

# Instability of Isolated Equilibria of Dynamical Systems With Invariant Measure in Spaces of Odd Dimension

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UDC 517.925

**ABSTRACT.** We discuss the conjecture asserting that isolated equilibrium states of autonomous systems admitting invariant measures are unstable in spaces of odd dimension. This conjecture is proved for systems for which quasihomogeneous truncations with isolated singularities can be found. We consider a counterexample in the class of systems with infinitely differentiable right-hand sides and zero Maclaurin series at the equilibrium state.

**KEY WORDS:** dynamical system, invariant measure, homogeneity, equilibrium, stability in the sense of Lyapunov, quasihomogeneous truncation, invariant torus.

## §1. Introduction

Let

$$\dot{x} = v(x), \quad x \in \mathbb{R}^n, \quad (1)$$

be an autonomous system of differential equations admitting an invariant measure with smooth density:

$$\operatorname{div}(\rho v) = 0, \quad x \in \mathbb{R}^n. \quad (2)$$

Let  $x = 0$  be an equilibrium state:  $v(0) = 0$ .

V. V. Ten conjectured that if  $n$  is odd and the state of equilibrium  $x = 0$  is isolated, then this equilibrium state is unstable in the sense of Lyapunov. There is an important corollary to this conjecture: all isolated equilibrium states of steady-state fluid flows in the three-dimensional Euclidean space are unstable.

In the typical situation system (1) has the form

$$\dot{x} = Ax + o(|x|), \quad \det A \neq 0. \quad (3)$$

We rewrite the "continuity equation" (2) as follows:  $\dot{\nu} = -\operatorname{div} v$ , where  $\nu = \ln \rho$ . By setting  $x = 0$  in this equation, we obtain the relation  $\operatorname{tr} A = 0$ . Hence the sum of all eigenvalues of the matrix  $A$  is equal to zero. At least one of the eigenvalues lies in the right-hand half-plane. Indeed, otherwise, the spectrum of the matrix  $A$  lies on the imaginary axis. Since  $A$  is a real matrix, the sum of all its eigenvalues is zero, and  $n$  is odd, zero is necessarily an eigenvalue. However, this contradicts the assumption that the matrix  $A$  is nondegenerate. Hence, by the Lyapunov theorem,  $x = 0$  is an unstable equilibrium of system (3).

In a similar way, one can prove that a nondegenerate periodic trajectory of a dynamical system with invariant measure is always unstable in a space of even dimension. Recall that a periodic orbit is said to be *nondegenerate* if its multipliers are not equal to 1. This observation also remains valid for nondegenerate reducible invariant tori of odd codimension filled with conditionally periodic trajectories.

## §2. Semiquasihomogeneous systems

This instability assertion can be generalized to systems with semihomogeneous right-hand side

$$v = v_m + v_{m+1} + \dots, \quad \text{where} \quad v_k(\lambda x) = \lambda^k v_k(x), \quad m \geq 1.$$

The only additional condition is that  $x = 0$  is the unique equilibrium of a homogeneous field  $v_m$ .

We even consider a more general case of semiquasihomogeneous vector fields. Recall that a field  $v(x)$  is called a *quasihomogeneous field of degree  $m$  with quasihomogeneity exponents  $g_1, \dots, g_n > 0$*  if

$$v_i(\lambda^{g_1} x_1, \dots, \lambda^{g_n} x_n) = \lambda^{g_i+m-1} v_i(x_1, \dots, x_n),$$

where  $v_i$  is the  $i$ th component of the field  $v$ . A smooth field  $v$  is called *semiquasihomogeneous* if one can write this field as a formal series

$$v_m + \sum_{\alpha > m} v_\alpha, \quad (4)$$

where  $v_k$  are quasihomogeneous fields of degree  $k$  with the same quasihomogeneity exponents. For homogeneous fields one can set  $g_1 = \dots = g_m = 1$ .

A field  $v_m$  is called a *quasihomogeneous truncation* of the original field  $v$ . Note that quasihomogeneous truncations can be chosen in different ways for the same system. A typical example of this situation is the following system on the plane

$$\dot{x}_1 = x_2^2, \quad \dot{x}_2 = x_1^3. \quad (5)$$

If we set  $g_1 = g_2 = 1$ , then the system  $\dot{x}_1 = x_2^2$ ,  $\dot{x}_2 = 0$  is a quasihomogeneous truncation. If we set  $g_1 = 3/5$ ,  $g_2 = 4/5$ , then the quasihomogeneous truncation coincides with the original system (5). In the first case the origin  $x_1 = x_2 = 0$  is a nonisolated equilibrium, while in the second case it is isolated. Algorithms for obtaining quasihomogeneous truncations are discussed in [1].

**Theorem 1.** *Suppose that a field  $v$  can be represented in the form (4), so that  $x = 0$  is an isolated zero of the field  $v_m$ . If  $n$  is odd and the system admits an invariant measure, then the equilibrium  $x = 0$  is unstable.*

**Proof.** Since  $n$  is even and  $x = 0$  is a unique zero of the quasihomogeneous field  $v_m$ , there exists (as was shown in [2]) a nonzero vector  $z$  that satisfies one of the equations

$$v_m(z) = -Gz \quad \text{or} \quad v_m(z) = Gz,$$

where  $G = \text{diag}(g_1, \dots, g_n)$  is a diagonal matrix. As was proved in [2], in this case system (1) admits solutions with asymptotics  $zt^{-G}$  or  $zt^{+G}$  as  $t \rightarrow +\infty$  or  $t \rightarrow -\infty$ , respectively. These solutions tend to the equilibrium  $x = 0$  as  $t \rightarrow \pm\infty$ . If there exists a solution of the second kind (i.e., "issuing from the point  $x = 0$ "), then, obviously, the equilibrium  $x = 0$  is unstable. It remains to consider the case in which there exists a solution that asymptotically tends to the equilibrium  $x = 0$  as  $t \rightarrow +\infty$ .

We use the following statement, which is of interest in itself.

**Lemma 1.** *Suppose that system (1) with invariant measure has a nontrivial solution  $t \mapsto x(t)$  that tends to zero as  $t \rightarrow +\infty$ . Then the equilibrium  $x = 0$  is unstable.*

Indeed, suppose that a point  $x_0 = x(0)$  lies outside some  $\varepsilon_0$ -neighborhood of zero. For any  $\varepsilon > 0$  there exists a small neighborhood  $U$  of the point  $x_0$  with the following property. After a time, under the action of the phase flow, the entire neighborhood  $U$  will be mapped into the  $\varepsilon$ -neighborhood of the point  $x = 0$ . Since the phase flow preserves the measure, it follows from the Schwarzchild–Littlewood Theorem [3] that almost all trajectories issuing from  $U_\varepsilon$  leave the  $\varepsilon_0$ -neighborhood of the point  $x = 0$ . This proves that the equilibrium  $x = 0$  is unstable, since the issuing trajectories intersect the  $\varepsilon$ -neighborhood of zero.

### §3. A counterexample in the smooth case

Now we show that in the smooth case Ten's conjecture without any additional nondegeneracy assumptions does not hold.

First, we note that it suffices to have an example in  $\mathbb{R}^3$ , since in this case, having an example in  $\mathbb{R}^{2n+1}$ , we can construct an example in  $\mathbb{R}^{2n+3}$  by using the direct product with the system

$$\dot{x} = -y, \quad \dot{y} = x, \quad (x, y) \in \mathbb{R}^2.$$

By construction, in a sufficiently small neighborhood of the equilibrium there are no periodic solutions whose periods are less than any constant given in advance. This means that, by adding the equation  $\dot{\psi} = 1$ ,  $\psi \in \mathbb{R}/(2\pi\mathbb{Z})$ , we obtain an example of a stable isolated periodic solution for a divergence-free vector field in a space of even dimension.

First, we describe the main idea. We construct a sequence  $\mathbb{R}^3 \supset S_0 \supset S_1 \supset \dots$  of embedded closed solid tori whose diameter (treated as the diameter of sets in  $\mathbb{R}^3$ ) tends to zero. A vector field  $v$  with a unique singular point  $O = \bigcap_{n=0}^{\infty} S_n$  is chosen on  $S_0$ . Since  $v$  is chosen so that the boundaries  $\sigma_k = \partial S_k$  of these solid tori are invariant with respect to  $v$ , the singular point  $O$  is stable in the sense of Lyapunov.

Our construction is based on induction. Therefore, it suffices to find a vector field  $v_0$ , which is the restriction of  $v$  to the domain  $S_0 \setminus S_1$ . By using a diffeomorphism  $F$ , we map this domain into the space  $\mathbb{R}^3/G$ , where  $G$  is the group of parallel translations  $(r, \varphi, h) \mapsto (r, \varphi, h + 2\pi k)$ ,  $k \in \mathbb{Z}$ . Here  $(r, \varphi, h)$  are the cylindrical coordinates in  $\mathbb{R}^3$ . The space  $\mathbb{R}^3/G$  is diffeomorphic to the direct product of a two-dimensional plane by a circle. In what follows, it is convenient to interpret  $\mathbb{R}^3/G$  as the slab  $\{(r, \varphi, h) : -\pi \leq h \leq \pi\}$  in which the lower and upper planes are identified. The torus  $M = F(\sigma_0)$  has the form  $M = \{(r, \varphi, h) : r = a\}$ . We assume that

$$F(\sigma_1) = N = \{(r, \varphi, h) : (r - a/2)^2 + h^2 = a^2/16\}.$$

The domain  $D = F(S_0 \setminus S_1)$  lies between the tori  $M$  and  $N$  (see Fig. 1).

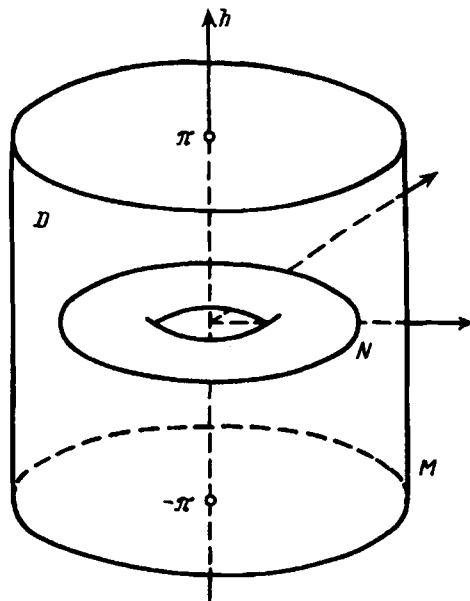


FIG. 1. Domain  $D$

The surfaces  $M$  and  $N$  are invariant under rotations about the axis  $h$ . The vector field  $u = F_*(v_0)$  is also invariant under rotations. We set  $u = u' + u''$ , where  $u' = \partial/\partial\varphi$  and  $u''$  is tangent to the tori  $M$  and  $N$  of the “vertical cylinders”  $\{(r, \varphi, h) : r = \text{const}\}$ . We assume that the projection of  $u''$  on the axis  $h$  is positive at the points of the circle  $\{(r, \varphi, h) : r = 0\}$ . Then the vector field  $u$  has no singular points on  $D$ . The field  $u''$  can be chosen to be divergence free.

The vector fields  $v_k = v|_{S_k \setminus S_{k+1}}$ ,  $k > 0$ , are constructed in a similar way. It remains to match the fields  $v_k$  and  $v_{k+1}$  and to take care that  $|v_k|$  tend to zero as  $k \rightarrow \infty$  sufficiently fast.

Now we pass to the accurate construction. To obtain smoothness, the construction must be made somewhat more complicated. Namely, we take the following sequence of solid tori:

$$\mathbb{R}^3 \supset S_0 \supset T_1 \supset S_1 \supset T_2 \supset \dots$$

All tori  $\sigma_k = \partial S_k$ ,  $\tau_k = \partial T_k$  are invariant. The above sequence of vector fields  $v_k$  is defined on the sequence of domains  $S_k \setminus T_{k+1}$ , and a smooth transition from  $v_k$  to  $v_{k+1}$  occurs on the domains  $T_{k+1} \setminus S_{k+1}$  that are diffeomorphic to the direct product of a two-dimensional torus by a half-interval.

Let us describe the main element of the construction, the vector field  $v_k$ , in more detail. First, we construct its preimage, the vector field  $u = u' + u''$ , in the domain  $D$ . The field  $u'$  has already been constructed. This field preserves the standard volume on  $\mathbb{R}^3/G$ . It suffices to construct the field  $u''$  on the cylinder

$$Z = \{(r, \varphi, h) : \varphi = 0, 0 < r < 2a\}.$$

In the definition of the cylinder  $Z$ , we write  $r < 2a$  instead of  $r < a$ , since, in the following, it is convenient to assume that  $u$  can be smoothly continued to a neighborhood of the closure  $\bar{D}$  of the domain  $D$ .

The standard volume form on  $\mathbb{R}^3$  is written as  $r dr \wedge d\varphi \wedge dh$ . Therefore, the condition that the field  $u''$  is volume-preserving turns into the condition that the area form  $r dr \wedge dh$  is preserved on  $Z$ .

Let us consider a smooth function  $H: Z \rightarrow \mathbb{R}$  such that

- i)  $H = r^2$  in the neighborhood of the circle  $\{r = 0\}$ ,
- ii)  $H = r^2 + \text{const}$  in the neighborhood of the circle  $\{(r, \varphi, h) : \varphi = 0, r = a\}$ ,
- iii) the function  $H$  is constant on the circle

$$\Lambda = \{(r, \varphi, h) : \varphi = 0, (r - a/2)^2 + h^2 = a^2/16\}.$$

For  $u''|_Z$  we take a Hamiltonian vector field with Hamiltonian  $H$  in the symplectic structure  $r dr \wedge dh$ . Obviously,  $u''|_Z$  preserves the form  $r dr \wedge dh$ . We have  $u''|_Z = 2\partial/\partial h$  for small  $r$  and for  $r$  close to  $a$ . Moreover,  $u''|_Z$  is tangent to the circles  $\{r = a\}$  and  $\Lambda$ . So we have constructed the vector field  $u$  on the domain  $D$  (and even on a neighborhood of its closure). This field is volume-preserving and has no singular points. Indeed, outside the circle  $\{r = 0\}$  the projection of  $u$  on the plane  $\{h = 0\}$  is nonzero, while on the circle  $\{r = 0\}$  we have  $u = u'' = 2\partial/\partial h$ .

Next, we assume that the number  $a$  in the definition of the domain  $D$  is small.

We consider the mapping  $f: D \rightarrow \mathbb{R}^3$  given by the formulas  $f(r, \varphi, h) = (x, y, z)$ , where

$$x = (1 + \rho \cos \varphi) \cos h, \quad y = (1 + \rho \cos \varphi) \sin h, \quad z = \rho \sin \varphi.$$

Here  $\rho = \rho(r, \varphi)$  is the least nonnegative root of the equation

$$\rho^2 + \frac{2}{3}\rho^3 \cos \varphi = r^2.$$

Since  $r$  attains values on the interval  $[0, a]$  where  $a > 0$  is small (even if it is necessary to continue the mapping  $f$  to a larger domain), the function  $\rho(r, \varphi)$  is smooth.

Straightforward calculations show that

$$dx \wedge dy \wedge dz = (\rho + \rho^2 \cos \varphi) d\rho \wedge d\varphi \wedge dh = r dr \wedge d\varphi \wedge dh,$$

i.e., the mapping  $f$  is volume-preserving. Hence the vector field  $v_0 = f_* u$  preserves the standard volume on the domain  $f(D)$ .

We set  $\sigma_0 = f(M)$  and  $\tau_1 = f(N)$ . Since  $a$  is small, the torus  $\sigma_0$  is close to the circle

$$\lambda_0 = \{(x, y, z) : x = \cos h, y = \sin h, z = 0\}.$$

On the other hand, if  $a$  is small, the circle

$$\lambda_1 = \left\{ (x, y, z) : x = 1 + \frac{a}{2} \cos \varphi, y = 0, z = \frac{a}{2} \sin \varphi \right\}$$

lies in the interior of the torus  $\tau_1$ . Hence in the interior of  $\tau_1$  there is a torus similar to the torus  $\sigma_0$  and close to the circle  $\lambda_1$ .

By  $S_0, T_1$ , and  $S_1$  we denote closed solid tori with boundaries  $\sigma_0, \tau_1$ , and  $\sigma_1$ , respectively. Obviously,  $S_1 \subset T_1 \subset S_0$ . The domain  $T_1 \setminus S_1$  is diffeomorphic to a two-dimensional torus multiplied by a half-interval.

Let  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a similarity transformation of  $\sigma_0$  into  $\sigma_1$ . We set

$$\begin{aligned} \tau_2 &= g(\tau_1), & \sigma_2 &= g(\sigma_1), & \tau_3 &= g(\tau_2), & \dots, \\ T_2 &= g(T_1), & S_2 &= g(S_1), & T_3 &= g(T_2), & \dots \end{aligned}$$

For the vector fields  $v_k = v|_{S_k \setminus T_{k+1}}$  we take

$$v_k = \vartheta_k \left( \underbrace{g \circ \dots \circ g}_{k \text{ times}} \circ f \right)_* u, \quad 0 < \vartheta_k \in \mathbb{R}.$$

So the field  $v$  is defined in the union of the domains  $(S_0 \setminus T_1) \cup (S_1 \setminus T_2) \cup \dots$ .

One can readily continue  $v$  to a smooth field in the domain  $S_0$ . Indeed, by using a diffeomorphism  $Q_k$ , we can transform a neighborhood of the domain  $T_k \setminus S_k$  to the domain

$$\{(\mu, \alpha, \beta) : -\varepsilon < \mu < 1 + \varepsilon, \alpha \bmod 2\pi, \beta \bmod 2\pi\}, \quad 0 < \varepsilon < \frac{1}{3},$$

so that

$$Q_k(\tau_k) = \{(\mu, \alpha, \beta) : \mu = 0\}, \quad Q_k(\sigma_k) = \{(\mu, \alpha, \beta) : \mu = 1\}.$$

Moreover, we can assume that the restriction of the field  $(Q_k)_* v$  to the tori  $Q_k(\tau_k)$  and  $Q_k(\sigma_k)$  has a positive coordinate with respect to  $\partial/\partial\alpha$  in the basis  $\partial/\partial\mu, \partial/\partial\alpha, \partial/\partial\beta$ . The existence of a smooth vector field coinciding with  $(Q_k)_* v$  in the domains

$$\{(\mu, \alpha, \beta) : -\varepsilon < \mu < 0\} \quad \text{and} \quad \{(\mu, \alpha, \beta) : 1 < \mu < 1 + \varepsilon\}$$

follows from the fact that  $(Q_k)_* v$  can be smoothly continued to the domains

$$\{(\mu, \alpha, \beta) : \varepsilon < \mu < \varepsilon\} \quad \text{and} \quad \{(\mu, \alpha, \beta) : 1 - \varepsilon < \mu < 1 + \varepsilon\}.$$

The smoothness at the equilibrium  $O$  can be achieved by letting the sequence  $\vartheta_k$  tend to 0 sufficiently fast. In this case the derivatives of all orders of the field  $v$  are zero at the point  $O$ . At the same time, we can prove that in the neighborhood of the point  $O$  there are no periodic solutions of small periods (for instance, of periods  $\leq 2\pi$ ).

It is not improbable that the conjecture studied here remains valid in the analytical case.

#### §4. Potential fields

With hydrodynamical applications in mind, we consider the potential vector field (1):  $v = \partial\varphi/\partial x$ .

**Theorem 2.** *Suppose that  $\varphi$  is a nonconstant analytic function. Then all equilibrium states of system (1) are unstable.*

We stress that here we do not assume that the dimension  $n$  is odd.

**Proof.** Suppose that  $x = 0$  is a critical point of the function  $\varphi$  and  $\varphi(0) = 0$ . We expand the potential in its Maclaurin series with respect to homogeneous forms

$$\varphi = \varphi_m + \varphi_{m+1} + \dots$$

Since  $\varphi \neq \text{const}$ , we have  $\varphi_m \neq 0$  for some  $m \geq 2$ . We expand the density of the invariant measure in the Maclaurin series  $\rho = \rho_0 + \dots$ ,  $\rho_0 = \text{const} > 0$ . It follows from Eq. (2) that  $\varphi_m$  is a harmonic function. By the mean value theorem, this function attains values of different sign on the unit sphere  $S^{n-1} = \{x : \|x\| = 1\}$ . Let  $z$  be the maximum point of the function  $\varphi_m$  on  $S^{n-1}$ ; obviously, we have  $\varphi_m(z) > 0$ . According to the Lagrange multiplier rule,

$$\frac{\partial\varphi_m}{\partial x} = \mu x$$

at the point  $x = z$ . The Euler formula for homogeneous functions implies

$$m\varphi_m = \left( \frac{\partial\varphi_m}{\partial x}, x \right) = \mu(x, x) = \mu.$$

Hence  $\mu > 0$ . So the algebraic equation  $v_{m-1}(x) = \mu x$  ( $v_{m-1}(x) = \partial\varphi_m/\partial x$ ) has a nontrivial solution with  $\mu > 0$ . Thus, according to [2], the original system (1) admits an asymptotic solution "issuing" from the equilibrium  $x = 0$ . The proof of the theorem is complete.  $\square$

The authors wish to thank V. V. Ten for useful discussions.

This research was supported by the Russian Foundation for Basic Research under grant No. 96-01-00747 and by INTAS.

#### References

1. A. D. Bryuno, "Power-law asymptotics of solutions to nonlinear systems," *Izv. Akad. Nauk SSSR Ser. Mat. [Math. USSR-Izv.]*, **29**, No. 2, 329–364 (1965).
2. V. V. Kozlov and C. D. Furta, "The first Lyapunov method for strongly nonlinear systems," *Prikl. Mat. Mekh. [J. Appl. Math. Mekh.]*, **60**, No. 1, 10–22 (1996).
3. K. L. Siegel, *Vorlesungen über Himmelsmechanik*, Springer, Berlin (1956).

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