrow x(t)$, $t_1 \leq t \leq t_2$, is the extremal of the variational problem (11) if and only if for all variations of this path with fixed endpoints inequality*

$$\int_{t_1}^{t_2} A^{-1}([L] + A\delta x) \cdot ([L] + A\delta x) dt \geq \int_{t_1}^{t_2} A^{-1}[L] \cdot [L] dt. \quad (13)$$

holds.

Here the covector $[L]$ and the elements of the matrix A are calculated along the path $x(t)$.

Proof. Let $t \rightarrow x(t)$ be the extremal. Then

$$\delta \int_{t_1}^{t_2} L dt = - \int_{t_1}^{t_2} [L] \cdot \delta x dt = 0 \quad (14)$$

for all variations with the fixed endpoints which satisfy (12). From this inequality (13) readily follows.

The opposite statement can be derived from the condition of minimum of the functional

$$J[\delta x] = \int_{t_1}^{t_2} A^{-1}([L] + A\delta x) \cdot ([L] + A\delta x) dt$$

when $\delta x = 0$ on the linear space of functions δx that satisfy (12). We use the method of Lagrange multipliers: considering λ being new coordinates we write down the Euler-Lagrange variational equations with the Lagrangian

$$\mathcal{L} = A^{-1}([L] + A\delta x) \cdot ([L] + A\delta x)/2 - \lambda\delta\Phi.$$

These equations take the following form:

$$\left(\frac{\partial\mathcal{L}}{\partial(\delta x)} \right) - \frac{\partial\mathcal{L}}{\partial\delta x} = 0, \quad \left(\frac{\partial\mathcal{L}}{\partial\lambda} \right) = \frac{\partial\mathcal{L}}{\partial\lambda}.$$

The second equation gives the equality $\delta\Phi = 0$, which is already known, and the first one can be transformed to the following form (where $\delta x = 0$):

$$[L] = - \left(\lambda \frac{\partial\Phi}{\partial\dot{x}} \right) + \lambda \frac{\partial\Phi}{\partial x}. \quad (15)$$

But this equation together with equality (1) is the equation for the extremals of the Lagrange variational problem.

The theorem is proved now.

Since the equations for variations (12) contain derivatives $(\delta x)'$, then (in contrast to theorem 1) one cannot reduce theorem 2 to local variational principle.

4. In conclusion we make some remarks.

a) As one can notice, in inequalities (6) and (13) the matrix A can be replaced by any positively defined symmetric matrix B . Then, for example, the local Gauss principle takes the form of the general inequality

$$\begin{aligned} B^{-1} (A(\ddot{x}_r - \ddot{x}_f) + B(\ddot{x}_c - \ddot{x}_r)) \cdot (A(\ddot{x}_r - \ddot{x}_f) + B(\ddot{x}_c - \ddot{x}_r)) &\geq \\ \geq B^{-1} A(\ddot{x}_r - \ddot{x}_f) \cdot A(\ddot{x}_r - \ddot{x}_f). \end{aligned}$$

It reduces to (5) when $B = A$.

b) When the forces are potential, nonholonomic equations (3) cannot be reduced to variational equations (15). However as [3] states, extremals of the Lagrange problem come out of some passage to limit in ordinary equations of motion of the "free" mechanical system. Being more precise, we consider the equations of motion with the modified kinetic energy

$$[T_N] = F, \quad (16)$$

where $T_N = T + N\Phi^2/2$, N being a positive parameter. For example, if Φ is a linear function of \dot{x} , then T_N is a positively defined quadratic form. In problems of dynamics the modification of the kinetic energy is usually connected with the effect of the adjoint masses.

It turns out that as $N \rightarrow \infty$ solutions of equation (16) tend to solutions of the following variational equations:

$$[T] = F - \left(\lambda \frac{\partial\Phi}{\partial\dot{x}} \right) + \lambda \frac{\partial\Phi}{\partial x}, \quad \Phi = 0.$$

The mathematical model of a motion based on these equations of motion is called vakonomic model by the author (see also discussion in [4]).

c) The passage to limit in (b) assumes the generalization. We substitute equation (16) by the more general one:

$$[T_N] = F - \frac{\partial\Phi_N}{\partial\dot{x}}, \quad (17)$$

where $T_N = T + \alpha N \Phi^2/2$, $\Phi_N = \beta N \Phi^2/2$ ($\alpha, \beta \geq 0$). The function Φ_N has the sense of the Rayleigh's dissipative function for anisotropic friction. It turns out (see [3]) that as $N \rightarrow \infty$ solutions to equations (17) tend to solutions of the following system:

$$[T] = F - \alpha \left(\lambda \frac{\partial \Phi}{\partial \dot{x}} \right)' + \alpha \lambda \frac{\partial \Phi}{\partial x} - \beta \lambda \frac{\partial \Phi}{\partial \dot{x}}, \quad \Phi = 0. \quad (18)$$

These equations depend on the parameter $k = \beta/\alpha$. When $\alpha = 0$ ($k \rightarrow \infty$), we get the usual nonholonomic model, and when $\beta = 0$ ($k = 0$) we have the vakonomic model.

Theorem 2 is valid also for equations (18), equations (12) for variations δx being substituted by

$$\alpha \frac{\partial \Phi}{\partial \dot{x}}(\delta x)' + \alpha \frac{\partial \Phi}{\partial x} \delta x + \beta \frac{\partial \Phi}{\partial \dot{x}} \delta x = 0.$$

If, for example, $\alpha = 0$, then we get theorem 1.

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INTEGRALNA ANALOGIJA GAUSOVOG PRINCIPA

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Čuveni Gausov princip kaže da je stvarno kretanje ono shvatljivo kretanje koje najmanje odstupa od oslobođenog kretanja. Herc je zasnovao svoju dinamiku bez sila na tom principu. Gaus je nazvao odstupanje shvatljivog kretanja od oslobođenog kretanja ograničenjem. Eksplicitni izraz ograničenja u generalisanim koordinatama prvi je dobio Lipshitz.

U ovom radu se ističu dve nove teoreme.