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Polynomial integrals of reversible mechanical systems with a two-dimensional torus as the configuration space

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Abstract. The problem considered here is that of finding conditions ensuring that a reversible Hamiltonian system has integrals polynomial in momenta. The kinetic energy is a zero-curvature Riemannian metric and the potential a smooth function on a two-dimensional torus. It is known that the existence of integrals of degrees 1 and 2 is related to the existence of cyclic coordinates and the separation of variables. The following conjecture is also well known: if there exists an integral of degree n independent of the energy integral, then there exists an additional integral of degree 1 or 2. In the present paper this result is established for $n = 3$ (which generalizes a theorem of Byalyi), and for $n = 4$, 5, and 6 this is proved under some additional assumptions about the spectrum of the potential.

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§ 1. Introduction

In this paper we consider natural mechanical systems with two degrees of freedom that have a two-dimensional torus as the configuration space and admit an additional first integral that is a polynomial with respect to momenta. Such systems are, of course, completely integrable. Their polynomial integrals are representable as sums of homogeneous polynomials in momenta with coefficients that are smooth single-valued functions on the configuration space.

Birkhoff [1] considered a local problem of the existence of linear and quadratic integrals that are polynomial in velocities. He discovered that the existence of a linear conditional integral is related to the existence of a 'hidden' cyclic coordinate, while a quadratic conditional integral allowed one to separate the canonical variables. Global versions of these results are known ([2], [3]) in the case when the configuration space of the system is a two-dimensional torus. The problem of polynomial integrals of degree at most 2 was discussed also in [4] and [5].

The problem of polynomial integrals of the geodesic flows on two-dimensional tori has been studied in [6] for metrics that have a trigonometric polynomial as the conformal factor. It is shown there that if the geodesic flow admits an irreducible

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additional integral polynomial in momenta, then the degree of this polynomial is at most two. This result was improved in [3].

In [7] the authors discussed the problem of the existence of a complete collection of independent polynomial integrals for systems with configuration space diffeomorphic to the torus $\mathbb{T}^m = \{x_1, \dots, x_m \bmod 2\pi\}$, the kinetic energy

$$T = \frac{1}{2}\sum_{i,j}^{m} a_{ij}\dot{x}_i\dot{x}_j, \qquad a_{ij} = \text{const}, \tag{1.1}$$

and a potential $V\colon \mathbb{T}^m \to \mathbb{R}$ that is a trigonometric polynomial. They found that there exists a complete collection of polynomial integrals if and only if the spectrum of the trigonometric polynomial V lies on $k \leqslant m$ mutually orthogonal straight lines through the origin. It that case it was shown that there exist m independent integrals of degrees at most 2. By Weierstrass's theorem, trigonometric polynomials are dense in the space of smooth functions on the torus. Nevertheless, the approach of [7] cannot be applied to systems with potential of general form.

As pointed out in [8], a natural mechanical system on a two-dimensional torus with kinetic energy

$$T = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) \tag{1.2}$$

can have no irreducible integrals of degree three or four: in these cases it necessarily has integrals of degrees one or two, respectively. This result has gained recognition and is often cited (see, for instance, [9]–[11], where this range of problems is discussed). However, only the proof for integrals of degree three can be found in [8]; moreover, it has a gap, which is fortunately not very essential for metrics of the form (1.2). Still, if we have a metric of the general form (1.1), then repairing this gap requires additional effort (we discuss this in greater length in § 4). For integrals of degree four the method of [8] does not work even in the simplest case of the metric (1.2).

Polynomial integrals for systems of interacting particles with potentials

$$\sum_{i<j} f(x_i - x_j), \tag{1.3}$$

have been studied in [12]. In all the integrable cases discovered the periodic function f has poles on the real axis. As shown in [13], if the potential of pairwise interaction is periodic and has no singularities, then the system with potential energy (1.3) cannot be completely integrable.

In the present paper we study the problem of polynomial integrals of degree n with kinetic energy of the general form (1.1) and an arbitrary analytic potential. We generalize Byalyi's theorem in the case $n = 3$ and prove similar results for $n = 4$, 5, and 6 under additional assumptions about the spectrum of the potential.

§ 2. Auxiliary results

We consider now a Hamiltonian system whose configuration space is the two-dimensional torus $\mathbb{T}^2 = \{q_1, q_2 \bmod 2\pi\}$. This Hamiltonian system has the following form:

$$H = \frac{ap_1^2 + 2bp_1p_2 + cp_2^2}{2} + W(q_1, q_2), \tag{2.1}$$

where the potential W is a 2π-periodic function in q_1 and q_2; a, b, and c are real constants such that $a > 0$ and $ac-b^2 > 0$. We shall look for a potential W such that there exists an integral F independent of the energy integral H that is a polynomial of degree n in momenta, with 2π-periodic coefficients.

Let

$$W = \sum_{-\infty}^{+\infty} [W]_{w_1 w_2} e^{i(w_1 q_1 + w_2 q_2)}$$

be the Fourier series of W. We mean by the *spectrum* of W the following (generally speaking, infinite) subset of the integer lattice:

$$S = \bigl\{ w = (w_1, w_2) \in \mathbb{Z}^2 : [W]_{w_1 w_2} \neq 0 \bigr\}.$$

The map $w \to -w$ takes this subset into itself.

Assume that there exists a polynomial (in the momentum variables) integral

$$F = F_n + F_{n-1} + F_{n-2} + \cdots + F_0$$

of degree n, where F_k is a homogeneous (in momentum variables) polynomial of degree k with coefficients that are smooth functions of q_1 and q_2. Note that the Poisson bracket of two homogeneous polynomials of degrees r and s in p_1, p_2 is a homogeneous polynomial of degree $r + s - 1$. Hence if F is an integral of a Hamiltonian system with Hamiltonian (2.1), and then the functions

$$\Phi_1 = F_n + F_{n-2} + \cdots \quad \text{and} \quad \Phi_2 = F_{n-1} + F_{n-3} + \cdots$$

are also integrals of this system. Hence we can assume that F is a sum of homogeneous polynomials that have only even or only odd degrees.

Conditions for the existence of linear and quadratic integrals are well known ([1]–[3]). For $n = 1$ the spectrum of W must lie on a straight line passing through the origin. If $n = 2$, then the spectrum must lie on two lines orthogonal in the intrinsic metric generated by the kinetic energy. Note that this is not a standard situation for metrics of the general form: there do not necessarily exist orthogonal lines passing through nodes of the integer lattice.

Assume that the spectrum of the potential lies on some straight line. Then $W = f(mq_1 + nq_2)$, where m and n are coprime integers and f is a 2π-periodic function. In that case the system has the integral $F_1 = mp_2 - np_1$, which is linear in momenta.

Assume now that W lies on two lines intersecting at right angles at the origin. Then

$$W = f_1(m_1 q_1 + n_1 q_2) + f_2(m_2 q_1 + n_2 q_2),$$

where we assume that the orthogonality condition is

$$a m_1 m_2 + b(m_1 n_2 + m_2 n_1) + c n_1 n_2 = 0,$$

and the functions f_1 and f_2 are, of course, 2π-periodic. Then there exists a quadratic integral

$$\begin{aligned} F_2 + F_0 = \bigl(a(r_1 + r_2) + 2b\bigr)p_1^2 + 2(c - a r_1 r_2) p_1 p_2 \\ - \bigl(c(r_1 + r_2) + 2b r_1 r_2\bigr) p_2^2 + 2(r_1 - r_2)(f_1 - f_2), \end{aligned}$$

where $r_i = m_i / n_i$.

Let (2.1) be a natural mechanical system admitting an additional independent integral F of degree $n \geqslant 3$. If n is even, then

$$F = F_n + F_{n-2} + \cdots + F_2 + F_0,$$

where F_0 is a function of q_1 and q_2. If n is odd, then

$$F = F_n + F_{n-2} + \cdots + F_3 + F_1.$$

The kinetic energy can be conveniently put in diagonal form by means of a linear transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Gamma^T \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad |\Gamma| \neq 0.$$

We now pass to the new variables x_1 and x_2, which are conformal in the covering plane of the torus. We extend this to a canonical transformation $q, p \to x, y$, where

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Gamma^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Then

$$H = H_2 + V(x_1, x_2), \qquad H_2 = \frac{y_1^2 + y_2^2}{2}, \tag{2.2}$$
$$V = \sum_{-\infty}^{+\infty} [W]_w e^{i(\Gamma^{-1} w, x)} = \sum_{-\infty}^{+\infty} [W]_w e^{i(w,q)}, \quad w \in \mathbb{Z}^2.$$

Clearly, $\Gamma^{-1} w \notin \mathbb{Z}^2$ in general: the spectrum of W lies after this transformation in the nodes of some lattice of parallelograms.

Let

$$\begin{aligned} F_n &= \sum_{i=0}^{n} b_i^{[n]} y_1^{n-i} y_2^i, \\ F_{n-2} &= \sum_{i=0}^{n-2} b_i^{[n-2]} y_1^{n-2-i} y_2^i, \\ F_{n-4} &= \sum_{i=0}^{n-4} b_i^{[n-4]} y_1^{n-4-i} y_2^i, \quad \ldots . \end{aligned} \tag{2.3}$$

Poincaré showed ([14], see also [11]) that one can assume the functions F_n and H_2 to be independent — otherwise there exists an integral of lower degree. As proved in [4], F_n and H_2 are dependent if and only if

$$b_0^{[n]} - b_2^{[n]} + b_4^{[n]} - \cdots = 0, \qquad b_1^{[n]} - b_3^{[n]} + b_5^{[n]} - \cdots = 0.$$

We shall call the sums on the right-hand sides of equalities (2.3) the *Birkhoff sums* (see [1]). They are equal to the real and the imaginary parts of F_n for $y_1 = 1$ and $y_2 = i$.

Using the Poincaré method one can prove the following results (see [11]; Chapter IV). First, the leading polynomial F_n has constant (in other words, independent of x_1 and x_2) coefficients $b_i^{[n]}$. Next, assume that a point with coordinates m_1 and m_2, $m_1^2+m_2^2\neq 0$, lies in the spectrum of the potential. Then

$$y_2\frac{\partial F_n}{\partial y_1}-y_1\frac{\partial F_n}{\partial y_2}=0\quad\text{for } m_1y_1+m_2y_2=0. \tag{2.4}$$

An equivalent result: the polynomial

$$m_1\frac{\partial F_n}{\partial y_1}+m_2\frac{\partial F_n}{\partial y_2}$$

is a multiple of $m_1y_1+m_2y_2$. Let $G_{m_1m_2}$ be the ratio of these polynomials; this is a homogeneous polynomial of degree $n-2$.

Our central result is as follows.

Theorem 1. *The following equality holds on the line* $m_1y_1+m_2y_2=0$:

$$m_1\frac{\partial G_{m_1m_2}}{\partial y_1}+m_2\frac{\partial G_{m_1m_2}}{\partial y_2}=0. \tag{2.5}$$

This supplements Poincaré's classical result (2.4). For better understanding of conditions (2.4) and (2.5) we consider the special case when the spectrum of the potential lies on the vertical line ($m_1=0$). Then the Poincaré condition (2.4) is equivalent to $b_1^{[n]}=0$, while (2.5) becomes the equality $b_3^{[n]}=0$.

Remark. It makes sense to conjecture a more general result: all coefficients of the polynomial F_n with odd subscripts are equal to zero. Unfortunately, we could not prove this; in any case this would not enable us to establish completely the conjecture concerning irreducible integrals that we mentioned in the introduction (see § 5).

§ 3. Proof of the central result

From the equation $\{F,H\}=0$ we deduce relations on the coefficients of the integral F. We write down the equations corresponding to the monomials $y_1^iy_2^j$ such that $i+j=n+1$. This is a group of $n+2$ equations:

$$\frac{\partial b_j^{[n]}}{\partial x_1}+\frac{\partial b_{j-1}^{[n]}}{\partial x_2}=0,\qquad j=0,\dots,n+1,\quad b_{-1}^{[n]}=b_{n+1}^{[n]}=0. \tag{3.1.n}$$

As already mentioned, these equations have only constant solutions; let

$$b_i^{[n]}=a_i^{[n]}=\text{const},\qquad i=0,\dots,n.$$

Setting equal to zero the coefficients of monomials $y_1^iy_2^j$ such that $i+j=n-1$, we obtain the following group of n equations:

$$\frac{\partial b_j^{[n-2]}}{\partial x_1}+\frac{\partial b_{j-1}^{[n-2]}}{\partial x_2}=(n-j)a_j^{[n]}\frac{\partial V}{\partial x_1}+(j+1)a_{j+1}^{[n]}\frac{\partial V}{\partial x_2}\,,$$
$$j=0,\dots,n-1,\qquad b_{-1}^{[n-2]}=b_{n-1}^{[n-2]}=0. \tag{3.1.$n-2$}$$

The remaining equations are similar:

$$\frac{\partial b_j^{[n-4]}}{\partial x_1} + \frac{\partial b_{j-1}^{[n-4]}}{\partial x_2} = (n-2-j) b_j^{[n-2]} \frac{\partial V}{\partial x_1} + (j+1) b_{j+1}^{[n-2]} \frac{\partial V}{\partial x_2}\,,$$
$$j = 0, \ldots, n-3, \qquad b_{-1}^{[n-4]} = b_{n-3}^{[n-4]} = 0; \tag{3.1.$n-4$}$$

. . . .

$$\frac{\partial b_j^{[n-2l]}}{\partial x_1} + \frac{\partial b_{j-1}^{[n-2l]}}{\partial x_2} = (n-2(l-1)-j) b_j^{[n-2(l-1)]} \frac{\partial V}{\partial x_1} + (j+1) b_{j+1}^{[n-2(l-1)]} \frac{\partial V}{\partial x_2}\,,$$
$$j = 0, \ldots, n-2l+1, \qquad b_{-1}^{[n-2l]} = b_{n-2l+1}^{[n-2l]} = 0; \tag{3.1.$n-2l$}$$

. . . .

Now, if n is even, then the last group contains two equations:

$$\frac{\partial b_0^{[0]}}{\partial x_1} = 2 b_0^{[2]} \frac{\partial V}{\partial x_1} + b_1^{[2]} \frac{\partial V}{\partial x_2}\,, \qquad \frac{\partial b_0^{[0]}}{\partial x_2} = b_1^{[2]} \frac{\partial V}{\partial x_1} + 2 b_2^{[2]} \frac{\partial V}{\partial x_2}\,. \tag{3.1.0}$$

If n is odd, then the last equation is as follows:

$$0 = b_0^{[1]} \frac{\partial V}{\partial x_1} + b_1^{[1]} \frac{\partial V}{\partial x_2}\,. \tag{3.1.-1}$$

Equations (3.1.$n-2$) are linear; we can solve them by the Fourier method. Write

$$V = \sum_{-\infty}^{+\infty} [V]_{uv} e^{i(ux_1+vx_2)}, \qquad b_j^{[n-2]} = \sum_{-\infty}^{+\infty} [b_j^{[n-2]}]_{uv} e^{i(ux_1+vx_2)}, \quad j = 0, \ldots, n-2,$$

where the $[V]_{uv}$ and the $[b_j^{[n-2]}]_{uv}$ are the Fourier coefficients. It is easy to show that for each choice of u and v we obtain the system

$$\begin{pmatrix} -u & 0 & 0 & \ldots & 0 & 0 & na_0^{[n]}u + a_1^{[n]}v \\ -v & -u & 0 & \ldots & 0 & 0 & (n-1)a_1^{[n]}u + 2a_2^{[n]}v \\ 0 & -v & -u & \ldots & 0 & 0 & (n-2)a_2^{[n]}u + 3a_3^{[n]}v \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & -v & -u & 2a_{n-2}^{[n]}u + (n-1)a_{n-1}^{[n]}v \\ 0 & 0 & 0 & \ldots & 0 & -v & a_{n-1}^{[n]}u + na_n^{[n]}v \end{pmatrix} \begin{pmatrix} [b_0^{[n-2]}]_{uv} \\ [b_1^{[n-2]}]_{uv} \\ [b_2^{[n-2]}]_{uv} \\ \ldots \\ [b_{n-2}^{[n-2]}]_{uv} \\ [V]_{uv} \end{pmatrix} = 0. \tag{3.2}$$

The system (3.2) has non-trivial solutions only for u and v such that the determinant of the system vanishes. We calculate the determinant of the $(n \times n)$-matrix expanding it along the last column:

$$0 = (na_0^{[n]}u + a_1^{[n]}v)v^{n-1} - ((n-1)a_1^{[n]}u + 2a_2^{[n]}v)uv^{n-2}$$
$$+ \cdots + (-1)^n(2a_{n-2}^{[n]}u + (n-1)a_{n-1}^{[n]}v)u^{n-2}v + (-1)^{n-1}(a_{n-1}^{[n]}u + na_n^{[n]}v)u^{n-1}.$$

We can transform this identity. Let $z = u/v$; then z is a zero of a polynomial of degree n:

$$\sum_{i=0}^{n} (-1)^i \big((i+1)a_{i+1}^{[n]} - (n-i+1)a_{i-1}^{[n]}\big) z^i = 0. \tag{3.3}$$

Here the constants $a_j^{[n]}$ (where $a_{-1}^{[n]} = a_{n+1}^{[n]} = 0$) are such that (3.3) has at least one real root $z_m = u_m/v_m$. Let V be a function of the following form:

$$V = \sum_{m=1}^{n} f_m(u_m x_1 + v_m x_2),$$

where

$$f_m = \sum_{\lambda=-\infty}^{+\infty} [V]_{\lambda u_m, \lambda v_m} e^{i\lambda(u_m x_1 + v_m x_2)},$$

that is, assume that the spectrum of V lies on at most n straight lines in the real plane. We shall call these lines containing the spectrum of V the *spectrum lines* of V. Consider, for instance, the kth line. Rotating the plane about the origin we can make this line vertical. Extending this to a canonical transformation we obtain a Hamiltonian function of the kind (2.2), but the potential energy now has the following form:

$$V = f_k(v_k x_2) + \sum_{\substack{m=1 \\ m\neq k}}^{n} f_m(u_m x_1 + v_m x_2). \tag{3.4}$$

Note that

$$\frac{\partial f_m}{\partial x_2} = v_m f_m' = \frac{v_m}{u_m}\frac{\partial f_m}{\partial x_1}, \qquad m = 1, \ldots, n, \quad m \neq k.$$

Here the dash indicates the derivative with respect to the argument of the function.

From the first three equations in $(3.1.n-2)$ we shall determine the functions $b_0^{[n-2]}$ and $b_1^{[n-2]}$. To this end we write down explicitly the first three equations in the group $(3.1.n-2)$, where $a_1^{[n]} = 0$:

$$\frac{\partial b_0^{[n-2]}}{\partial x_1} = n a_0^{[n]} \frac{\partial V}{\partial x_1}, \tag{3.5}$$

$$\frac{\partial b_1^{[n-2]}}{\partial x_1} + \frac{\partial b_0^{[n-2]}}{\partial x_2} = 2a_2^{[n]} \frac{\partial V}{\partial x_2}, \tag{3.6}$$

$$\frac{\partial b_2^{[n-2]}}{\partial x_1} + \frac{\partial b_1^{[n-2]}}{\partial x_2} = (n-2)a_2^{[n]} \frac{\partial V}{\partial x_1} + 3a_3^{[n]} \frac{\partial V}{\partial x_2}. \tag{3.7}$$

We can express $b_0^{[n-2]}$ from (3.5):

$$b_0^{[n-2]} = n a_0^{[n]} V + c_0^{[n-2]}(x_2),$$

where $c_0^{[n-2]}$ is a function of x_2 alone. We can find this function from (3.6):

$$\frac{\partial b_1^{[n-2]}}{\partial x_1} + n a_0^{[n]} \left(\frac{df_k}{dx_2} + \sum_{\substack{m=1 \\ m\neq k}}^{n} \frac{v_m}{u_m}\frac{\partial f_m}{\partial x_1} \right) + \frac{dc_0^{[n-2]}}{dx_2} = 2a_2^{[n]} \left(\frac{df_k}{dx_2} + \sum_{\substack{m=1 \\ m\neq k}}^{n} \frac{v_m}{u_m}\frac{\partial f_m}{\partial x_1} \right).$$

This is an equation of the following form:

$$\frac{\partial\Delta}{\partial x_1} + \left(\frac{dc_0^{[n-2]}}{dx_2} - (2a_2^{[n]} - na_0^{[n]})\frac{df_k}{dx_2}\right) = 0, \tag{3.8}$$

where

$$\Delta = b_1^{[n-2]} - (2a_2^{[n]} - na_0^{[n]})\Bigl(\sum_{\substack{m=1\\m\neq k}}^{n} \frac{v_m}{u_m} f_m\Bigr).$$

It is clear that the only functions in (3.8) containing harmonics of the form $e^{i\nu x_2}$ are $c_0^{[n-2]}$ and f_k'. They satisfy the equation

$$\frac{dc_0^{[n-2]}}{dx_2} - (2a_2^{[n]} - na_0^{[n]})\frac{df_k}{dx_2} = 0.$$

This gives us $c_0^{[n-2]}$:

$$c_0^{[n-2]} = (2a_2^{[n]} - na_0^{[n]})f_k + a_0^{[n-2]},$$

where $a_0^{[n-2]} = \text{const}$. Thus, we have found $b_0^{[n-2]}$:

$$b_0^{[n-2]} = na_0^{[n]}V + (2a_2^{[n]} - na_0^{[n]})f_k + a_0^{[n-2]}. \tag{3.9}$$

From equation (3.6) we can determine $b_1^{[n-2]}$. Recall that we have transformed the function (3.6) to the form (3.8), where

$$\frac{\partial\Delta}{\partial x_1} = 0.$$

Here

$$b_1^{[n-2]} = (2a_2^{[n]} - na_0^{[n]})\Bigl(\sum_{\substack{m=1\\m\neq k}}^{n} \frac{v_m}{u_m} f_m\Bigr) + c_1^{[n-2]}(x_2),$$

where $c_1^{[n-2]}$ is a function of x_2 alone. We get it from equation (3.7) after writing it down explicitly:

$$\frac{\partial b_2^{[n-2]}}{\partial x_1} + (2a_2^{[n]} - na_0^{[n]})\Bigl(\sum_{\substack{m=1\\m\neq k}}^{n} \Bigl(\frac{v_m}{u_m}\Bigr)^2 \frac{\partial f_m}{\partial x_1}\Bigr) + \frac{dc_1^{[n-2]}}{dx_2}$$
$$= (n-2)a_2^{[n]}\frac{\partial V}{\partial x_1} + 3a_3^{[n]}\Bigl(\frac{df_k}{dx_2} + \sum_{\substack{m=1\\m\neq k}}^{n} \frac{v_m}{u_m}\frac{\partial f_m}{\partial x_1}\Bigr).$$

This is an equation of the following form:

$$\frac{\partial\Upsilon}{\partial x_1} + \left(\frac{dc_1^{[n-2]}}{dx_2} - 3a_3^{[n]}\frac{df_k}{dx_2}\right) = 0, \tag{3.10}$$

where

$$\Upsilon = b_2^{[n-2]} + (2a_2^{[n]} - na_0^{[n]})\Bigg(\sum_{\substack{m=1\\m\neq k}}^{n}\left(\frac{v_m}{u_m}\right)^2 f_m\Bigg) - (n-2)a_2^{[n]}V - 3a_3^{[n]}\Bigg(\sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}f_m\Bigg).$$

The only functions in equation (3.10) containing harmonics of the form $e^{i\nu x_2}$ are $c_1^{[n-2]}$ and f_k'. For these functions we have the equation

$$\frac{dc_0^{[n-2]}}{dx_2} - 3a_3^{[n]}\frac{df_k}{dx_2} = 0.$$

This gives us $c_1^{[n-2]}$:

$$c_1^{[n-2]} = 3a_3^{[n]}f_k + a_1^{[n-2]},$$

where $a_1^{[n-2]} = \text{const}$. Thus, we have found $b_1^{[n-2]}$:

$$b_1^{[n-2]} = 3a_3^{[n]}f_k + (2a_2^{[n]} - na_0^{[n]})\Bigg(\sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}f_m\Bigg) + a_1^{[n-2]}. \tag{3.11}$$

We now proceed to the first equation in the group $(3.1.n-4)$. Substituting for $b_0^{[n-2]}$ and $b_1^{[n-2]}$ in this equation their expressions (3.9) and (3.11) we obtain

$$\begin{aligned}
\frac{\partial b_0^{[n-4]}}{\partial x_1} &= (n-2)\big(na_0^{[n]}V + (2a_2^{[n]} - na_0^{[n]})f_k + a_0^{[n-2]}\big)\frac{\partial V}{\partial x_1}\\
&\quad + \Bigg(3a_3^{[n]}f_k + (2a_2^{[n]} - na_0^{[n]})\sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}f_m + a_1^{[n-2]}\Bigg)\Bigg(\frac{df_k}{dx_2} + \sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}\frac{\partial f_m}{\partial x_1}\Bigg)\\
&= n(n-2)a_0^{[n]}\frac{1}{2}\frac{\partial V^2}{\partial x_1} + (n-2)(2a_2^{[n]} - na_0^{[n]})\frac{\partial f_k V}{\partial x_1} + (n-2)a_0^{[n-2]}\frac{\partial V}{\partial x_1}\\
&\quad + (3a_3^{[n]}f_k + a_1^{[n-2]})\frac{df_k}{dx_2} + (2a_2^{[n]} - na_0^{[n]})\frac{df_k}{dx_2}\Bigg(\sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}f_m\Bigg)\\
&\quad + 3a_3^{[n]}\sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}\frac{\partial f_k f_m}{\partial x_1} + (2a_2^{[n]} - na_0^{[n]})\frac{1}{2}\frac{\partial}{\partial x_1}\Bigg(\Bigg(\sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}f_m\Bigg)^2\Bigg)\\
&\quad + a_1^{[n-2]}\sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}\frac{\partial f_m}{\partial x_1}.
\end{aligned}$$

This equation has the following form:

$$\frac{\partial\Psi}{\partial x_1} + (3a_3^{[n]}f_k + a_1^{[n-2]})\frac{df_k}{dx_2} + (2a_2^{[n]} - na_0^{[n]})\frac{df_k}{dx_2}\Bigg(\sum_{\substack{m=1\\m\neq k}}^{n}\frac{v_m}{u_m}f_m\Bigg) = 0, \tag{3.12}$$

where

$$\Psi = \frac{n(n-2)}{2} a_0^{[n]} V^2 + (n-2)(2a_2^{[n]} - na_0^{[n]}) f_k V + (n-2) a_0^{[n-2]} V$$
$$+ (3a_3^{[n]} f_k + a_1^{[n-2]}) \Biggl(\sum_{\substack{m=1 \\ m \neq k}}^{n} \frac{v_m}{u_m} f_m \Biggr) + \frac{1}{2}(2a_2^{[n]} - na_0^{[n]}) \Biggl(\sum_{\substack{m=1 \\ m \neq k}}^{n} \frac{v_m}{u_m} f_m \Biggr)^2 - b_0^{[n-4]}.$$

Clearly the only functions in (3.12) containing harmonics of the form $e^{i\nu x_2}$ are f_k' and $f_k f_k'$. For f_k we have the following equation:

$$(3a_3^{[n]} f_k + a_1^{[n-2]} + \varkappa) v_k f_k' = 0, \tag{3.13}$$

where $\varkappa$ is the mean value of

$$(2a_2^{[n]} - na_0^{[n]}) \Biggl(\sum_{\substack{m=1 \\ m \neq k}}^{n} \frac{v_m}{u_m} f_m \Biggr).$$

By assumption f_k is a non-constant analytic function. Since the ring of analytic functions contains no zero divisors, it follows that $a_3^{[n]} = 0$, as required.

§ 4. Integrals of degree three

Theorem 2. *If a system with Hamiltonian* (2.1) *admits an integral* $F_3 + F_1$ *of degree three with* $F_3 \neq 0$, *then the spectrum of the potential energy lies on a single line passing through the origin.*

Corollary. *Under the assumptions of Theorem* 2 *the Hamiltonian equations have an integral linear in momenta.*

For $a = c$ and $b = 0$, Theorem 2 becomes Byalyi's theorem [8].

Proof of Theorem 2. Since $F_3 \neq 0$, the algebraic equation (3.3) has at most three real roots. Thus, the number of spectrum lines of the potential energy is at most three.

The case where the spectrum lies on two lines has been considered in [11]; Chapter IV. It is shown there that the Hamiltonian equations admit an additional polynomial integral if and only if these lines are orthogonal (with respect to the metric generated by the kinetic energy). We emphasize that this holds for integrals of arbitrary degree. If the spectrum lines are orthogonal, then there exists a polynomial integral of degree two. It remains to show that there are no non-trivial integrals of degree three in this case.

Assume the contrary: let $F_3 + F_1$ be an integral of degree three and let

$$F_3 = a_0 y_1^3 + a_1 y_1^2 y_2 + a_2 y_1 y_2^2 + a_3 y_2^3. \tag{4.1}$$

Rotating, we can make one spectrum line vertical. Then the other line is horizontal, because of orthogonality. By (2.4), $a_1 = a_2 = 0$. Using Theorem 1 we obtain two additional relations: $a_0 = a_3 = 0$. Hence $F_3 \equiv 0$, which is a contradiction.

Remark. The case where the spectrum lies on two straight lines was not discussed in [8] — probably for reasons of space.

Now it remains to consider the case of three distinct spectrum lines. We assume that one of these lines is vertical. Then $a_1 = a_3 = 0$ in (4.1). Let (u, v) be a point in the spectrum with $u \neq 0$. Relations (2.4) and (2.5) bring us to the equalities

$$3a_0v^2 + a_2(u^2 - 2v^2) = 0 \quad \text{and} \quad a_0u^2 + a_2v^2 = 0. \tag{4.2}$$

Since a_0 and a_2 must be distinct from zero (otherwise $F_3 = 0$), the determinant of this linear system vanishes:

$$3v^4 + 2u^2v^2 - u^4 = 0, \quad \text{or} \quad 3z^4 + 2z^2 - 1 = 0, \tag{4.3}$$

where $z = v/u$. The roots of this biquadratic equation are $\pm\sqrt{3}/3$. Thus, the angles between the spectrum lines are $\pi/3$. Now, each equation in (4.2) gives rise to the following relation between the remaining coefficients a_0 and a_2:

$$3a_0 + a_2 = 0. \tag{4.4}$$

Remark. This is the case left out in [8], where Byalyi introduces the quantities

$$k_{ii} = u_i(a_1 + 3a_3 + z_i(3a_0 + a_2)),$$

where the $z_i = v_i/u_i$ are the zeros of (3.3), and he assumes that only one of these quantities may vanish. In the above case, however, $k_{ii} = 0$ for $1 \leqslant i \leqslant 3$. Still, it must be pointed out that the metric considered in [8] has the standard form (1.2), therefore (since the biquadratic equation (4.3) has irrational roots) oblique spectrum lines cannot pass through nodes of the integral lattice.

Thus, it remains to consider the exceptional case when the angles between the three spectrum lines of the potential energy are multiples of $\pi/3$. The Hamiltonian function can be reduced to the following form:

$$H = \frac{(y_1^2 + y_2^2)}{2} + f_1(\sqrt{3}\,x_1 + x_2) + f_2(x_2) + f_3(-\sqrt{3}\,x_1 + x_2),$$

where the f_s are non-constant periodic analytic functions.

The additional third-degree integral F has the form $F_3 + b_0y_1 + b_1y_2$, where b_0 and b_1 are analytic functions on the configuration torus. In our case $a_1 = a_3 = 0$ and (4.4) holds. Since $F_3 \neq 0$, we can set $a_0 = 1$ and $a_2 = -3$.

The functions f_s are defined up to arbitrary additive constants. We shall assume that $\langle f_2 \rangle = 0$, where $\langle \cdot \rangle$ denotes the mean value on the torus $\mathbb{T}^2 = \{x_1, x_2 \bmod 2\pi\}$. Since f_2 is a periodic function, there exists x_2^0 such that $f_2'(x_2^0) = 0$. We set

$$\langle f_1 \rangle = -\langle f_3 \rangle, \qquad \frac{\langle f_3 \rangle}{\sqrt{3}} = -f_2(x_2^0) + \frac{\langle b_0 \rangle}{6}\,. \tag{4.5}$$

The condition $\{H, F\} = 0$ is equivalent to the following system of equations for b_0 and b_1:

$$\begin{aligned} &\frac{\partial b_0}{\partial x_1} + 3\sqrt{3}\,(f_3' - f_1') = 0, \qquad \frac{\partial b_0}{\partial x_2} + \frac{\partial b_1}{\partial x_1} + 6(f_1' + f_2' + f_3') = 0,\\ &\frac{\partial b_1}{\partial x_2} - 3\sqrt{3}\,(f_3' - f_1') = 0, \quad \sqrt{3}\,b_0(f_3' - f_1') - b_1(f_1' + f_2' + f_3') = 0. \end{aligned} \tag{4.6}$$

The first and the third equations give us the equalities

$$b_0 = 3[f_1 + f_3 + a_0(x_2)] \quad \text{and} \quad b_1 = 3\sqrt{3}\,[f_3 - f_1 + a_1(x_1)], \tag{4.7}$$

where a_0 and a_1 are periodic functions of one variable that are not yet known. Substituting (4.7) in the second equation in (4.6) we obtain

$$\frac{da_0}{dx_2} + \sqrt{3}\,\frac{da_1}{dx_1} + 2f_2' = 0.$$

Averaging with respect to x_1 and x_2 we see that $a_0 = -2f_2 + c_0$, $c_0 = \text{const}$, and $a_1 = c_1 = \text{const}$.

Now, equalities (4.7) take the following form:

$$b_0 = 3(f_1 - 2f_2 + f_3 + c_0), \qquad b_1 = 3\sqrt{3}\,(f_3 - f_1 + c_1). \tag{4.8}$$

Substituting (4.8) in the last equation in (4.6) we obtain a non-trivial relation between the functions f_s:

$$2f_1'\Bigl(f_2 - f_3 - \frac{c_0}{2}\Bigr) + f_2'(f_1 - f_3 - c_1) + 2f_3'\Bigl(f_1 - f_2 + \frac{c_0}{2} - \frac{c_1}{2}\Bigr) = 0. \tag{4.9}$$

We set $x_2 = x_2^0$ and

$$\begin{aligned} g_1(x_1) &= f_1(\sqrt{3}\,x_1 + x_2^0) - f_2(x_2^0) + \frac{c_0}{2} - \frac{c_1}{2}\,, \\ g_3(x_1) &= f_3(-\sqrt{3}\,x_1 + x_2^0) - f_2(x_2^0) + \frac{c_0}{2}\,. \end{aligned} \tag{4.10}$$

Since x_2^0 is a critical point of f_2, equality (4.9) takes the following form:

$$\frac{\partial}{\partial x_1}(g_1 g_3) = 0.$$

Hence

$$g_1 g_3 = c = \text{const}\,. \tag{4.11}$$

Averaging the first equality in (4.8) over $\mathbb{T}^2$ we obtain

$$\langle b_0\rangle = 3(\langle f_1\rangle - 2\langle f_2\rangle + \langle f_3\rangle + c_0).$$

By assumption $\langle f_1\rangle + \langle f_3\rangle = 0$ (see (4.5)) and $\langle f_2\rangle = 0$. Hence $c_0 = \langle b_0\rangle/3$. Clearly,

$$\langle g_3\rangle = -\frac{\langle f_3\rangle}{\sqrt{3}} - f_2(x_2^0) + \frac{\langle b_0\rangle}{6}\,.$$

In view of (4.5), the right-hand side of this equality vanishes. Since g_3 is a periodic function of mean value zero, it must have zeros. Hence the constant c in equality (4.11) is zero. The ring of analytic functions contains no zero divisors, therefore either g_1 or g_3 vanishes identically. Then, however, (4.10) shows that one of the functions f_1 and f_3 is constant. Hence (4.9) has no non-trivial periodic solutions, which completes the proof.

Remark. Equation (4.9) has aperiodic solutions. One simple example is as follows:

$$f_1 = \exp(\sqrt{3}\,x_1 + x_2), \quad f_2 = \exp(-2x_2), \quad f_3 = \exp(-\sqrt{3}\,x_1 + x_2),$$

and the constants c_0 and c_1 are equal to zero. This corresponds to the case of an integrable Toda lattice for three particles with centre of mass separated out.

§ 5. Integrals of degree four

Assume that equations with Hamiltonian (2.1) admit a four-degree integral $F_4 + F_2 + F_0$, where

$$F_4 = a_0 y_1^4 + a_1 y_1^3 y_2 + a_2 y_1^2 y_2^2 + a_3 y_1 y_2^3 + a_4 y_2^4. \tag{5.1}$$

We can assume without loss of generality that one of the spectrum lines is vertical. Then the results of § 2 show that $a_1 = a_3 = 0$.

Let (u, v), $u \neq 0$, be a point in the spectrum. Conditions (2.4) and (2.5) give us the relations

$$\begin{aligned} 2a_0 v^3 u + a_2(u^3 v - uv^3) - 2a_4 u^3 v = 0,\\ 2a_0 v u^3 + a_2(uv^3 - u^3 v) - 2a_4 uv^3 = 0. \end{aligned} \tag{5.2}$$

They are trivially satisfied if $v = 0$. Hence the existence of a horizontal spectrum line is not an obstacle to the integrability of the system in question.

What other straight lines can be spectrum lines in the case of an integral of degree four? Dividing the left-hand sides of equations (5.2) by uv and adding up we obtain

$$2(a_0 - a_4)(u^2 + v^2) = 0.$$

Hence $a_0 = a_4$. Now, each equation in (5.2) can be brought into the following form:

$$(2a_0 - a_2)(u^2 - v^2) = 0,$$

therefore $2a_0 = a_2$ or $u = \pm v$. In the first case the Birkhoff sum $a_0 - a_2 + a_4$ is equal to zero. Bearing in mind the equalities $a_1 = a_3 = 0$ we see that the polynomial F_4 is a multiple of the kinetic energy $(y_1^2 + y_2^2)/2$. However, we have already excluded this case (see § 2). Hence $u = \pm v$. This means that the bisectors of the coordinate sectors are also 'admissible' spectrum lines.

Thus, we have proved the following result.

Proposition 1. *If a system with Hamiltonian* (2.1) *admits an irreducible integral of degree four, then its spectrum lies on three or four straight lines making angles* $\pi/4$ *or* $\pi/2$ *with one another.*

This result leaves out the system with Hamiltonian

$$H = \frac{(y_1^2 + y_2^2)}{2} + f_1(x_1) + f_2(x_1 + x_2) + f_3(x_2) + f_4(x_2 - x_1).$$

Here the f_s are 2π-periodic functions at least three of which are not constant. An integral of degree four independent of the energy integral can be brought to the following form:

$$F = \frac{(y_1^4 + y_2^4)}{4} + F_2 + F_0.$$

From the condition $\{H, F\} = 0$ we deduce (as in § 4) the following relation for the functions f_s, which is similar to (4.9):

$$\begin{aligned} f_1'' f_2 + 3f_1' f_2' + 2f_1 f_2'' - f_1'' f_4 + 3f_1' f_4' - 2f_1 f_4'' - f_2 f_3'' - 3f_2' f_3' - 2f_2'' f_3\\ + f_3'' f_4 + 3f_3' f_4' + 2f_3 f_4'' + c_1(f_1'' - f_3'') + c_2(f_2'' - f_4'') = 0. \end{aligned} \tag{5.3}$$

Here the dash indicates the derivative with respect to the argument of the function and c_1 and c_2 are constants. Unfortunately, we have not managed to show that equation (5.3) has no non-trivial periodic solutions. Note that (5.3) has aperiodic solutions.

§ 6. Integrals of degree five or six

Assume now that a system with Hamiltonian (2.1) admits a fifth-degree integral $F_5 + F_3 + F_1$, where

$$F_5 = a_0 y_1^5 + a_2 y_1^3 y_2^2 + a_4 y_1 y_2^4 + a_5 y_2^5. \tag{6.1}$$

As before, we assume that one of the spectrum lines is vertical (so that we set from the start $a_1 = a_3 = 0$ in (6.1)). If the integral is irreducible, then there must be at least one extra spectrum line. Assume that it contains a point in the spectrum with coordinates u, v ($u \neq 0$). Relations (2.4) and (2.5) give us two equations:

$$\begin{gathered} \gamma z^4 + \delta z^2 + a_4 + 5a_5 z = 0, \\ 6\gamma z^2 + \delta(3z^4 - 2z^2 + 1) + a_4(10z^2 + 4) + 30a_5 z^3 = 0. \end{gathered} \tag{6.2}$$

Here $\gamma = 5a_0 - 2a_2$, $\delta = 3a_2 - 4a_4$, $z = v/u$.

In fact, if the spectrum is irreducible, then there must be a third line. For it is shown in [11] that if the spectrum lies on two lines that are not orthogonal, then there can be no additional polynomial integrals of any degree whatsoever. On the other hand, if the two spectrum lines are orthogonal, then there exists a quadratic integral, and the above integral of degree five must be reducible.

Assume that the third spectrum line contains a point with coordinates u_1 and v_1 ($u_1 \neq 0$) and let $z_1 = v_1/u_1$. Then $z_1 \neq z$ satisfies the same equations (6.2). Thus, we obtain a system of homogeneous linear equations with respect to γ, δ, a_4, and a_5. If the determinant Δ of this system does not vanish, then $F_5 = 0$ and the integral is reducible.

The determinant Δ is as follows:

$$30(z^2+1)(z_1^2+1)(z-z_1)^2(z+z_1)[z^2 z_1^2 + (2zz_1+1)^2 - (2zz_1+1)(z^2+z_1^2)]. \tag{6.3}$$

Since $z \neq z_1$, it vanishes in the following two cases:

$$\text{(a)} \quad z_1 = -z, \qquad \text{(b)} \quad z^2 z_1^2 + (2zz_1+1)^2 - (2zz_1+1)(z^2+z_1^2) = 0.$$

In case (a) the additional spectrum lines are symmetric with respect to the first vertical line. The equation in case (b) is easily soluble if one sets $2zz_1 + 1 = x$. Then $x^2 - x(z^2 + z_1^2) + z^2 z_1^2 = 0$. Hence $x = z^2$ or $x = z_1^2$, and therefore

$$2zz_1 + 1 = z^2 \quad \text{or} \quad 2zz_1 + 1 = z_1^2. \tag{6.4}$$

In the first case

$$z_1 = -\left(\frac{2z}{1-z^2}\right)^{-1}. \tag{6.5}$$

Assume that the second (the third) spectrum line runs at angle φ (φ_1) to the horizontal axis. Then (6.5) is equivalent to the relation

$$\varphi_1 = 2\varphi \pm \pi/2.$$

This gives us a simple construction of the third spectrum line: we rotate the second line by angle φ counterclockwise and draw the orthogonal line. The second equation in (6.4) has a similar geometric interpretation.

Note that each of the last two spectrum lines can also be distinguished (and set to be vertical). Then the mutual position of the remaining two lines must also have the above properties. In particular, assume that the vertical and the horizontal axes contain points from the spectrum. Then the third spectrum line makes angle $\pi/4$ with the horizontal axis.

Let l_0, l_1, and l_2 be straight lines through some point; we assume that l_0 is vertical. Let z and z_1 be the slopes of l_1 and l_2 (the tangents of their angles with the horizontal axis). We shall say that l_2 is *conjugate to* l_1 *with respect to* l_0 (and write $l_1 \to l_2$) if z and z_1 are related by the equality

$$z_1 = \frac{z^2 - 1}{2z}.$$

Note that this conjugacy relation is neither reflexive nor transitive.

We can state the above results on the structure of the structure of the spectrum of a Hamiltonian system possessing an irreducible integral of degree 5 as follows. Let l_0, l_1, and l_2 be three arbitrary spectrum lines. Then l_1 and l_2 are symmetric relative to l_0, or $l_1 \to l_2$, or $l_2 \to l_1$.

As one example we consider the case when the spectrum lies on five lines, each making angle $\pi/5$ with the preceding one. We denote them by $l_0, l_1, \dots, l_4$ (Fig. 1). We claim that the necessary condition for the existence of an irreducible integral of degree 5 is satisfied in this case. For the table of the conjugacy relations existing between these lines (with respect to l_0, for definiteness) looks as follows:

$$l_0 \to l_0, \quad l_1 \to l_2, \quad l_2 \to l_4, \quad l_3 \to l_1, \quad l_4 \to l_3.$$

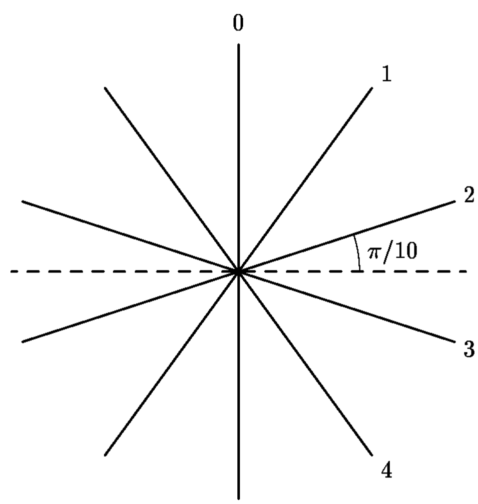

Figure 1

We fix l_0. Then we can choose two other spectrum lines by one of the six ways:

$$l_1 \text{ and } l_4, \quad l_2 \text{ and } l_3, \quad l_1 \text{ and } l_2, \quad l_1 \text{ and } l_3, \quad l_2 \text{ and } l_4, \quad l_3 \text{ and } l_4.$$

In the first two cases the two lines are symmetric relative to l_0, and in the remaining cases one line in the pair is conjugate to the other.

We now discuss the problem of an integral of degree six. It must have the form $F_6 + F_4 + F_2 + F_0$, where

$$F_6 = a_0 y_1^6 + a_2 y_1^4 y_2^2 + a_4 y_1^2 y_2^4 + a_5 y_1 y_2^5 + a_6 y_2^6. \tag{6.6}$$

We assume that one spectrum line is vertical, so that (6.6) contains no coefficients a_1 and a_3.

We can also assume that, alongside the vertical line, there exist two additional distinct spectrum lines defined by their slopes z and z_1. By (2.4) and (2.5) the real-valued slope z satisfies the equations

$$6a_0 z^5 + (-2z^5 + 4z^3)a_2 + (-4z^3 + 2z)a_4 + (5z^2 - 1)a_5 - 6a_6 z = 0, \tag{6.7}$$

$$10a_0 z^3 + (2z^5 - 6z^3 + 2z)a_2 + (-2z^5 + 6z^3 - 2z)a_4 + (5z^4 - 5z^2)a_5 - 10a_6 z^3 = 0. \tag{6.8}$$

It is convenient to introduce the variables

$$\gamma_1 = 6a_0 - 2a_2, \quad \gamma_2 = 4a_2 - 4a_4, \quad \gamma_3 = 2a_4 - 6a_6. \tag{6.9}$$

In this notation equations (6.7) and (6.8) get a simpler form:

$$\gamma_1 z^5 + \gamma_2 z^3 + \gamma_3 z + a_5(5z^2 - 1) = 0,$$
$$10\gamma_1 z^3 + \gamma_2(3z^5 - 4z^3 + 3z) + 10\gamma_3 z^3 + a_5(30z^4 - 30z^2) = 0.$$

Obviously, the same relations hold for z_1. Thus, we obtain a homogeneous linear system with respect to γ_1, γ_2, γ_3, and a_5. If its determinant is distinct from zero, then these parameters must vanish. Since $a_1 = a_3 = 0$ and we have (6.9), both Birkhoff sums

$$a_1 - a_3 + a_5 \quad \text{and} \quad a_0 - a_2 + a_4 - a_6$$

vanish in that case, which means that F_6 is a multiple of $(y_1^2 + y_2^2)/2$ (the kinetic energy of the system).

The above determinant is equal to

$$z z_1 (z z_1 + 1)\Delta, \tag{6.10}$$

where Δ is defined by (6.3). Hence we can determine the possible position of the two remaining spectrum lines from the condition that the product (6.10) vanishes. We have already analyzed the condition $\Delta = 0$ in our discussion of integrals of degree five. In the present case we have two additional relations: $zz_1 = 0$ and $zz_1 + 1 = 0$. The first shows that one spectrum line can be horizontal, and the second is the orthogonality condition for two spectrum lines.

Now let l_0, l_1, and l_2 be three arbitrary spectrum lines of a Hamiltonian system. If it possesses an irreducible integral of degree six, then one of the following conditions must hold:

(a) l_1 or l_2 is orthogonal to l_0,
(b) l_1 is orthogonal to l_2,
(c) l_1 and l_2 are symmetric relative to l_0,
(d) $l_1 \to l_2$ or $l_2 \to l_1$ (with respect to l_0).

We consider now the example when six spectrum lines make angle $\pi/6$ with one another. We fix one of them, say l_0. Then there exist 10 ways to select l_1 and l_2. It is easy to verify that in each case one of conditions (a)–(d) is satisfied.

§ 7. On integrals of arbitrary degrees

There exists another approach to the problem of integrals polynomial in momenta. Assume that the spectrum of the potential energy lies on n distinct straight lines passing through the origin. As pointed out in § 2, the dynamical system admits no non-trivial additional integrals of degree $k < n$ in this case. Hence the minimum possible degree of an additional polynomial integral is n.

On the other hand we showed in § 4 that if a system admits an integral of degree three and there are three distinct spectrum lines, then they make angles $\pi/3$ with one another. A similar observation holds for integrals of degree four: the four possible spectrum lines make angles $\pi/4$ with one another. It turns out that this observation can be generalized.

Theorem 3. *If there exists a polynomial integral of degree n independent of the energy integral, then the n spectrum lines of the potential energy make angles*

$$\frac{\pi}{n}, \quad \frac{2\pi}{n}, \quad \dots, \quad \frac{(n-1)\pi}{n}$$

with one another.

This indicates an interesting connection between a continuous symmetry group (related to the additional integral) and discrete symmetries of a dynamical system.

Theorem 3 has an interesting application to systems with 'standard' metric (1.2).

Corollary. *Assume that a Hamiltonian system with kinetic energy* (1.2) *has spectrum lying on $n \neq 4$ distinct straight lines passing through the origin. Then the system has no additional polynomial integral of degree n.*

In fact it is well known that the quantities $\tan(\pi/n)$ are irrational for $n \geqslant 3$, $n \neq 4$. The difficulties arising in the special case $n = 4$ were expounded in § 5.

Proof of Theorem 3. Let

$$F_n = a_0 y_1^n + a_1 y_1^{n-1} y_2 + \dots + a_n y_2^n$$

be the leading homogeneous component of the assumed polynomial integral of degree n. Suppose that the spectrum contains a point with coordinates u, v. We can always assume that $v \neq 0$ (by rotating the plane about the origin). The ratio $z = u/v$ defines the spectrum line containing the point (u, v). By assumption, we are given n distinct quantities $z_1, \dots, z_n$ obtained in this way. By (2.4) and (2.5) all these quantities are roots of the following two polynomials of degree n:

$$0 = \sum_{i=0}^{n} z^i \big((-1)^i C_{i+1}^1 a_{i+1} + (-1)^{i-1} C_{n-i+1}^1 a_{i-1}\big), \tag{7.1}$$

$$0 = \sum_{i=0}^{n} z^i \big((-1)^i C_{i+3}^3 a_{i+3} + (-1)^{i-1} C_{n-i-1}^1 C_{i+1}^2 a_{i+1} + (-1)^{i-2} C_{n-i+1}^2 C_{i-1}^1 a_{i-1} + (-1)^{i-3} C_{n-i+3}^3 a_{i-3}\big). \tag{7.2}$$

Hence their coefficients are proportional. Let ω be the ratio of the corresponding coefficients of (7.1) and (7.2).

Let $n = 2m$. The case of odd n can be considered in a similar way (and is even simpler). The condition that the coefficients are proportional can be expressed as the combination of two closed systems of linear equations:

$$\omega(-C_{2j+2}^1 a_{2j+2} + C_{n-2j}^1 a_{2j}) = -C_{2j+4}^3 a_{2j+4} + C_{n-2j-2}^1 C_{2j+2}^2 a_{2j+2} - C_{n-2j}^2 C_{2j}^1 a_{2j} + C_{n-2j+2}^3 a_{2j-2}, \tag{7.3}$$

where $j = 0, \dots, m-1$, and

$$\omega(C_{2j+1}^1 a_{2j+1} - C_{n-2j+1}^1 a_{2j-1}) = C_{2j+3}^3 a_{2j+3} - C_{n-2j-1}^1 C_{2j+1}^2 a_{2j+1} + C_{n-2j+1}^2 C_{2j-1}^1 a_{2j-1} - C_{n-2j+3}^3 a_{2j-3}, \tag{7.4}$$

where $j = 0, \dots, m$.

We multiply each equation in (7.3) by $(-1)^j$ and add them, arriving at the relation

$$(\omega C_n^1 + C_n^3)[a_0 - a_2 + a_4 - \dots + (-1)^m a_{2m}] = 0. \tag{7.5}$$

In a similar way, by (7.4) we obtain

$$(\omega C_n^1 + C_n^3)[a_1 - a_3 + a_5 - \dots + (-1)^{m-1} a_{2m-1}] = 0. \tag{7.6}$$

The expressions in the square brackets in (7.5) and (7.6) are the Birkhoff sums for F_n. Since F_n is not a multiple of $(y_1^2 + y_2^2)/2$ (by assumption), at least one of these sums is distinct from zero. Hence

$$\omega = -C_n^3 / C_n^1. \tag{7.7}$$

Substituting this in (7.4) with $j = 0$ we obtain

$$a_1 = \alpha C_n^1, \qquad a_3 = -\alpha C_n^3,$$

where α is a real constant distinct from zero. Using relations (7.4) and taking account of (7.7) we successively find the a-coefficients with odd indices:

$$a_{2j+1} = (-1)^j \alpha C_n^{2j+1}, \qquad 0 \leqslant j \leqslant m-1.$$

Hence

$$C^1_{2j+1}a_{2j+1} - C^1_{n-2j+1}a_{2j-1} = (-1)^j \alpha n C^{2j}_n. \tag{7.8}$$

Relation (7.3) with $j = 0$, in view of (7.7), gives us

$$-C^1_2 a_2 + C^1_n a_0 = \beta C^1_n,$$

where β is a real coefficient distinct from zero.

By (7.3) and (7.7) we successively obtain

$$-C^1_{2j+2}a_{2j+2} + C^1_{n-2j}a_{2j} = (-1)^j \beta C^{2j+1}_n. \tag{7.9}$$

Relations (7.8) and (7.9) enable us to bring algebraic equation (7.1) into the following form:

$$n\alpha \sum_{j=0}^{m} (-1)^j C^{2j}_n z^{2j} + \beta \sum_{j=0}^{m-1} (-1)^j C^{2j+1}_n z^{2j+1} = 0. \tag{7.10}$$

Bearing in mind that $z = \tan\varphi$, where φ is the angle between the spectrum line and the horizontal axis, by (7.10) we obtain the equation

$$n\alpha \cos n\varphi - \beta \sin n\varphi = 0. \tag{7.11}$$

Since α and β are distinct from zero, its solutions are the n angles cited in Theorem 3, as required.

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