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Hydrodynamic Theory of a Class of Finite-Dimensional Dissipative Systems

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Finite-dimensional systems with viscous friction are studied in the case when the Rayleigh function modeling this friction is proportional to the kinetic energy. Hydrodynamic analogies are presented for these systems.

1. SYSTEMS WITH VISCOUS FRICTION

The energy dissipation in the dynamics of a viscous fluid gives rise to many characteristic phenomena, for example, to the diffusion of vortices (see [1]). In the dynamics of systems with a finite number of degrees of freedom at low velocities of motion, the viscous-friction forces are usually modeled by the Rayleigh dissipation function [2].

Suppose that M is the n-dimensional configuration space of a mechanical system with n degrees of freedom; $(x_1, \dots, x_n) = x$ are the generalized coordinates; and T and V are the kinetic and potential energies of the system, respectively. The equations of motion of the mechanical system with viscous friction are as follows:

$$\left(\frac{\partial L}{\partial \dot{x}}\right)^{\cdot} - \frac{\partial L}{\partial x} = -\frac{\partial \Phi}{\partial \dot{x}}\,, \tag{1.1}$$

where $L = T - V$ is the Lagrangian and Φ is the Rayleigh function—a positive definite quadratic form in the generalized velocity $\dot{x}$. Let $H = T + V$ be the total mechanical energy. Equation (1.1) implies the known equation $\dot{H} = -2\Phi$, which shows that energy is really dissipated.

In this paper, we consider the case when $\Phi = \nu T$, where ν is a known positive function of time (for example, $\nu = \text{const}$). Applying the Legendre transformation, we pass from the Lagrange equations (1.1) to the generalized Hamilton canonical equations

$$\dot{x} = \frac{\partial H}{\partial y}\,, \qquad \dot{y} = -\frac{\partial H}{\partial x} - \nu y\,. \tag{1.2}$$

Here $(y_1, \dots, y_n) = y$, $y = \partial T/\partial \dot{x}$ are canonical momenta, $H(x, y, t)$ is the Hamiltonian, which can be assumed to be an arbitrary known function of x, y, and time t.

By the substitution

$$X = x\,, \qquad Y = y\mu(t)\,, \qquad \mu = \exp\left[-\int \nu(t)\,dt\right],$$

we can reduce the dissipative system (1.2) to the conventional Hamilton differential equations

$$\dot{X} = \frac{\partial K}{\partial Y}\,, \qquad \dot{Y} = -\frac{\partial K}{\partial X}\,, \qquad K = \mu H\left(X, \frac{Y}{\mu}, t\right). \tag{1.3}$$

Thus, system (1.2) can be analyzed by the methods of the Hamiltonian mechanics. However, we prefer a direct method of analysis, without reducing (1.2) to system (1.3). In the time-independent case (when the functions T and V do not explicitly depend on t), system (1.2) is autonomous; however, this property is lost when the system is reduced to (1.3).

2. THE GROMEKI–LAMB-TYPE EQUATIONS

We will seek n-dimensional invariant surfaces Σ of equations (1.2) in the following form:

$$y = u(x,t)\,, \qquad y_i = u_i(x_1, \dots, x_n, t)\,, \qquad 1 \le i \le n\,. \tag{2.1}$$

The invariance is understood as follows: if the phase trajectory of system (1.2) intersects the surface Σ, this trajectory completely lies in Σ. Thus, we can represent Σ as a surface that is entirely woven from the n-dimensional family of the trajectories of (1.2). By virtue of (2.1), the surface Σ is uniquely projected onto the configuration space M for any t. Let

$$h(x,t) = H(x, u(x,t), t)\,, \qquad \dot{x} = v(x,t) = \left.\frac{\partial H}{\partial y}\right|_{y=u}. \tag{2.2}$$

If the field u is known, the motion of system (1.2) is determined as a solution to (2.2)—a system of first-order differential equations on M.

It can be shown that the covector field u, the function h, and the vector field v are related by

$$\frac{\partial u}{\partial t} + (\operatorname{rot} u)v = -\frac{\partial h}{\partial x} - \nu u\,, \qquad \operatorname{rot} u = \left\| \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right\|. \tag{2.3}$$

Here, $\operatorname{rot} u$ is a skew-symmetric matrix of order n, called the curl of the field u. Let $n = 3$ and $T = \sum \dot{x}_i^2/2$. Then, $u = v$, and $(\operatorname{rot} u)v$ is equal to the vector product of a conventional curl multiplied by the flow velocity v. The term $-\nu u$ can be interpreted as the external-friction force. Equations (2.3) resemble the Gromeki–Lamb equations [1]; we will call them the generalized Gromeki–Lamb equations. The properties of equations (2.2) and (2.3) for $\nu = 0$ were analyzed in [3].

Using external differential forms, we can represent the Gromeki–Lamb equations in an equivalent form. Let

$$\omega = \sum u_i\, dx_i\,, \qquad \Omega = d\omega\,.$$

Then,

$$\frac{\partial \omega}{\partial t} + i_v \Omega = -dh - \nu\omega\,, \tag{2.4}$$

where i is the symbol of the internal multiplication of a field by a differential form.

Applying the operation of external multiplication to both sides of (2.4), we obtain

$$\frac{\partial \Omega}{\partial t} + d i_v \Omega = -\nu \Omega\,.$$

Using the known homotopy formula

$$L_v = d i_v + i_v d \tag{2.5}$$

and the closedness of the 2-form Ω ($d\Omega = 0$), we arrive at the equation

$$\frac{\partial \Omega}{\partial t} + L_v \Omega = -\nu \Omega\,. \tag{2.6}$$

The left-hand side of this equation, $\dot{\Omega}$, is the total derivative of the form Ω by virtue of (2.2). Equation (2.6) is an analogue of the vortex-evolution equation.

Using (2.5), we can rewrite (2.4) as follows:

$$\frac{\partial \omega}{\partial t} + L_v \omega = -dg - \nu \omega\,, \tag{2.7}$$

where $g = h - \omega(v)$ is the Lagrangian of the problem under consideration restricted onto the invariant surface Σ.

3. POTENTIAL FLOWS

Let g^t be the flow of system (2.2), γ be a closed contour on M. The integral

$$I = \int\limits_{g^t(\gamma)} \omega$$

is a function of t. Let us apply the formula

$$\dot{I}(t) = \int\limits_{g^t(\gamma)} \dot{\omega}\,, \qquad \dot{\omega} = \frac{\partial \omega}{\partial t} + L_v \omega\,.$$

Since γ is a closed contour, we obtain the following equations from (2.7):

$$\dot{I} = -\nu(t) I\,, \qquad I(t) = I(0)\mu(t)\,. \tag{3.1}$$

Thus, if the integral

$$\int\limits_0^\infty \nu(t)\, dt$$

is divergent (for example, $\nu = \mathrm{const} \neq 0$), then $I(t)$ monotonically tends to zero as t indefinitely increases. This property can be interpreted as the diffusion of vortices.

A convective field u is called potential if $u = \partial\varphi/\partial x$. Obviously, the potentiality criterion consists in the vanishing of the contour integral of a 1-form ω over any closed path. Equations (3.1) entail an analogue of the Lagrange theorem on potential flows: if the field u is potential for a certain $t = t_0$, then it is potential for any t.

Substituting $u = \partial\varphi/\partial x$ into (2.3), we obtain an analogue of the Lagrange–Cauchy integral:

$$\frac{\partial\varphi}{\partial t} + H\left(x, \frac{\partial\varphi}{\partial x}, t\right) + \nu\varphi = f(t)\,. \tag{3.2}$$

Let $\vartheta(t)$ be a solution to the ordinary differential equation

$$\dot{\vartheta} + \nu(t)\vartheta = f(t)\,.$$

The gauge transformation $\varphi \to \varphi + \vartheta(t)$ does not change the gradient field and allows us to set $f = 0$ in (3.2). When $\nu = 0$, the equation obtained coincides with the known Hamilton–Jacobi equation.

In the time-independent case (when φ, ν, and f do not depend on t), equation (3.2) was first obtained by Arzhanykh [4]; however, he did not associate the substitution $u = \partial\varphi/\partial x$ with the Lagrange theorem. Arzhanykh also proved the following theorem, which generalizes the famous Jacobi theorem on the complete integral. Let $\varphi(t, x_1, \ldots, x_n, c_1, \ldots, c_n)$ be the complete integral of equation (3.2) (where $f = 0$). Then, the general solution to (1.2) is determined from the relations

$$\frac{\partial\varphi}{\partial c} = -\alpha\mu(t)\,, \qquad \frac{\partial\varphi}{\partial x} = y\,, \qquad \alpha = (\alpha_1, \ldots, \alpha_n) = \text{const}\,.$$

When $\nu = 0$, we obtain the Jacobi theorem. The sign "$-$" in the first relation is written according to tradition.

By the way, the Arzhanykh theorem can be readily obtained from the Jacobi theorem. For this purpose, we substitute $\varphi = \psi(x,t)/\mu(t)$, into (3.2) (where $f = 0$). The function ψ satisfies the equation

$$\frac{\partial\psi}{\partial t} + \mu H\left(x, \frac{1}{\mu}\frac{\partial\psi}{\partial x}, t\right) = 0\,, \tag{3.3}$$

which explicitly contains only the derivatives of ψ. Equation (3.3) represents the Hamilton–Jacobi equation for a system with the Hamiltonian given by (1.4).

4. VORTEX MANIFOLDS AND THE GENERALIZED CLEBSCH POTENTIALS

A nonzero vector w such that $i_w\Omega = \Omega(w, \cdot) = 0$ is called a vortex vector. For fixed t, the vortex vectors form a linear subspace of the tangent n-dimensional space T_xM at every point $x \in M$. Assuming that the rank of the 2-form Ω (of the matrix $\operatorname{rot} u$) is constant, we obtain an m-dimensional ($m = n - \operatorname{rank}\Omega$) distribution of vortex vectors. This distribution proves to be integrable: there is a unique m-dimensional submanifold $N_t \subset M$ passing through every point $x \in M$ such that the tangent vectors to the submanifold are vortex vectors [5]. It is natural to refer to the manifolds N_t as vortex manifolds. When $n = 3$, these manifolds are ordinary lines.

Suppose that $w_1(t)$ and $w_2(t)$ are tangent vectors to M (at a common point) that are carried by the flow of system (2.2). Equation (2.6) entails

$$\Omega(w_1(t), w_2(t)) = \Omega(w_1(0), w_2(0))\mu(t)\,.$$

Thus, if the left-hand side of this equation vanishes for $t = 0$, then it is identically equal to zero for any t. Hence, the vortex vectors of the 2-form Ω are frozen into the flow of system (2.2). This, in turn, entails the generalized Helmholtz–Thomson theorem: the flow of the system of differential equations (2.2) translates the vortex manifolds into the vortex manifolds. For the case $\nu = 0$, this fact was established in [3].

If the 1-form ω has a constant class, this form can be locally reduced to

$$\omega = dS + x_1 dx_2 + \ldots + x_{2k-1} dx_{2k} \,,$$

where $2k = n - m$ is the rank of the 2-form Ω (which is always even) and S is a certain smooth function of $x_1, \ldots, x_n, t$ [6].

In these variables, the vortex manifolds are defined by the equations

$$x_{2k+1} = a_{2k+1}\,, \ldots, \; x_n = a_n\,, \qquad a = \mathrm{const}\,, \tag{4.1}$$

while the components of the covector field u are given by

$$u_1 = \frac{\partial S}{\partial x_1}\,,\; u_2 = \frac{\partial S}{\partial x_2} + x_1\,, \ldots, \; u_{2k+1} = \frac{\partial S}{\partial x_{2k+1}}\,, \ldots, \; u_n = \frac{\partial S}{\partial x_n}\,.$$

These formulas allow us to write out the generalized Gromeki–Lamb equations in the explicit form:

$$\begin{aligned} \dot{x}_1 &= -\frac{\partial \varkappa}{\partial x_2} - \nu x_1\,, & \dot{x}_2 &= \frac{\partial \varkappa}{\partial x_1}\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots & & \\ \dot{x}_{2k-1} &= -\frac{\partial \varkappa}{\partial x_{2k}} - \nu x_{2k-1}\,, & \dot{x}_{2k} &= \frac{\partial \varkappa}{\partial x_{2k-1}}\,, \end{aligned} \tag{4.2}$$

$$\frac{\partial \varkappa}{\partial x_{2k+1}} = \ldots = \frac{\partial \varkappa}{\partial x_n} = 0\,, \qquad \varkappa = \frac{\partial S}{\partial t} + \nu S + h\,. \tag{4.3}$$

From (4.1) and (4.3), we obtain the generalized Bernoulli theorem: for fixed t, the function $\varkappa$ is constant on vortex manifolds. Hence, equations (4.2) represent a closed system of ordinary differential equations that generalize the Hamilton canonical equations. Since the vortex manifolds are numbered by the coordinates $x_1, \ldots, x_{2k}$, we again obtain the Helmholtz–Thomson theorem.

The coordinates $x_1, \ldots, x_{2k}$ and the function S represent a natural generalization of the known Clebsch potentials, which were defined in fluid dynamics [7]. If $k = 0$ (the field u is potential), formulas (4.3) yield the known relation (3.2).

5. THE EULER GYROSCOPE WITH VISCOUS FRICTION

Let us illustrate some of the results obtained through an example of the problem of rotation of a solid body about a fixed point under the viscous-friction forces described in Section 1. In this case, the group $SO(3)$ with the Euler angles ψ, ϑ, and φ serves as the configuration space. The Euler dynamic equations are given by

$$\mathcal{I}\dot{\omega} + \omega \times \mathcal{I}\omega = -\nu \mathcal{I}\omega\,. \tag{5.1}$$

Here, ω is the angular velocity and $\mathcal{I}$ is the inertia tensor.

Let α, β, and γ be the unit vectors of a fixed trihedron. Equations (5.1) combined with the Poisson equations

$$\dot{\alpha} + \omega \times \alpha = \dot{\beta} + \omega \times \beta = \dot{\gamma} + \omega \times \gamma = 0$$

yield the relations

$$(\mathcal{I}\omega, \alpha) = c_1\mu\,, \qquad (\mathcal{I}\omega, \beta) = c_2\mu\,, \qquad (\mathcal{I}\omega, \gamma) = c_3\mu\,, \qquad c = \text{const}\,. \tag{5.2}$$

Without loss of generality, we can assume that $c_1 = c_2 = 0$. Setting $c_3\mu = k(t)$, we obtain the following equation from (5.2):

$$\mathcal{I}\omega = k\gamma\,. \tag{5.3}$$

This equation allows us to determine the angular velocity of the body as a single-valued function of its coordinates and time. Using the Euler kinematic formulas, we obtain the following explicit form for the system of equations (2.2) from (5.3):

$$\dot{\psi} = k\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right), \qquad \dot{\vartheta} = k\left(\frac{1}{A} - \frac{1}{B}\right)\sin\vartheta \cdot \sin\varphi \cdot \cos\varphi\,,$$
$$\dot{\varphi} = k\cos\vartheta\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right). \tag{5.4}$$

Here, A, B, and C are the principal moments of inertia of the solid body.

Let p_ψ, p_ϑ, and p_φ be the canonical momenta that are conjugate to the Euler angles. The covector field corresponding to the invariant three-dimensional surface (5.3) and the vectors ω and Ω are expressed as

$$p_\psi = k\,, \qquad p_\vartheta = 0\,, \qquad p_\varphi = k\cos\vartheta\,,$$
$$\omega = k\,d\psi + k\cos\vartheta\,d\varphi\,, \qquad \Omega = k\sin\vartheta\,d\varphi \wedge d\vartheta\,.$$

Therefore, the vector fields $\psi' = \lambda$, $\vartheta' = 0$, and $\varphi' = 0$ are vortex fields. They initiate the rotation of the solid body at the angular velocity $\omega = \lambda\gamma$. In particular, the vortex fields are left-invariant, and all vortex lines are closed. The fibration of the group $SO(3)$ by the vortex lines coincides with the known Hopf fibration. Since the right-hand sides of (5.4) do not contain the precession angle ψ, the phase flow of system (5.4) translates the vortex lines into vortex lines.

Having an internal metric—the kinetic energy $\frac{1}{2}(\mathcal{I}\omega, \omega)$—on the three-dimensional group $SO(3)$, we can calculate the curl of the vector field (5.4). Its components are found to be $-k/\sqrt{ABC}$, 0, 0. This field is a vortex field and commutes with the field (5.4) for any t. One can show that these properties of the curl of the field are not coincidental; however, we will not dwell on this fact. The fluid-dynamic theory of the Euler gyrator (when $\lambda = 0$) was considered in detail in [5].

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