Doklady Physics, Vol. 44, No. 7, 1999, pp. 466–468. Translated from Doklady Akademii Nauk, Vol. 367, No. 2, 1999, pp. 191–193.

MECHANICS

Chaotization of Oscillations of Coupled Pendulums

N. V. Denisova and Corresponding Member of the RAS V. V. Kozlov

Received March 15, 1999

1. Equations of motion. The configuration space of two coupled mathematical pendulums is a two-dimensional torus, and the generalized coordinates x_1 and x_2 are the angular deflections of the pendulums from the vertical. The kinetic energy is specified by the flat metric on the configuration torus as

$$T = \frac{1}{2}\left(\frac{y_1^2}{m_1 l_1^2} + \frac{y_2^2}{m_2 l_2^2}\right).$$

Here, y_1 and y_2 are the canonical momenta, m_1 and m_2 are the point masses, and l_1 and l_2 are the rod lengths. The potential energy is the sum of the gravitational energy

$$-\sum_{k=1}^{2} m_k g l_k \cos x_k$$

(g is the gravitational acceleration) and the elastic-strain energy f of the spring.

If the points of suspension do not coincide, the Fourier series for the potential energy contains essentially all harmonics. Therefore, according to the Poincaré classical results [1, 2], this system exhibits a chaotic behavior provided that the total energy $h = T + V$ is sufficiently large.

We consider the more complex case when the points of suspension of the pendulums coincide; in this case, the potential interaction energy f depends on the phase difference $x_1 - x_2$.

The problem of motion of a system with large h is equivalent to studying the Hamilton differential equations

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}, \quad k = 1, 2, \qquad (1.1)$$

with the Hamiltonian $H = T + \varepsilon V$, where ε is a small parameter [2].

Moscow State University, Vorob'evy gory, Moscow, 119899 Russia

e-mail: ndenis@nw.math.msu.su (Denisova) and vako@nw.math.msu.su (Kozlov)

2. Elastic interaction. In the case of an elastic spring, the potential energy V is a trigonometric polynomial and has the spectrum (indices of zero harmonics) shown in the figure.

The perturbation theory of such systems was studied in [3]. Let

$$\alpha = (0, 1), \quad \beta = (-1, 1)$$

be two adjoining vertices of the convex envelope of the spectrum, which are nonorthogonal to each other in the intrinsic metric $\langle \cdot, \cdot \rangle$ defined by the kinetic energy T. As shown in [3], the invariant tori of the nonperturbed problem (when $\varepsilon = 0$), which are defined by the equations

$$\langle y, k\alpha + \beta \rangle = 0, \quad k = 0, 1, 2, \ldots, \qquad (2.1)$$

disappear provided that a perturbation is added. These tori are resonance ones. When they disappear, pairs of isolated long-period solutions originate [2]. For $k \longrightarrow \infty$, they accumulate near the resonance torus $y_2 = 0$: the second pendulum is at rest, but the first rotates steadily. One more set of disappearing resonance tori is obtained by an interchange of the pendulums.

Let the adjoining vertices α and β be also interchanged. Then, equation (2.1) takes the form

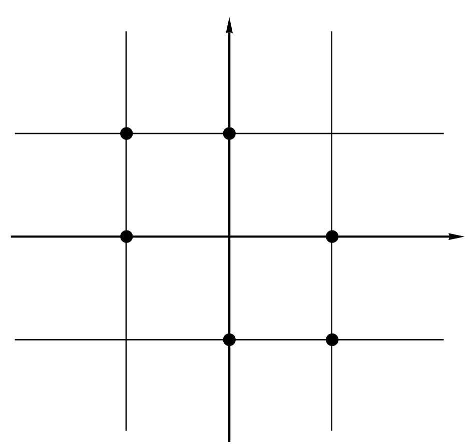

Figure.

$\langle y, k\beta + \alpha\rangle = 0$. For $k \longrightarrow \infty$, these resonance tori accumulate near the tori defined by the resonance relations

$$\omega_1 = \omega_2, \quad \omega_k = \frac{\partial T}{\partial y_k}.$$

A set of steady rotations of pendulums with the same phase corresponds to this torus.

3. Inelastic interaction. The case when the Fourier series for the interaction energy f contains an infinite number of harmonics is of the most interest for us.

Let us consider the more general system (1.1) with

$$T = \frac{1}{2}\sum a_{ij} y_i y_j, \quad a_{ij} = \text{const},$$

and an infinite part of the spectrum of the potential energy V lying along a vertical straight line. The problem of coupled pendulums under consideration is reduced to this problem through rotation by the angle $\pi/4$ in the plane x_1, x_2.

In the case of system (1.1), the existence of the first analytical integral, independent of the integral $T + \varepsilon V$ and taking the form of a series in ε, was shown in [2] to be equivalent to the existence of an additional integral in the form of a polynomial in momenta, with its coefficients periodic in x_1 and x_2.

The following theorem is our principal result.

Theorem. *System* (1.1) *admits an additional polynomial integral if and only if the remaining points of the spectrum lie in a horizontal straight line passing through the origin of coordinates.*

Corollary. *If system* (1.1) *admits of a new polynomial integral, then there necessarily exists an independent polynomial integral of a degree no larger than two.*

The theorem is proved by using the results of [2, 3].

The Poincaré sets are vital to this proof [2]. Let us recall their definition.

Let

$$V = \sum v_\tau e^{i(\tau, x)}, \quad \tau \in \mathbb{Z}^2.$$

The Poincaré set $\mathbb{P}^1$ of the first order is a set of the momenta $y \in \mathbb{R}^2$ such that

$$\langle y, \tau\rangle = 0, \quad v_\tau \neq 0, \quad \tau \neq 0.$$

The Poincaré set $\mathbb{P}^2$ of the second order is defined as a set of the momenta $y \in \mathbb{R}^2 \backslash \mathbb{P}^1$ satisfying the conditions

$$\langle y, \tau\rangle = 0, \quad v'_\tau \neq 0, \quad \tau \neq 0,$$

where

$$v'_\tau = \sum_{p+q=\tau} \frac{\langle p, q\rangle v_p v_q}{\langle y, p\rangle\langle y, q\rangle}. \tag{3.1}$$

Under the hypotheses of the theorem, the set $\mathbb{P}^1$ consists of a finite number of straight lines passing through the origin of coordinates. If the number of straight lines composing $\mathbb{P}^2$ is infinite, the perturbed system is nonintegrable [2].

Let α be the maximal element of the spectrum V with respect to the standard lexicographic order. Let β be the maximal element from V that is linearly independent of α. It is possible that the vector β does not exist. In that case, a finite part of the spectrum V lies in the same straight line passing through the origin of coordinates. In this case, the theorem was previously proved in [2].

Thus, let β exist. By using the method reported in [3], it is easy to prove that, for an integer $m \geq 0$, the vectors α and β satisfy the relation

$$m\langle\alpha, \alpha\rangle + 2\langle\alpha, \beta\rangle = 0. \tag{3.2}$$

Therefore, $\langle\alpha, \beta\rangle \leq 0$, and the angle between the vectors α and β is no less than $\pi/2$.

Let Γ be a vertical straight line in the right-hand half-plane that contains the points of the spectrum and is most distant from the origin of coordinates. Let the points of the spectrum in Γ have the radius vectors $\alpha_1, \ldots, \alpha_s$, with $s \geq 1$. Let these points be arranged in decreasing lexicographic order, so that $\alpha_1 = \alpha$. We now prove that $s \leq 2$.

Let $s \geq 3$. Then, $\alpha_2 = \beta$. After the substitution $x_2 \longrightarrow -x_2$, α_s will be the largest element of the spectrum V. The adjoining vector α_{s-1} is no less than α_2 (because it is assumed that $s \geq 3$). The angles between the vectors α_1 and α_2 and the vectors α_s and α_{s-1} are no less than $\pi/2$. This holds only if $s \leq 2$.

We now prove that, actually, $s = 1$ and the corresponding vector α is horizontal. Let us assume the opposite for the vectors α and β. Sufficiently lengthy vectors, with their ends in Γ, can be represented only in two ways as the sum of two vectors of the spectrum V; i.e.,

$$\tau = \alpha + \gamma_u = \beta + \gamma_v, \tag{3.3}$$

where γ_u and γ_v are vertical vectors. It may be assumed that $u = |\gamma_u|$ and $v = |\gamma_v|$. If there exists only a finite number of the nonzero numbers v'_τ of form (3.1), then, beginning with some index, the following relations will hold:

$$au_p + bu_q = 0.$$

Here,

$$u_p = \frac{v_{\gamma_p}}{p}, \quad u_q = \frac{v_{\gamma_q}}{q},$$

and a and b are complex-valued nonzero numbers.

For instance, let $p > q$. Then,

$$u_p = -\frac{b}{a}u_q. \tag{3.4}$$

If $|b/a| \geq 1$, then either $|u_n| \longrightarrow \infty$ when $n \longrightarrow +\infty$ or $u_n = 0$ when $n \geq n_0$. In the second case, the spectrum V is represented by a trigonometric polynomial, and this contradicts our assumptions.

Thus, $|b/a| < 1$. However, equality (3.4) is also valid for large negative p and q and, in this case, should be rewritten in the form

$$u_q = -\frac{a}{b}u_p.$$

Since $|a/b| > 1$, $|u_n| \longrightarrow \infty$ when $n \longrightarrow -\infty$. Therefore, we reach a contradiction.

Thus, $s = 1$. In this case, representation (3.3) of the vector τ is unique, and the sum in equality (3.1) consists of a single term proportional to $\langle\alpha, \gamma_u\rangle$. If this scalar product is nonzero, the Poincaré set $\mathbb{P}^2$ consists of an infinite number of different straight lines; this implies that the Hamiltonian is nonintegrable. Therefore, the vector α is horizontal.

It remains to prove that the rest of the nonvertical vectors of the spectrum V are collinear with α. Let β be the maximal vector from those noncollinear with α. Since the spectrum V is invariant under the reflection in the origin of coordinates, the vector β lies on the right-hand half-plane. According to [3], the vectors α and β satisfy relation (3.2). Thus, the angle between α and β is no less than $\pi/2$, and the vector β is vertical.

4. Comments.

(i). Let us consider pendulums with the same lengths (sympathetic pendulums), which can elastically collide. In this case, the potential interaction energy can be represented by the Dirac function $\delta(x_1 - x_2)$. Such a system can be referred to as a sympathetic billiard. We can formally apply the discussion in Section 3 to this problem and state that an additional one-valued integral is nonexistent. It would be desirable to prove rigorously that the sympathetic billiard is chaotic.

(ii). The results presented can be extended to a system of n pendulums ($n \geq 3$) coupled consecutively to each other. If there is at least a single pair of coupled pendulums, this system with n degrees of freedom does not admit n independent polynomial (in momenta) integrals with periodic coefficients.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 99-01-01096), INTAS (grant no. 93-0570-ext), and the program "Universities of Russia" (project no. 5581).

REFERENCES

1. Poincaré, H., *New Methods of Celestial Mechanics*, Washington (DC): NASA, 1967.
2. Kozlov, V.V., *Simmetrii, topologiya i rezonansy v gamil'tonovoi mekhanike* (Symmetries, Topology, and Resonances in Hamiltonian Mechanics), Izhevsk, 1995.
3. Kozlov, V.V. and Treshchev, D.V., *Mat. Sb.*, 1988, vol. 135(177), pp. 119–138.

Translated by V. Chechin