Symmetries and regular behavior of Hamiltonian systems

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The behavior of the phase trajectories of the Hamilton equations is commonly classified as regular and chaotic. Regularity is usually related to the condition for complete integrability, i.e., a Hamiltonian system with n degrees of freedom has n independent integrals in involution. If at the same time the simultaneous integral manifolds are compact, the solutions of the Hamilton equations are quasiperiodic. In particular, the entropy of the Hamiltonian phase flow of a completely integrable system is zero. It is found that there is a broader class of Hamiltonian systems that do not show signs of chaotic behavior. These are systems that allow n commuting "Lagrangian" vector fields, i.e., the symplectic 2-form on each pair of such fields is zero. They include, in particular, Hamiltonian systems with multivalued integrals. [S1054-1500(95)00604-1]

I. INTRODUCTION

Let (M^{2n},ω) be a symplectic manifold and ω a closed nondegenerate 2-form (symplectic structure). Each smooth function

$$H:M\to\mathrm{R}$$

can be placed in correspondence with a Hamiltonian vector field v_H in accordance with the rule

$$i_{v_H}\omega=dH,\quad i_v\omega=\omega(v,\cdot).$$

The field v_H generates a Hamiltonian dynamic system on M, namely,

$$\dot z=v_H(z),\quad z:\{t\}\to M. \tag{1.1}$$

In terms of local canonical coordinates

$$z=(x_1,\ldots,x_n,y_1,\ldots,y_n),$$

the symplectic structure takes the form

$$\omega=\sum dx_k\wedge dy_k,$$

and (1.1) assumes the familiar canonical form

$$\dot x_k=\frac{\partial H}{\partial y_k},\quad \dot y_k=-\frac{\partial H}{\partial x_k};\quad 1\leqslant k\leqslant n. \tag{1.2}$$

We recall that the Hamiltonian system (1.1) is called *completely integrable* if there are n functions

$$F_1=H,\ F_2,\ldots,F_n:M^{2n}\to\mathrm{R} \tag{1.3}$$

that are almost everywhere independent and are in pairs in involution. The involution property of the set of functions given by (1.3) is described by

$$\omega(v_{F_i},v_{F_j})=0,\quad 1\leqslant i,j\leqslant n. \tag{1.4}$$

Of course, each of the functions in (1.2) is an integral of (1.1):

$$L_{v_H}F_k=i_{v_H}dF_k=i_{v_H}\omega(v_{F_k},\cdot)=\omega(v_{F_k},v_{F_1})=0,$$

where L_w is the Lie derivative along the vector field w.

The concept of complete integrability has two aspects: local and global. The former is related to the famous *Liouville theorem*: if there are n independent integrals in involution of the system given by (1.2), the system is integrable in quadratures. The proof of this relies on the complete integral of the Hamilton–Jacobi equation. Let

$$F_1(x,y)=c_1,\ldots,F_n(x,y)=c_n. \tag{1.5}$$

If

$$\frac{\partial(F_1,\ldots,F_n)}{\partial(y_1,\ldots,y_n)}\neq 0,$$

then (1.5) can be used to express (locally) the momenta y in terms of the coordinates x and parameters c:

$$y_1=f_1(x,c),\ldots,y_n=f_n(x,c).$$

Since the paired Poisson brackets of the functions (1.3) are zero, the 1-form

$$f_1(x,c)dx_1+\cdots+f_n(x,c)dx_n$$

is the complete differential of a certain function $S(x,c)$. The next step is to show that this function is a complete integral of the Hamilton–Jacobi equation:

$$H\left(x,\frac{\partial S}{\partial x}\right)=h(c_1,\ldots,c_n),\qquad \det\left\|\frac{\partial^2 S}{\partial x_i\partial c_j}\right\|\neq 0.$$

The global aspect is related to the structure of the n-dimensional integral surfaces:

$$\Lambda_c=\{z:F_1(z)=c_1,\ldots,F_n(z)=c_n\}$$

and the structure of the phase flow of (1.1) on Λ_c. It is found that if the surface Λ_c is regular, then each of its compact connected components is an n-dimensional torus T^n, where in terms of certain angular coordinates $\varphi_1,\ldots,\varphi_n$ mod 2π, the equations on Λ_c have the simple form

$$\dot\varphi_k=\omega_k=\text{ const},\quad 1\leqslant k\leqslant n. \tag{1.6}$$

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Dynamics are thus reduced to quasiperiodic motions on n-dimensional invariant tori.

If Λ_c is not compact, then under a certain additional condition, it is a diffeomorphism of the direct product $T^m \times R^{n-m}$, where it is possible to introduce m angular $(\varphi_1,...,\varphi_m \bmod 2\pi)$ and $n-m$ linear $(\varphi_{m+1},...,\varphi_n)$ coordinates that vary uniformly with time. The additional condition reduces to the requirement that the n Hamiltonian fields

$$v_{F_1},...,v_{F_n} \tag{1.7}$$

satisfy the condition of completeness on Λ_c, i.e., their phase flows are determined along the entire number axis.

The proof of the geometric aspect of the Liouville theorem is based on the following idea: the Hamiltonian fields (1.7) are tangent to Λ_c and commute in pairs:

$$[v_{F_i}, v_{F_j}] = 0, \quad 1 \leq i,j \leq n. \tag{1.8}$$

A discussion of these problems can be found in Refs. 1–4.

The local Liouville theorem admits of a generalization noted in Ref. 5. Let us suppose that the Hamilton equations (1.2) have n integrals (1.3) such that

$$\{F_i, F_j\} = \sum c_{ij}^k F_k.$$

If (1) the functions given by (1.3) are independent on Λ_c and (2) the Lie algebra with structure constants c_{ij}^k is solvable, then Eqs. (1.2) can be integrated in quadratures on integral surfaces Λ_c satisfying the additional conditions

$$\sum c_{ij}^k c_k = 0 \quad \text{for all} \quad 1 \leq i,j \leq n.$$

This proposition has become relatively well known and is discussed, for example, in Refs. 1–4.

II. HAMILTONIAN SYSTEMS WITH TWO DEGREES OF FREEDOM

In the simplest nontrivial case, when $n=2$, we lack only one integral that is independent of H in order to achieve complete integrability.

Let h be a noncritical value of the Hamiltonian H. In that case,

$$N_h = \{z : H(z) = h\}$$

will be a regular three-dimensional manifold. The Hamiltonian field v_H must, clearly, be tangent to N_h and has no singular points.

Let us fix the total energy h and let us denote N_h and the bounded v_H on $N=h$ by N and v, respectively. The dynamic system on N

$$\dot{z} = v(z), \quad z:\{t\} \rightarrow N, \tag{2.1}$$

then has an invariant nondegenerate 3-form Ω that is generated by the invariant Liouville 4-form $\omega \wedge \omega$ specified on the enveloping manifold M.[4] The construction of Ω and the corresponding explicit formulas can be found, for example, in Ref. 6.

Since Ω is nondegenerate, it defines the orientation of N. In the compact case, it is usually assumed that

$$\int_N \Omega > 0.$$

Hence the invariant volume form is a smooth invariant measure.

On three-dimensional manifolds with invariant measure, systems such as (2.1) have many properties typical of Hamiltonian systems. The study of such systems is of interest in its own right.

III. COMPLETELY REGULAR HAMILTONIAN SYSTEMS

The behavior of the phase trajectories of the Hamiltonian equations can be divided into regular and chaotic. Rigorous general definitions are difficult and are not readily formulated despite the intuitive clarity of this subdivision.

Chaotic behavior has a number of related characteristics. The following are some of them:

(a) There is an infinite number of nondegenerate periodic trajectories (their multiplicators are different from unity);
(b) the characteristic Lyapunov indices are nonzero;
(c) asymptotic manifolds intersect transversally;
(d) the Kolmogorov–Sinai entropy is positive

Symbolic dynamics can be used to derive (a) from (c) (see, for example, Ref. 7). Pesin's formula[8] relates (b) and (d).

Poincaré has shown that when the involution integrals (1.3) of a completely integrable system are independent at points of a periodic trajectory, all its multiplicators are equal to unity. Poincaré used this as the basis for a rigorous proof of the fact that Hamiltonian systems are nonintegrable (see Ref. 9; a modern presentation is given in Ref. 10). Poincaré also noted that the asymptotic surfaces of integrable systems are double. Paternain has recently obtained rigorous results on the vanishing of the entropy of completely integrable Hamiltonian systems.[11]

Remark. An example of geodesic flow on a smooth two-dimensional surface of genus >1 with nonconstant infinitely differentiable integral is given in Ref. 10. At the same time, according to Ref. 12, the topological entropy of such systems is positive. The situation is different in the analytic case: the geodesic flow on an analytic surface of genus >1 does not admit nonconstant analytic integrals.[13]

Thus, completely integrable systems do not display signs of chaotic behavior and their dynamics is therefore considered regular. It is found (and this is our observation) that analogous properties are exhibited by a wider class of Hamiltonian systems that includes completely integrable systems.

Definition. The Hamiltonian system (1.1) will be defined as *completely regular* if there are n vector fields

$$v_1 = v_H, \quad v_2, ..., v_n, \tag{3.1}$$

that are linearly independent at almost all points of M^{2n} with the properties

(i) $[v_k, v_l]=0, \quad 1 \leq k,l \leq n,$

(ii) $\omega(v_k, v_l)=0, \quad 1 \leq k,l \leq n.$

Completely integrable systems are obviously completely regular. In fact, Hamiltonian vector fields generated by the integrals given by (1.3) must be taken in (3.1). Properties (i) and (ii) are satisfied by virtue of (1.8) and (1.4), respectively.

To elucidate further the utility of completely regular systems, let us consider the illustrative example of a system of the Hamiltonian "type" on a three-dimensional manifold (see Sec. II). Let N be a three-dimensional torus with angular coordinates

$$z_1, z_2, z_3 \mod 2\pi,$$

and suppose that the system (2.1) is specified by the simple equations

$$\dot{z}_k = \omega_k, \quad k=1,2,3; \quad \omega_k = \text{const}. \tag{3.2}$$

This system has the standard invariant measure

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3.$$

It is well known that if the frequencies ω_1, ω_2, ω_3 are nonresonant, the system (3.2) is ergodic. All its phase trajectories densely fill N everywhere. However, the characteristic Lyapunov indices of all the solutions are zero, so that the entropy of the system (3.2) is zero. The reason for this is the existence of a "rich" group of symmetries: any vector field on N with constant components will be a symmetry field.

We shall see later that completely regular systems can be transitive on $(2n-1)$-dimensional energy manifolds.

IV. SOME PROPERTIES OF COMPLETELY REGULAR SYSTEMS

Let $t \to z_0(t)$ be a solution of the set of equations

$$\dot{z} = v(z). \tag{4.1}$$

The linear system

$$\dot{Z} = A(t)Z, \quad A = \frac{\partial v}{\partial x}\bigg|_{z_0(t)} \tag{4.2}$$

is usually referred to as variational equations.

Lemma 1. If $u(z)$ is a symmetry field for (4.1), then

$$u(t) = u(z_0(t))$$

is a solution of the variational equations (4.2).

Actually, the commutation relation for u and v is

$$\frac{\partial u}{\partial z} v = \frac{\partial v}{\partial z} u.$$

Corollary. If the vectors in (3.1) are linearly independent at a point $z' \in M^{2n}$, then they are linearly independent at all points of the phase trajectory of (1.1) passing through z'.

Theorem 1. If at a certain point on a periodic trajectory of (1.1) the vectors (3.1) are linearly independent, all the multiplicators of the trajectory are equal to unity.

The proof of this relies on well-known results in symplectic geometry, which can be found in Ref. 14 and elsewhere. We note, to begin with, that the mapping of S over a period is a symplectic transformation: it preserves the 2-form ω. According to Lemma 1, the transformation of S converts $v_1, \dots, v_n$ into themselves. Since they are linearly independent, the n eigenvalues of the linear mapping of S are equal to unity. We shall use the following fact: if u_1, u_2 are eigenvectors of S with eigenvalues λ_1, λ_2 such that $\lambda_1\lambda_2 \neq 1$, we have $\omega(u_1, u_2)=0$. Let us suppose that the spectrum of S contains an eigenvalue $\lambda \neq 1$ that corresponds to nonzero eigenvector u. We then have

$$\omega(u, v_k)=0$$

for all $k=1,\dots,n$. However, in the $2n$-dimensional symplectic space there is then a $(n+1)$-dimensional zero plane (spanning the vectors $u, v_1, \dots, v_n$). However, it is well known that the dimensionality of a zero plane does not exceed n, which was to be proved.

Corollary. Almost all periodic trajectories of a completely regular Hamiltonian system are degenerate.

Theorem 2. Let h be a regular value of the Hamiltonian H, where the set

$$N_h = \{z \in M^{2n} : H(z)=h\} \tag{4.3}$$

is compact. If the vectors (3.1) are linearly independent almost everywhere on N_h, the characteristic Lyapunov indices all vanish for trajectories on N_h with almost all initial values.

Let $cv_k(t)$, c=const be a solution of the variational equations. Its Lyapunov index is defined by

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \ln |cv_k(t)| = \lambda_k. \tag{4.4}$$

Because the energy surface (4.3) is compact, the function $|v_k(t)|$ is bounded. Consequently, $\lambda_k \leq 0$. On the other hand, according to the Poincaré recurrence theorem, for almost all initial data, the value of $v_k(t)$ will be as close as desired to $v_k(0)$ for a certain sequence of instants of time that increases without bound. Consequently, the upper limit in (4.4) is non-negative for almost all trajectories laying on N_h. Hence $\lambda_k=0$ for $k=1,\dots,n$. The fact that the remaining n Lyapunov indices are all zero can be proved with the help of condition (ii) (as in the case of Theorem 1).

Corollary. If the surface (4.3) is regular and compact, the entropy of a completely regular Hamiltonian system on N is zero.

The proof of this relies on the Pesin formula[8] according to which the Kolmogorov–Sinai entropy is given by $\int_N[\Sigma_i \lambda_i(z)]\Omega$, where λ_i are non-negative Lyapunov indices and Ω is the measure on N, generated by the Liouville measure on M^{2n}.

Completely regular Hamiltonian systems would not, probably, allow a transverse intersection of asymptotic manifolds. So far, a rigorous proof of this has been given only for systems with two degrees of freedom. Let us suppose that, for a certain value of the total energy h, there are two hyperbolic periodic trajectories γ_1, γ_2 on a three-dimensional surface N_h, whose stable and unstable asymptotic surfaces W_1^+ and W_2^+ intersect without coinciding. In the analytic case, these surfaces are analytic. In respect of continuity, this picture persists for close values of the total energy. The case where $\gamma_1=\gamma_2$ is not excluded.

Theorem 3. Let u be an analytic vector field on M^4 such that

$$\text{(i)}\ [u,v]=0,\quad v=v_H,$$

$$\text{(ii)}\ \omega(u,v)=0.$$

If the asymptotic surfaces W_1^+ and W_2^+ cross, but do not coincide, then

$$u=f(H)v,$$

where f is an analytic function.

We note, first, the symmetry field u is tangent to the three-dimensional energy manifold N. Actually, by virtue of (ii),

$$\frac{\partial H}{\partial z}u=i_u dH=i_u(i_v\omega)=0. \tag{4.5}$$

Thus, on the three-dimensional manifold N, the Hamiltonian system has the symmetry field u. However, it is shown in Ref. 15 that if, in addition,

$$W_1^+\cap W_2^-\neq\phi,\quad W_1^+\neq W_2^-,$$

then

$$u=cv,\quad c=\text{const}.$$

It would be interesting to extend Theorem 3 to the multidimensional case.

V. STRUCTURE OF PHASE FLOW

Let us suppose that (4.3) is a regular compact $(2n-1)$-dimensional manifold. According to (4.5), the vector fields (3.1) are tangent to N_h. We shall suppose that the vectors (3.1) are linearly independent at all points. Let $\Pi_z\subset T_zM$ be a tangent plane generated by linear combinations of the vectors (3.1). It is clear that

$$\dim \Pi_z=n.$$

Since the fields (3.1) commute, the Frobenius theorem shows that the distribution of the n-dimensional planes Π_z, $z\in N_h$ is integrable: each point of N_h is crossed by a single n-dimensional integral manifold Λ of this distribution (at each point $z\in\Lambda$ the plane tangential to Λ coincides with Π_z). The integral manifolds Λ are not in general compact and can be imbedded in N_h in an arbitrarily complex manner.

However, Λ has a simple "internal" topology and the phase flow of a completely regular Hamiltonian system on Λ is quite simply constructed.

Theorem 4. The integral manifolds Λ are diffeomorphic of the direct product $T^k\times R^{n-k}$ and we can choose upon them a set of coordinates

$$\varphi_1,\ldots,\varphi_k \bmod 2\pi,\varphi_{k+1},\ldots,\varphi_n,$$

that have a uniform variation with time:

$$\dot\varphi_j=\omega_j,\quad \omega_j=\text{const};\quad 1\leq j\leq n.$$

Actually, the vector fields (3.1) are tangent to Λ, they are linearly independent at all points of Λ, and they commute and are unconstrained (since N_h is compact by hypothesis).

The integral n-dimensional surfaces Λ are Lagrangian [by virtue of (ii)]. Let us suppose that $x_1,\ldots,x_n$, $y_1,\ldots,y_2$ are local symplectic coordinates. We shall also suppose that the surface Λ is uniquely projected (if only locally) on the "configuration" space $R^n=\{x\}$. We then find that Λ is determined by the equations

$$y_1=\frac{\partial S}{\partial x_1},\ldots,y_n=\frac{\partial S}{\partial x_n},\quad S=S(x). \tag{5.1}$$

Since the $2n$-dimensional space M^{2n} is completely filled with n-dimensional surfaces, these surfaces are given by (5.1) near Λ, but only the function S depends in addition on the n parameters $c_1,\ldots,c_n$. It may be shown that $S(x,c)$ satisfies the Hamilton–Jacobi equation and is a complete integral of this equation. These facts are consequences of the invariance of the surfaces Λ with respect to the phase flow of the system defined by (1.1). Problems of this kind are discussed in, for example, Ref. 16.

Unfortunately, and in contrast to the completely integrable case, here the complete integral of S cannot be found with the aid of a constructive algorithm. However, when $n=2$, the differential equation of a completely regular Hamiltonian system with compact energy surfaces can be solved with the help of a finite number of algebraic operations, quadratures, and differentiations. This somewhat unexpected result is a consequence of the general theorem according to which the equations can be integrated explicitly on a three-dimensional compact manifold with invariant measure, if they admit a nontrivial symmetry field.[17]

VI. MULTIVALUED INTEGRALS

A vector field u on M^{2n} will be called locally Hamiltonian if there is a closed (but not necessarily exact) 1-form φ, such that

$$i_u\omega=\varphi. \tag{6.1}$$

Locally, $\varphi=dF$. If the function F is then an integral of the Hamilton equations (1.1), the form φ will be referred to as a *multivalued integral*.

Lemma 2. If u is locally a Hamiltonian vector field, then

$$L_u\omega=0.$$

Actually, by evaluating the outer differential of both sides of (6.1), and using the homotopy formula (see Ref. 14), we obtain

$$0=d\varphi=di_u\omega=L_u\omega-i_u d\omega=L_u\omega.$$

Lemma 3. If u,w are locally Hamiltonian fields and $\omega(u,w)=0$, then they commute, i.e., $[u,w]=0$.

Proof. Let us evaluate

$$i_{[u,w]}\omega = L_u i_w \omega - i_w L_u \omega.$$

According to Lemma 2, the last term is zero. Since

$$i_u \omega = \psi, \quad d\psi = 0,$$

we have

$$L_u i_w \omega = L_u \psi = d i_u \psi = d\omega(w,u) = 0.$$

Thus

$$i_{[u,w]}\omega = 0.$$

Since the 2-form ω is nondegenerate, we have $[u,w]=0$.

Thus, if the fields (3.1) are locally Hamiltonian, then property (ii) follows from property (i).

Lemma 4. If u is locally a Hamiltonian vector field and $\omega(u,v)=0$, then φ is a multivalued integral.

Actually, by (6.1),

$$\frac{\partial F}{\partial z} v = i_v \varphi = \omega(u,v) = 0.$$

Thus, if we can find n independent locally Hamiltonian vector fields (3.1) and (ii) is satisfied, then the Hamiltonian system with Hamiltonian H will be completely regular. Moreover, in that case, in addition to H there are a further $n-1$ multivalued integrals in involution, where n-dimensional surfaces of their "simultaneous levels" coincide with the invariant Lagrangian surfaces Λ described in Sec. V.

Let us now consider an example of a completely regular Hamilton system with multivalued integrals whose trajectories everywhere densely cover the energy surface N_h. The system does not therefore have even continuous nonconstant single-valued integrals.

Let $n=2$ and

$$M^4 = T^2 \times R^2 = \{x_1, x_2 \bmod 2\pi, y_1, y_2\},$$

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2, \quad H = (\alpha y_1 + y_2)/F(x_1, x_2),$$

where $\alpha \in R$, and let F be a positive analytic function on T^2 (its Fourier series must contain all the harmonics). It is found that, when the number α is suitably chosen, this Hamiltonian system has the indicated properties (see Ref. 10). The closed 1-form

$$\varphi = dx_1 - \alpha dx_2$$

is then a multivalued integral.

The existence of this integral enables us to integrate the Hamilton equations in quadratures.

We must not assume that all the fields of symmetry of the Hamilton equations are locally Hamiltonian. Here is a simple example, borrowed from Ref. 18. Let us again suppose that $M^4 = T^2 \times R^2$ and let $H = (y_1^2 + y_2^2)/2$. This Hamiltonian defines the geodesic flow on a two-dimensional torus with flat geometry. The symmetry field u will be specified by

$$L_u = y_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial x_2}.$$

It is then readily verified that

$$[u,v]=0, \quad \omega(u,v)=0.$$

However, the field u is not Hamiltonian relative to the standard symplectic structure ω.

The symmetries of geodesic flows on closed surfaces were investigated in Refs. 17 and 18.

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