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Problemata Nova, ad Quorum Solutionem Mathematici Invitantur

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We would like to draw the attention of the reader to some mathematical problems of classical mechanics. They came out in connection with investigations at the Theoretical Mechanics Department at Moscow State University. It is worth mentioning that, beginning with Newton, classical mechanics was always a source of new mathematical problems. Recall the Kepler equation

$$u - e\sin u = \zeta\,,$$

relating the eccentric anomaly u of the orbit to the mean anomaly ζ, which is a linear function of time. Lagrange, when he was solving the Kepler equation, was one of the first to use Fourier series. He obtained the following expression

$$u = \zeta + 2\sum_{m=1}^{\infty} \frac{J_m(me)}{m} \sin m\zeta\,.$$

Here $J_m(z)$ is the Bessel function of mth order, which was first introduced precisely in this problem. Looking for a representation of the solution of the Kepler equation as a power series in eccentricity, Lagrange came to the general theorem on local inversion of holomorphic functions (it is known now as the Burman–Lagrange theorem). According to Lagrange,

$$u = \sum_{m=0}^{\infty} c_m(\zeta)\,\frac{e^m}{m!}\,, \qquad c_0 = \zeta\,,\ c_m = \frac{d^{m-1}}{d\zeta^{m-1}} \sin^m \zeta \quad (m \geqslant 1)\,.$$

Last, but not least, let us mention that the main motivation that led Cauchy to his discoveries in complex analysis was to determine rigorously the region of convergence for the Lagrange power series (it is convergent for $e \leqslant 0.6627434\ldots$). Of course, the problems presented below do not pretend to be of such global importance. They are intended in the first place for young mathematicians who would like to try their skills in this interesting field.

We shall start our discussion with some problems of stability theory.

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1. According to the famous *Lagrange–Dirichlet theorem*, a position of equilibrium is stable if the potential energy has a strict local minimum in this position. Unfortunately, the inverse theorem is not true. This is shown by a simple counterexample suggested by Painlevé (to find an example is a good problem for those who are not acquainted with this result). However, in the Painlevé example, the potential energy is of only finite smoothness. Later Wintner provided a counterexample with the infinitely smooth potential.

In 1892 Lyapunov formulated the inversion problem of the Lagrange–Dirichlet theorem in the analytic case. In spite of serious efforts of many mathematicians and experts in mechanics, a complete positive solution of the Lyapunov problem was obtained only hundred years later by Palamodov (see his article in this volume). To prove instability, he constructed a suitable Lyapunov function.

Merging Palamodov's theorem with the result of [1], it is possible to prove that if $x = (x_1, \dots, x_n) = 0$ is an *isolated* position of equilibrium that is not a point of local minimum for the potential energy, then the equations of motion have a solution $x(t)$ such that $x(t) \to 0$ as $t \to -\infty$.

Incidentally, since the dynamics equations are reversible, $x(-t)$ also is an asymptotic solution: it tends to the equilibrium $x = 0$ as $t \to \infty$.

a. *Does the existence of an asymptotic solution hold in general* (*without the assumption that the critical point $x = 0$ of the potential energy is isolated*)?

Earlier the problem of inversion of the Lagrange–Dirichlet theorem was treated by the author (in collaboration with Palamodov), using the first Lyapunov method. This method is based on constructing asymptotic solutions of dynamic equations in the form of a certain series (see the paper [2], where references to preceding publications can be found).

Let $V = V_2 + V_3 + \dots$ be the Maclaurin series for the potential energy; V_k is a homogeneous form in x of degree $k \geqslant 2$. If $x = 0$ is not a minimum for V_2, then, as was proved by Lyapunov, the equations of motion have a nontrivial asymptotic solution of the form

$$(1) \qquad \sum_{m=1}^{\infty} x_m(t) e^{-m\lambda t}, \qquad \lambda = \text{const} > 0,$$

where $x_m(t)$ is a polynomial in t with constant coefficients. In the analytic case this series is convergent for all sufficiently large values of t. By reversibility, the equilibrium $x = 0$ is unstable.

The more general case

$$V = V_2 + V_m + V_{m+1} + \dots, \qquad m \geqslant 3,$$

is considered in [2]. Here the form V_2 is nonnegative and the dimension of the plane $\Pi = \{x : V_2(x) = 0\}$ is positive. Let W_m be the restriction of the form V_m to the plane Π. It turns out that if $x = 0$ is not a minimum for W_m, the system has an asymptotic solution in the form of a series

$$(2) \qquad \sum_{m=1}^{\infty} \frac{x_m(\ln t)}{t^{m\mu}}, \qquad \mu = \text{const} > 0,$$

where $x_m(\cdot)$ are polynomials with constant coefficients.

b. *Is it always true that if a critical point of the potential energy is not a minimum, then the equations of motion have an asymptotic solution of the form* (1) *or* (2), *or series with multiple logarithms are needed?*

If the system is analytic and $V_2 \equiv 0$, the series (2) is convergent for sufficiently large t (Kozlov, Palamodov; 1982). However, if $V_2 \not\equiv 0$, the series (2) are, as a rule, divergent. Here is a simple model example:

$$\ddot{x} = -\frac{\partial V}{\partial x}, \quad \ddot{y} = \dot{x}^2 - \frac{\partial V}{\partial y}, \qquad V = -6x^3 + \frac{y^3}{2}. \tag{3}$$

The presence of the term $\dot{x}^2$ means that the kinetic energy is noneuclidean. Equations (3) have a formal solution

$$x = \frac{1}{2t^2}, \quad y = \frac{1}{t^6}\sum_{n=0}^{\infty}\frac{a_{2n}}{t^{2n}}, \qquad a_{2n} = \frac{(-1)^n(2n+5)!}{120}. \tag{4}$$

The radius of convergence of the series for y is zero.

However, according to the remarkable theorem by Kuznetsov [3], even in the case when the series (2) is divergent, the equations of motion have a solution such that the series (2) is its asymptotic representation. Incidentally, the work [3] was stimulated by investigations on the inversion of the Lagrange–Dirichlet theorem. As an illustration, we shall give the exact asymptotic solution of the system (3) corresponding to the formal series (4):

$$x(t) = \frac{1}{2t^2}, \qquad y(t) = -\sin t\int_t^{\infty}\frac{\cos s}{s^6}\,ds + \cos t\int_t^{\infty}\frac{\sin s}{s^6}\,ds. \tag{5}$$

Performing successive integration by parts, it is possible to obtain the series (4) from these formulas. The function $y(\cdot)$ in (5) can be regarded as a sum (in a generalized sense) of the divergent series (4).

c. *Is it always possible to represent asymptotic solutions corresponding to divergent series* (2) *in an integral form like* (5) *with finite "kernels"?*

c′. *Is it true that in the analytic case to every formal solution in the form of a series there corresponds a unique "real" solution such that this series is its asymptotic representation?*

2. Now let us turn to problems of another nature, connected with the existence of *tensor conservation laws*.

Let M be a two-dimensional closed analytic surface that serves as the configuration space for a mechanical system with two degrees of freedom. We shall study only inertial motion and assume that the kinetic energy H (Riemannian metric on M) is an analytic function on TM, quadratic in velocity. According to the Maupertuis principle, all trajectories of the system are geodesics. Hence the corresponding dynamical system on the invariant three-dimensional surface $H = 1$ is often called a geodesic flow.

In 1979 the author proved the following theorem: if the genus of the surface M is greater than one, the geodesic flow does not admits nonconstant analytic integrals.

This result allows us to speak about purely topological obstacles to the integrability of reversible systems.

a. *What multidimensional surfaces M admit geodesic flows with a complete set of independent analytic integrals?*

Significant progress in this problem was achieved by Taĭmanov [4]. He proved, in particular, that the Betti numbers of a connected n-dimensional configuration space with a complete set of first integrals satisfy the inequalities

$$b_k \leqslant \binom{n}{k}, \qquad 0 \leqslant k \leqslant n.$$

Unfortunately, this elegant result does not solve the whole problem.

In the two-dimensional case, geodesic flows with first integrals that are independent of H may exist only on the sphere S^2 and on the torus T^2 (we consider the orientable case). The case of the two-dimensional torus is the most interesting, since it is possible to introduce global isotermic angle coordinates q_1, q_2 on the torus, so that the Hamiltonian function takes the simple form:

$$H = \Lambda(q_1, q_2)(p_1^2 + p_2^2)/2. \tag{6}$$

Here p_s is canonical momentum dual to q_s.

Every analytic integral can be expanded in a Maclaurin series in powers of momenta. It is evident that every homogeneous part of the series is also an integral. Hence, following Birkhoff, we should study systems admitting first integrals that are polynomial in momentums. Birkhoff showed that the existence of linear integrals is related to hidden cyclic coordinates, and quadratic integrals are related to separable variables. For the case of the torus, global versions of these results are given in [5].

Bolotin suggested the following conjecture.

b. *If the system on the torus with the Hamiltonian* (6) *has an independent polynomial integral of degree k, then there exists an integral independent of H that is linear or quadratic in momentum.*

Is this true?

From the point of view of the Maupertuis principle, the problem of existence of polynomial integrals for Hamiltonian systems with the Hamiltonian function $H = (p_1^2 + p_2^2)/2 + V(q_1, q_2)$, is closely related to the previous problem. Here V is an analytic function on the torus $\mathbb{T}^2$. In [6] this problem is treated for systems on the n-dimensional torus with a Hamiltonian of a similar form. The potential energy is assumed to be a *trigonometric polynomial* on $\mathbb{T}^n$. It is proved that if the Hamiltonian equations admit n independent polynomial integrals, then there exist n independent integrals in involution of first or second degree. From the technical point of view, the proof is rather complicated. It is based on the classical approach of perturbation theory.

c. *Is this statement true for Hamiltonian systems with general analytic periodic potentials?*

For the case of integrals of third or fourth degree in momentum, the positive answer was obtained by Bialy [7].

The same questions can be formulated for reversible systems with the two-dimensional sphere as the configuration space. However, in this case the situation is completely different. There exist metrics on the two-dimensional sphere such that the corresponding geodesic flow admits homogeneous integrals of third and fourth degree that cannot be reduced to linear or quadratic integrals. The reader is challenged to find an example. This is an interesting and enlightening problem. Let us consider the Hamiltonian function on T^*S^n $(n \geqslant 1)$ of the form $T + V$, where T is the standard Riemannian metric on the standard n-dimensional sphere and V is an analytic function on S^n.

d. *Is it possible for this system to possess a complete "irreducible" set of integrals of arbitrary high order in momentum?*

Probably, as a first step, the case when S^n is the standard sphere in $\mathbb{R}^{n+1}$ should be considered. Note that in the integrable Neumann problem (V is quadratic on $\mathbb{R}^n$), all integrals are quadratic in velocity.

3. Integrals are the simplest tensor invariants (type $(0, 0)$). The next class (in simplicity) of tensor invariants are invariants of type $(1, 0)$, i.e., symmetry fields. A *symmetry field* is a vector field u commuting with the field v that defines the dynamical system.

For Hamiltonian systems the problem of symmetry fields includes the problem of integrals. The reason is that to any function F on the phase space there corresponds a Hamiltonian vector field v_F. If F is an integral, then $u = v_F$ is a symmetry field.

In the paper [8] it is proved that a reversible analytic system with a surface of genus greater than one as the configuration space admits no nontrivial symmetry fields: any symmetry field is of the form $u = \lambda v_H$, where λ is an analytic function of the Hamiltonian H. It follows that there are no analytic integrals independent of H (cf. §3). The field λv_H is Hamiltonian: the Hamiltonian function is

$$\int \lambda(H)\, dH.$$

a. *Is it true that if there exists a nontrivial symmetry field, then Hamiltonian equations admit an integral independent of H?*

Since a reversible system is homogeneous, we should look for symmetry fields of the form

$$L_u = Q_1 \frac{\partial}{\partial q_1} + Q_2 \frac{\partial}{\partial q_2} + P_1 \frac{\partial}{\partial p_1} + P_2 \frac{\partial}{\partial p_2},$$

where Q_s (P_s) are homogeneous polynomials in p_1, p_2 of degree $k-1$ (respectively k). It is natural to say that a field is homogeneous of degree k if its Lie operator is of the above form (if F is a homogeneous polynomial in impulses of degree k, then the field v_F is homogeneous of degree k).

Recently, Denisova obtained a positive answer to problem **a** in the case $k \leqslant 2$.

b. *Investigate problem* **a**, *replacing the two-dimensional torus with the two-dimensional sphere.*

c. *Find conditions for the existence of nontrivial symmetry fields for nonreversible systems (with gyroscopic forces) with two degrees of freedom.*

Topological restrictions for the existence of first integrals of nonreversible systems were obtained by Bolotin [9].

4. Every Hamiltonian system with two degrees of freedom has the tensor invariants H, ω, $\omega^2 = \omega \wedge \omega$. Here H is the Hamiltonian function and $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ is the symplectic form. The invariance of the form ω^2 is equivalent to the Liouville theorem on the conservation of phase volume. The tensor fields (8) are of the type $(0, 0)$, $(0, 2)$, and $(0, 4)$ respectively.

Describe all analytic tensor invariants of geodesic flows on surfaces with negative Euler characteristics.

It is possible that there are no other nontrivial invariants.

5. In applications we often encounter differential equations with quadratic right-hand side:

$$\dot{x}_i = v_i(x), \quad 1 \leqslant i \leqslant n, \qquad v(\lambda x) = \lambda^2 v(x). \tag{9}$$

The most important example is the *Euler–Poincaré equations* on Lie algebras:

$$\dot{m}_k = \sum c_{ki}^j m_j \omega_i, \qquad m_s = \sum I_{sp}\omega_p. \tag{10}$$

Here $\omega = (\omega_1, \dots, \omega_n)$ is the velocity of the the system, $m = (m_1, \dots, m_n)$ is the momentum, c_{ki}^j are the structure constants of the algebra, and $\|I_{sp}\|$ is the constant inertial tensor. If we substitute the expressions for m_i in terms of ω_i into equations (10), we obtain a system of equations on the Lie algebra. The inverse transformation yields a dynamical system on the dual space. In both cases we get equations of type (9).

Kovalevskaya and Lyapunov introduced a method yielding necessary conditions for solutions to be univalent or meromorphic on the complex time plane. We shall describe it briefly for systems of type (9). The first step is to find a solution of the form

$$x = c/t. \tag{11}$$

The complex vector c satisfies the algebraic equation $v(c) = -c$. Let us write out the first variation equation for the solution (11):

$$\dot{\xi} = t^{-1}A\xi, \qquad A = (\partial v/\partial x)(c).$$

This is a Fuchsian system. It has particular solutions of the form $\xi = \varphi t^{\rho-1}$, where ρ is an eigenvalue and φ an eigenvector of the matrix $K = A + E$. This matrix is called *Kovalevskaya matrix*, and its eigenvalues are *Kovalevskaya exponents*. It can be proved that if the general solution of system (9) is represented by univalent (respectively meromorphic) functions of complex time, then the Kovalevskaya exponents are integers (respectively nonnegative integers except one, which is equal to -1).

Ioshida proved in 1983 that if $f(x)$ is a homogeneous integral of the system (9) and $df(c) \neq 0$, then $\rho = m$ is a Kovalevskaya exponent. This result establishes a remarkable connection between the univalence property of the general solution and existence of nonconstant integrals. An extension of Ioshida's theorem to the case of tensor invariants with homogeneous components appears in [10].

a. *Apply the Kovalevskaya–Lyapunov method to the Euler–Poincaré equations.*

Conditions for the general solution to be univalent or meromorphic will include restrictions on the structure of the Lie algebra and on the inertia tensor. For certain Lie algebras (for example, *so* (4)) this method was applied in the work of Adler and van Moerbeke (see, for example, [11]). It is possible to consider other simplified versions of problem **a**. For example, one can try to find all algebras such that solutions of the Euler–Poincaré equations are univalent for any choice of the inertia tensor (one of these algebras is *so* (3)).

b. *Apply the Kovalevskaya–Lyapunov method to the Euler–Poincaré–Suslov equations.*

The EPS-equations appear, for example, in the article by Fedorov and the author in this volume. They also are of the form (9).

Now let us discuss some variational problems. The first belongs to the realm of Morse theory.

6. Let $M = N \times S^1$ be a Riemannian manifold diffeomorphic to the Cartesian product of a compact manifold N and the circle S^1. We look for closed geodesics that are homotopic to the curves $n \times S$, where n is a point in N.

Find a lower bound for the number of these geodesics in terms of topological invariants of the surface N.

Probably this lower bound is at least $\operatorname{cat}(N)$. The *category* of the manifold N is defined as the smallest number of closed subsets of N retractable to a point, that cover N. For example, $\operatorname{cat}\mathbb{T}^k = k + 1$. The category of N provides a lower bound for the number of critical points of a smooth function on N.

An example from dynamics is provided by the problem of periodic oscillations of an n-link pendulum. The configuration space M is the n-dimensional torus $\mathbb{T}^n$. According to the Maupertuis principle, for fixed energy h greater then the maximum of the potential energy (when all the rods are pulled up), trajectories of the pendulum are geodesics in M. In this case, for a given h there exist at least n different periodic motions of energy h in any homotopy class. This means that each link makes the prescribed number of rotations in a period.

7. Let M be the configuration space, T the kinetic energy, $V\colon M \to \mathbb{R}$ the potential energy. The equations of motion have the energy integral $T + V = h$. Since $T \geqslant 0$, trajectories with energy h are contained in the *region of possible motion*

$$\Sigma_h = \{x \in M : V(x) \leqslant h\}.$$

We assume that Σ is compact, its boundary $\partial\Sigma$ is nonempty, and $\partial\Sigma$ contains no positions of equilibrium. The latter is equivalent to the assumption that h is not a critical value of the function V.

According to the Maupertuis principle, trajectories of energy h contained in the interior of Σ are geodesics in the Jacobi metric $(h - V)\,T$. This metric is degenerate on the boundary $\partial\Sigma$.

There are two types of periodic orbits with energy h: *rotations* and *librations*. Trajectories of rotations have no points on the boundary. For the trajectory of a libration there are exactly two such points. The velocity periodically become zero (as for librations of the ordinary pendulum).

Bolotin proved that if these assumptions are satisfied, there exists at least one libration [12]. Much earlier (1948) this result was proved by the well-known topologist Herbert Seifert for the case when Σ is diffeomorphic to the ball. He stated the following conjecture: *if Σ is diffeomorphic to the n-dimensional ball, then there exist at least n different librations.* This conjecture is not proved yet. It is easy to give an example where there exist exactly n librations (try!).

Let $\Sigma = I \times N$, where I is the segment $[0, 1]$ and N is a closed manifold.

The problem is to find a lower estimate for the number of librations in terms of topological invariants of the surface N. Is it true that in this case there are at least two librations?

For a start, one may consider the simplest case of the two-dimensional annulus, when $N = S^1$. In [13] the number of librations was estimated for the not simply connected case by the rank of the fundamental group of the space $\Sigma/\partial\Sigma$.

8. In a paper written when he was still a student, Chaplygin (1890) studied the motion of a heavy plate in a boundless ideal fluid. He obtained the elegant equation

$$\ddot{x} = t^2 \sin x\,. \tag{12}$$

The coordinate x is the double rotation angle of the plate and t is a parameter proportional to physical time. Later we shall discuss some properties of generic solutions of equation (12). Here we consider special doubly asymptotic solutions $x(t)$ that tend to the unstable equilibrium $x = 0 \mod 2\pi$ as $t \to \pm\infty$.

It is relatively easy to prove the existence of a doubly asymptotic solution $x(t)$ that performs exactly one full rotation:

$$\lim_{t\to+\infty} x(t) - \lim_{t\to-\infty} x(t) = 2\pi\,. \tag{13}$$

This result was obtained in [14] by means of the Hamilton variational principle. Bolotin showed that it is possible to modify the proof in [14] and to prove the existence of a doubly asymptotic solution such that the difference (13) is equal to $2\pi n$ with arbitrary integer n.

It is possible to generalize equation (12). Let M be a compact configuration space, T the kinetic energy, and $p(t)V$ the potential energy. Here V is a smooth function on M that has a strict nondegenerate maximum at $a \in M$ and $p(\cdot)$ is a nonnegative function of time. Suppose that $p(t)$ is monotone for $|t| > \text{const}$ and tends to infinity as $t \to \pm\infty$. In the local coordinates $x_1, \dots, x_n$ the motion is described by the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} - \frac{\partial T}{\partial x_i} = -p(t)\frac{\partial V}{\partial x_i}\,, \qquad 1 \leqslant i \leqslant n\,.$$

It is clear that $x = a$ is an unstable equilibrium.

The question is, *does there exist a nontrivial solution $x(t)$, such that $x(t) \to a$ as $t \to \pm\infty$?* Probably the number of these solutions is always infinite.

Note that for the ordinary pendulum (when there is no multiplier t^2 in (13)), there exist only two different doubly asymptotic trajectories (they correspond to rotation numbers $n = \pm 1$).

9. Now let us discuss some questions connected with the existence of periodic orbits for nonreversible systems. Again let M be the configuration space, ω a closed 2-form on M, and V the potential energy. In the nonreversible case, the equations of motion are:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = \omega(\dot{x}\,,\,\cdot\,) - \frac{\partial V}{\partial x}\,. \tag{14}$$

It is natural to interpret the term $\omega(\dot{x}\,,\,\cdot\,)$ as an additional force acting on the mechanical system. Since it does not prevent the conservation of full energy $T + V = h$, it is usually called a *gyroscopic force*. Hence the 2-form ω is said to be the form of gyroscopic forces. The nature of these forces can be quite diverse. For example, they appear when we use a rotating frame of reference, after the reduction of systems with symmetries, or when we study the motion of a charged particle in a magnetic field. The assumption that the form ω is closed implies that locally solutions of equation (14) are extremals of the variational problem with the Lagrangian $L = T - \varphi - V$, $d\varphi = \omega$.

Suppose that M is compact and $h > \max_M V$. Then the velocity of the system is never zero.

The question is: *does there exist a periodic motion with fixed energy* $h > \max V$?

S. P. Novikov formulated sufficient conditions. For example, it is sufficient to assume that M is simply connected and $H^2(M) \neq 0$ (see the survey [15], where there are references to preceding publications). The case of two-dimensional sphere $M = S^2$, important for applications, is included here.

However, full and rigorous proofs of these results are unknown as of now. In the recent paper [16], Taĭmanov proved the existence of a closed orbit on the two-dimensional sphere under the additional assumption that the isoenergetic action functional

$$F^* = \int_{\partial\Pi} \sqrt{2(h-V)T} - \iint_{\Pi} \omega$$

assumes a negative value on some two-dimensional surface Π (with boundary), imbedded into M. The orientation of Π is induced by the orientation of M. This assumption is satisfied if the 2-form ω changes sign somewhere on the sphere. An analogous result is announced in [16] for all two-dimensional closed surfaces M. Earlier in [17] the opposite case was considered, when the form of gyroscopic forces does not vanish anywhere on the two-dimensional torus. This case is one of the most interesting for physical applications. Using the generalization of the last geometric theorem of Poincaré suggested by Arnold, the existence of three different closed orbits for the inertial motion of a particle on a flat torus (the curvature of the Riemannian metric is zero), was proved [17].

a. *It is interesting to extend this method to the case of an arbitrary metric on the two-dimensional torus.*

Suppose now that $h < \max_M V$ and h is a regular value of the potential energy.

b. *Does a periodic trajectory with a given energy h always exist?*

This problem seems more difficult then the problem considered by Novikov.

10. Once again consider the motion on a compact manifold M under the action of a potential force field. Let a , b be different points of M.

The question is, *does there exist an orbit with energy h, joining these points?*

If $h > \max_M V$, the positive answer is well known (it follows from the Maupertuis principle and the Hopf–Rinov theorem in Riemannian geometry). If $h < \max_M V$, for arbitrary points in the connected region of possible motion

$$\Sigma_h = \{x \in M : V(x) \leqslant h\},$$

the answer is negative (give an example!).

Suppose that V has no critical points on the boundary of Σ_h. Then it is easy to show that any two points $a, b \in \Sigma$ can be joined by a piecewise smooth broken trajectory (each link is a trajectory of energy h). In other words, it is possible to get from the point a to the point b by applying a finite number of isoenergetic impulses. Hence our problem is connected with control theory.

Let $k(a, b)$ be the least possible number of impulses needed for getting from the position a to the position b. By reversibility, $k(a, b) = k(b, a)$. It turns out that $K = \max_{a, b \in \Sigma} k(a, b)$ is finite. This is an important geometric characteristic of the system. Of course, it depends on the energy h.

We have the *maximum principle*:

$$K = \max_{a, b \in \partial\Sigma} k(a, b).$$

This a simple corollary of the *boundary hit theorem* (see [17]): for every point $a \in \Sigma$ there exists a motion $x(t)$, $0 \leqslant t \leqslant \tau$, such that $x(0) \in \partial\Sigma$ and $x(\tau) = a$.

a. *Obtain upper and lower estimates for K. Is it true that always $K \leqslant 2$?*

Let us make the problem more complicated by adding a linear nonintegrable constraint

$$(c(x), \dot{x}) = 0, \qquad c \neq 0. \tag{15}$$

The equations of motion are replaced by the nonholonomic equations with a multiplier λ:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = -\frac{\partial V}{\partial x} + \lambda c. \tag{16}$$

Equations (15), (16) form a complete system. Once again the energy $T + V$ is constant on every solution.

b. *Is the boundary hit theorem true in the case of a nonintegrable constraint* (15)?

For the nonholonomic problem it is also possible to introduce the number K. Once again it is finite. This fact is less evident than in the holonomic case. The proof uses the Rashevskiĭ–Chow theorem (every two points of a connected manifold M can be joined by an admissible curve satisfying equation (15)). The reader can try to prove that K is finite himself.

c. *Is the maximum principle true in the nonholonomic case?*

Note that the Hopf–Rinov theorem does not hold in the presence of a nonintegrable constraint (try to find an example). The reason is that solutions of the nonholonomic equations in general cannot be described as extremals of any smooth functional.

Now let us discuss some problems of celestial mechanics.

11. It is well known that equations of the famous restricted circular three-body problem in suitable units for length, time, and mass, are as follows

$$(17)\qquad \begin{gathered} \ddot{x} = 2\dot{y} + \partial W/\partial x\,, \quad . \quad \ddot{y} = -2\dot{x} + \partial W/\partial y\,, \\ W = (x^2+y^2)/2 + (1-\mu)/\rho_1 + \mu/\rho_2\,, \\ \rho_1^2 = (x+\mu)^2 + y^2\,, \qquad \rho_1^2 = (x+\mu)^2 + y^2. \end{gathered}$$

Here x, y are coordinates of the asteroid in the rotating coordinate system, and μ is the ratio of the masses of Jupiter and the Sun $(0 \leqslant \mu \leqslant 1/2)$. Equations (17) have the Jacobi integral

$$(\dot{x}^2 + \dot{y}^2)/2 - W = h = \text{const}\,.$$

For a fixed value of h, motion takes place in the region of admissible motions

$$\Sigma_h = \{x\,,\, y : -W \leqslant h\}\,,$$

called the *Hill region* in celestial mechanics. Its geometry is well known from analytic and numerical investigations. Since equations (17) are nonreversible, the boundary hit theorem is no longer true (see §11). Define the set K_h as the closure of the union of trajectories that start at the boundary of the Hill region.

a. *It is interesting to study numerically the structure of the set K_h for different values of h.*

The boundary of K_h includes envelopes for the family of trajectories starting on $\partial\Sigma_h$. For integrable nonreversible systems K usually differs from Σ. Since the three-body problem is nonintegrable, it is natural to ask the following question.

b. *Is it possible that for the restricted three-body problem we have $K_h = \Sigma_h$ for some h?*

12. Consider another variant of the spatial three-body problem, in which two points of equal masses move in the x, y plane along elliptic orbits that are symmetric with respect to z-axis while the third point of zero mass lies always on this axis. The motion of the third point is governed by the differential equation $\ddot{z} = -z[z^2 + r^2(t)]^{-3/2}$, where

$$r(t) = (1 + e\cos\varphi(t))^{-1}\,, \qquad \dot{\varphi} = (1 + e\cos\varphi)^2\,, \quad \varphi(0) = 0\,.$$

This problem was suggested by Kolmogorov in order to verify the Chazy conjecture on the existence of oscillating motions in the three-body problem. Alekseev [18] established the quasirandom character of oscillations for equation (18) if the amplitude is sufficiently large. In particular, there exist infinitely many long-periodic unstable motions.

If the eccentricity e of elliptic orbits of massive bodies is zero, equation (18) is autonomous and hence integrable. For negative values of energy the phase trajectories are closed curves. Thus it is possible to introduce the action-angle variables and for small e consider the Kolmogorov problem as a perturbation of a completely integrable system.

The perturbed system is nonintegrable and satisfies the assumptions of the well-known *Poincaré theorem*, i.e., the criterion for the existence of nondegenerate long-periodic orbits. For each sufficiently large integer n and small e equation (18) has a pair of nondegenerate periodic solutions. One of them is elliptic (the first variation equation is stable) and the other hyperbolic (unstable).

Poincaré periodic solutions depend on two parameters: continuous e and discrete n. The question arises about the behavior of these solutions when we increase e.

Is it true that if e increases up to the limit $\sim 1/n$, the multiplicators λ, λ^{-1} of the Poincaré periodic solution, starting from the point $\lambda = \lambda^{-1} = 1$ for $e = 0$, in the hyperbolic case move in the opposite directions along the real axes, and in the elliptic case revolve along the unit circle on the complex plane until the collision at the point $\lambda = \lambda^{-1} = -1$, and then move in the opposite directions along the negative real half-axes?

This conjecture is based on the result of Dovbysh [19] about the behavior of Poincaré periodic solutions near the split separatrices when the small parameter is increased. If the answer is positive, it will provide a connection between the Poincaré and Alekseev periodic solutions.

13. The potential of the gravitational interaction has two fundamental properties. On one hand, it is a harmonic function on the three-dimensional space (i.e., satisfies the Laplace equation), and on the other hand only this potential (and the potential of an elastic spring) generates the central field where all bounded orbits are closed (Bertrand theorem). It appears that in the more general situation of motion in a constant curvature space these properties remain true [20].

As a matter of convenience, we shall consider motion on the three-dimensional sphere with unit radius. Let a particle m of unit mass move in the force field with potential V depending only on the distance between the particle and some fixed point M. Let ϑ be the length of the arc of the great circle connecting m and M (ϑ is measured in radians). Then the function V depends only on ϑ. The Laplace equation is replaced by the Laplace–Beltrami equation:

$$\Delta V = \sin^{-2}\vartheta \frac{\partial}{\partial\vartheta}\left(\sin^2\vartheta \frac{\partial V}{\partial\vartheta}\right) = 0 .$$

Its solution is

$$V = -\gamma \tan^{-1}\vartheta + \alpha , \qquad \alpha , \gamma = \text{const} . \tag{19}$$

The constant α is irrelevant. The parameter γ plays the role of the gravitational constant.

The dual potential is

$$V = (k/2)\tan^2\vartheta , \qquad k = \text{const} . \tag{20}$$

This is the analog of the potential of an elastic string. It appears that all orbits in central fields with the potentials (19) and (20) are closed.

This discussion leads to the natural generalization of the n-body problem: n particles move in the three-dimensional constant curvature space and their interaction is governed by the potential (19). The two-body problem is especially interesting. Contrary to the plane case, it cannot be reduced to the generalized Kepler problem.

a. *Are the bounded orbits of the generalized two-body problem closed?*

b. *Is the analog of the Sundman theorem (on expansion of solutions in convergent series for all t) true for the generalized three-body problem?*

The main difficulty is to exclude triple collisions. In the flat case it is sufficient to assume that the angular momentum of gravitating particles with respect to their center of mass does not vanish.

Concluding the discussion of this topic, consider a Hamiltonian system with two degrees of freedom and the following Hamilton function:

$$H = \frac{\Lambda(q_1, q_2)}{2}(p_1^2 + p_2^2) - \frac{f(q_1, q_2)}{\sqrt{q_1^2 + q_2^2}}. \tag{21}$$

Here Λ and f are positive analytic functions. We already mentioned that locally the kinetic energy always can be transformed to the given form. Coordinates q_1, q_2 are isotermic coordinates. The potential in the Hamiltonian (21) is said to be a *Newtonian-type* potential.

c. *Find all functions Λ and f such that all bounded orbits of the system with the Hamiltonian function* (21) *are closed.*

This is a generalization of the Bertrand problem. *Are there any solutions except the constant curvature metric and the potential of type* (19)?

In conclusion let us formulate two separate problems.

14. Let us return to the nonlinear Chaplygin equation (12). In [14] it is proved that for almost all initial conditions the solutions tend to the stable equilibrium (to the point $x = \pi \mod 2\pi$) as $t \to \infty$. These solutions have the asymptotics

$$x(t) = \pi + 2\pi n_+ + \frac{a_+}{\sqrt{t}}\sin\frac{t^2}{2} + \frac{b_+}{\sqrt{t}}\cos\frac{t^2}{2} + O\left(\frac{1}{\sqrt{t}}\right), \qquad n_+ \in \mathbb{Z},\ a_+, b_+ \in \mathbb{R}. \tag{22}$$

a. *Is it true that the numbers n_+, a_+, b_+, define the solution of the Chaplygin equation?*

If we replace t by $-t$, the equation (12) does not change. Hence for almost all solutions, we have an asymptotic representation of type (22) as $t \to -\infty$ (the numbers n_+, a_+, b_+ must be replaced by n_-, a_-, b_-).

b. *Study the properties of the correspondence $S_n\colon (a_-, b_-) \to (a_+, b_+)$.*

This nonlinear scattering problem depends on the discrete parameter $n = n_+ - n_-$, which is the number of half-rotations of the falling plate while t changes from $-\infty$ to $+\infty$.

The map S is not determined for some solutions. Among them are the doubly asymptotic solutions considered in §9. The simplest is the solution $x_a(t)$ satisfying the condition

$$x_a(t) \to \begin{cases} 0 & \text{as } t \to -\infty, \\ 2\pi & \text{as } t \to +\infty. \end{cases}$$

Asymptotically, as $t \to -\infty$ and $t \to +\infty$, a certain point of the plate descends along vertical lines. The distance between these lines is given, up to a constant coefficient, by the integral

$$I = \int_{-\infty}^{+\infty} t \sin x_a(t)\, dt.$$

It is easy to see that the integral is convergent.

c. *Is it possible to express I in terms of known mathematical constants?*

This problem can be generalized to the case when the solution $x_a(t)$ is replaced by one of the infinite sequence of solutions considered in §9.

15. Let us recall that a dynamical system

$$\dot{x} = v(x) \tag{23}$$

is called *Hamiltonian* if there exist a closed nondegenerate 2-form ω and a function $H(x)$ such that $\omega(v, \cdot) = -dH$. In local *canonical* coordinates p, q such that $\omega = \sum dp \wedge dq$, equation (23) takes the usual form

$$\dot{p} = -\partial F/\partial q, \qquad \dot{q} = \partial F/\partial p. \tag{24}$$

Here F is the function H expressed in the coordinates p, q. The form ω is called the *symplectic structure*, and H is the Hamilton function.

It follows that in order to find out whether or not a given dynamical system is Hamiltonian, we need to search for two objects: the symplectic structure and the Hamiltonian. If the system does not have the form (24), this does not mean that it is not Hamiltonian: it might just be represented in noncanonical variables.

Let us give a simple example demonstrating a hidden Hamiltonian structure. Consider a linear system with constant coefficients

$$\dot{x} = Ax, \tag{25}$$

admitting a quadratic integral $f = (Bx, x)/2$. It turns out that if the matrices A and B are nondegenerate, the system (25) is Hamiltonian. The Hamiltonian function is the integral f. The reader can verify this by producing a suitable symplectic structure.

The problem of recognizing the Hamiltonian nature of a dynamical system is a difficult and, probably, unsolvable problem. It makes sense to consider it for dynamical systems from particular classes. One of the approaches is to consider systems that are close to completely integrable systems:

$$\dot{I}_i = \varepsilon F_i(I, \varphi) + \dots, \qquad \dot{\varphi}_j = \omega_j(I) + \varepsilon G(I, \varphi) + \dots, \tag{26}$$

$1 \leqslant i, j \leqslant n$. Here $I = (I_1, \dots, I_n)$ are the slow variables and $\varphi = (\varphi_1, \dots, \varphi_n)$ are the fast angle variables. The functions $F_i, G_j, \dots$ are 2π-periodic in $\varphi_1, \dots, \varphi_n$. Dots mean terms of order ε^2, where ε is a small parameter. Equations like (26) are often encountered in applications. Of course, it also makes sense to study systems with different numbers of fast and slow variables.

Let us consider the most important case when, for $\varepsilon = 0$, system (26) is nondegenerate:

$$\frac{\partial(\omega_1, \dots, \omega_n)}{\partial(I_1, \dots, I_n)} \neq 0.$$

When it is possible to find ω and H in the form of series in ε

$$\omega_\varepsilon = \omega_0 + \varepsilon\omega_1 + \dots, \qquad H_\varepsilon = H_0 + \varepsilon H_1 + \dots,$$

with 2π-periodic in φ coefficients, such that the Hamiltonian condition $\omega_\varepsilon(v_\varepsilon, \cdot) = -dH_\varepsilon$ *is satisfied*? Here v_ε is the vector field defined by system (26).

The idea of this problem goes back to Poincaré who was the first to consider the problem of existence of "univalued" (periodic in the angles φ) integrals for Hamiltonian equations of type (26), which are represented in the form of power series in ε. This problem is closely related to the so called *small denominators problem,* first encountered in celestial mechanics. Related problems of the existence of integral invariants and symmetry fields in the form of power series in ε for systems of type (26) were considered in [21] and [22]. In the symmetry fields problem, small denominators also play the central role.

References

1. V. V. Kozlov, *Conjecture on the existence of asymptotic motions in classical mechanics*, Funktsional. Anal. i Prilozhen. **16** (1992), no. 4, 72–73; English transl. in Functional Anal. Appl. **16** (1992), no. 4.
2. ———, *Asymptotic motions and the inversion problem of the Lagrange–Dirichlet theorem*, Prikl. Mat. Mekh. **50** (1986), no. 6, 928–937; English transl. in J. Appl. Math. Mech. **50** (1986).
3. A. N. Kuznetsov, *On the existence of solutions entering a critical point of an autonomous system possessing a formal solution*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 4, 63–74; English transl. in Functional Anal. Appl. **23** (1989), no. 4.
4. I. A. Taĭmanov, *On topological properties of integrable geodesic flows*, Mat. Zametki **44** (1988), no. 2, 283–284; English transl. in Math. Notes **44** (1988).
5. V. V. Kozlov, *Integrable and nonintegrable Hamiltonian systems*, Soviet Sci. Rev. Sect. C: Math. Phys. Rev., vol. 8, Part 1, Harood Academic, Chur, 1989, pp. 1–81.
6. V. V. Kozlov and D. V. Treshchev, *The integrability of Hamiltonian systems with configuration torus*, Mat. Sb. **135** (1988), no. 1, 119–138; English transl. in Math. USSR-Sb. **63** (1989).
7. M. L. Bialy, *On polynomial in integrals first integrals of a mechanical system on the two-dimensional torus*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 4, 64–65; English transl. in Functional Anal. Appl. **21** (1987), no. 4.
8. V. V. Kozlov, *On symmetry groups of geodesic flows on closed surfaces*, Mat. Zametki **48** (1990), no. 5, 62–67; English transl. in Math. Notes **48** (1990).
9. S. V. Bolotin, *On first integrals of systems with gyroscopic forces*, Vestnik Moskov. Univ. Ser. I Matem. Mekh. **1984**, no. 6, 75–82; English transl. in Moscow Univ. Math. Bull. **39** (1984).
10. V. V. Kozlov, *Tensor invariants of quasihomogeneous systems of differential equations and the asymptotic method of Kovalevskaya–Lyapunov*, Mat. Zametki **51** (1992), no. 2, 46–52; English transl. in Math. Notes **51** (1992).
11. M. Adler and P. van Moerbeke, *Kowalewski's asymptotic method, Kac-Moody Lie algebras and regularization*, Comm. Math. Phys. **83** (1982), 83–106.
12. S. V. Bolotin, *Libration motions of natural dynamical systems*, Vestnik Moskov. Univ. Ser. I Matem. Mekh. **1978**, no. 6, 72–77; English transl. in Moscow Univ. Math. Bull. **32** (1978).
13. S. V. Bolotin and V. V. Kozlov, *Libration in systems with many degrees of freedom*, Prikl. Mat. Mekh. **42** (1978), no. 2, 245–250; English transl. in J. Appl. Math. Mech. **42** (1978).
14. V. V. Kozlov, *On the fall of a heavy rigid body in an ideal fluid*, Mekh. Tverd. Tela **5** (1989), 10–17. (Russian)
15. S. P. Novikov, *Hamiltonian formalism and a multivalued analog of Morse theory*, Uspekhi Mat. Nauk **37** (1982), no. 5, 3–49; English transl. in Russian Math. Surveys **37** (1982).
16. I. A. Taĭmanov, *Nonselfintersecting closed extremals of multivalued or non everywhere positive functionals*, Izv. Akad. Nauk SSSR Ser. Mat. **55** (1991), no. 2, 367–383; English transl. in Math. USSR-Izv. **38** (1992).
17. V. V. Kozlov, *Calculus of variations in the large and classical mechanics*, Uspekhi Mat. Nauk **40** (1985), no. 2, 33–60; English transl. in Russian Math. Surveys **40** (1985).
18. V. M. Alekseev, *Quasirandom dynamical systems*. I, Mat. Sb. **76** (1968), no. 1, 72–134 II, Mat. Sb. **77** (1968), no. 4, 545–601; III, Mat. Sb. **78** (1969), no. 1, 3–50; English transl. in Math. USSR-Sb **5** (1968); **6** (1968); **7** (1969).

19. S. A. Dovbysh, *Splitting of separatrices and creating of isolated periodic solutions in Hamiltonian systems with one and a half degrees of freedom*, Uspekhi Mat. Nauk **44** (1989), no. 2, 229–230; English transl. in Russian Math. Surveys **44** (1989).
20. V. V. Kozlov and A. Kharin, *Kepler's problem in constant curvature spaces*, Celestial Mech. (1992).
21. V. V. Kozlov, *On the existence of integral invariants for smooth dynamical systems*, Prikl. Mat. Mekh. **51** (1987), no. 4, 538–545; English transl. in J. Appl. Math. Mech. **51** (1987).
22. V. V. Kozlov, *On symmetry groups of dynamical systems*, Prikl. Mat. Mekh. **52** (1988), no. 4, 531–541; English transl. in J. Appl. Math. Mech. **52** (1988).

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