

Hydrodynamics of Noncommutative Integration of Hamiltonian Systems

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§1. Hydrodynamics of Hamiltonian systems

First we recall some known results from hydrodynamics. Let v be a stationary velocity field of a homogeneous ideal fluid. If the flow is barotropic and the external forces are potential, then

$$(1.1) \quad v \times \operatorname{curl} v = \operatorname{grad} f.$$

Here f is the *Bernoulli integral*: it is constant on the flow lines (the integral curves of the field v) and on vortex curves (the integral curves of the curl field $w = \operatorname{curl} v$). Equation (1.1) is called the *Lamb equation*. If the flow is rotational (i.e., the fields v and w are linearly independent), then the *Bernoulli surfaces* $B_c = \{f = c\}$ are regular. The compact surfaces B_c are diffeomorphic to two-dimensional tori and the motion of fluid particles are conditionally-periodic on these tori (since the fields v and w are tangent to B_c and their commutator $[v, w]$ vanishes).

It is shown in [1] that one can develop a multi-dimensional hydrodynamical theory for Hamiltonian systems.

Let N be the configuration space of a mechanical system with local coordinates $(x_1, \dots, x_n) = x$, let $M^{2n} = T^*N$ be the phase space, $(y_1, \dots, y_n) = y$ be the canonical momenta conjugated with the coordinates x and $H(x, y)$ be the Hamiltonian function. In the reversible case,

$$(1.2) \quad H = \frac{1}{2} \sum_{i,j=1}^n g_{ij}(x) y_i y_j + V(x).$$

Let $\Sigma \subset M^{2n}$ be an invariant n -dimensional surface for the Hamiltonian equations

$$(1.3) \quad \dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}.$$

We suppose that Σ is projected one-to-one on the configuration space N . Thus it can be represented by the equations

$$y_j = u_i(x_1, \dots, x_n), \quad 1 \leq i \leq n.$$

We put (for brevity) $u = (u_1, \dots, u_n)$; u is a covector field on N .

Let us consider the function $f(x) = H(x, u(x))$, the vector field

$$v(x) = \left. \frac{\partial H}{\partial y} \right|_{y=u(x)},$$

the 1-form $\omega = u(x) dx$, and its differential $\Omega = d\omega$. These objects have a sufficiently clear meaning. For example, the field v is the projection of the Hamiltonian vector field (1.3) (which is tangent to the surface Σ) on the configuration space N . The projection $T^*N \rightarrow N$ is natural: each point (x, y) is mapped to x .

If the n -dimensional invariant surface Σ exists, then the study of the Hamiltonian system (1.3) is reduced to the analysis of the dynamical system

$$(1.4) \quad \dot{x} = v(x)$$

on N . This system turns out to possess many properties typical for stationary flows of an ideal fluid.

It is shown in [1] that f, v, ω , and Ω are connected by the following relations:

$$(1.5) \quad i_v \Omega = -df, \quad L_v \Omega = 0.$$

Here i is the interior product ($i_v \Omega = \Omega(v, \cdot)$) and L_v is the differentiation along the vector field v . The second equation (1.5) is a consequence of the first one.

A vector w is called a *vortex* vector if $i_w \Omega = 0$. At each point $x \in N$ the vortex vectors form a linear space. We denote it by W_x . Let us suppose that $\dim W_x = m = \text{const}$. Then the family $\{W_x\}$ generates an m -dimensional distribution of tangent planes on the configuration space N . It is shown in the paper [2] that this distribution is always integrable. Consequently, every point of N lies on some m -dimensional integral manifold of the distribution $\{W_x\}$. It is natural to call them *vortex manifolds*. The second equation (1.5) implies the generalized *Thompson theorem*: the phase flow of system (1.4) transforms vortex manifolds into vortex manifolds (see [1]; a nonstationary version of this statement is obtained there too).

For Hamiltonian systems the following analog of the *Bernoulli theorem* holds: the function f is constant on flow lines (integral curves of the field v) and on vortex manifolds.

The first equation (1.5) can be written in a more usual coordinate form:

$$(1.6) \quad (\text{curl } u)v = -\frac{\partial f}{\partial x}.$$

Here $\text{curl } u$ denotes the skew-symmetric square matrix of order n with entries

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}.$$

The form of equation (1.6) coincides with that of equation (1.1); therefore, we call it the *Lamb equation* too.

For the Hamiltonian (1.2), the vector equation (1.6) has the following explicit form:

$$\sum_{j,k=1}^n g_{jk} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) u_k = -\frac{1}{2} \frac{\partial}{\partial x_i} \left(\sum_{j,k=1}^n g_{jk} u_j u_k \right) - \frac{\partial V}{\partial x_i}, \quad 1 \leq i \leq n.$$

As in hydrodynamics, we refer to the phase flow of system (1.4) (the stationary “stream” on N) as a *rotational (potential) flow* if $\Omega \neq 0$ (respectively $\Omega = 0$). We use analogous terms for the characterization of the field v and the surface Σ . In symplectic topology the potential surfaces Σ are called *Lagrangian*.

For a potential flow we have $\text{curl } u \equiv 0$. Consequently, locally, $\omega = dS(x)$ or, equivalently, $u = \partial S / \partial x$. In accordance with (1.6), we have $\partial f / \partial x \equiv 0$. Hence $f = h = \text{const}$. In this case system (1.6) can be replaced by the single Hamilton–Jacobi equation

$$(1.7) \quad H(x, \partial S / \partial x) = h.$$

The derivation of (1.7) from (1.6) follows exactly the derivation of the Lagrange–Cauchy integral from the hydrodynamical equations of a potential flow of ideal fluid. Unfortunately, the analogy between hydrodynamics and the theory of Hamiltonian systems is usually sketched very superficially in literature.

§2. An application to the problem on the geodesic lines on Lie groups with left-invariant metric

The nontrivial problem of the existence of stationary invariant surfaces Σ plays an essential role in the hydrodynamic theory of §1. Such surfaces actually exist, however, in a number of problems of Hamiltonian mechanics.

Our basic example is the problem of geodesic lines on an n -dimensional Lie group G with a Riemann metric invariant with respect to left shifts. The mechanical aspect of this problem is inertial motion. The simplest example is the Euler problem of the motion of a rigid body in three-dimensional space about a fixed point (here $G = SO(3)$).

Let v_1, \dots, v_n be independent right-invariant vector fields on G . The corresponding phase flows are families of left shifts on G . Since the metric is invariant with respect to all left shifts, the equations of the geodesic lines have n independent Noether integrals

$$(2.1) \quad (y, v_k) = c_k, \quad 1 \leq k \leq n.$$

These relations define n -dimensional invariant surfaces Σ_c , which are projected one-to-one on the group G .

The analysis of the corresponding stationary flows on Lie groups is an important and instructive problem. It was considered in detail in the paper [2] for the group $SO(3)$. Let us describe briefly the results of this paper.

The Noether integrals (2.1) imply that the angular momentum \vec{K} of a body is constant in the fixed space. Let $\vec{K} \neq 0$ (if $\vec{K} = 0$, then the body is at rest). We set $\vec{\gamma} = \vec{K} / |\vec{K}|$, $|\vec{K}| = k$. We see that $\vec{K} = k\vec{\gamma}$. Using this fact, one can obtain the

angular velocity of the body as a single-valued function of its position. Thus on the group $SO(3)$, the configuration space of a rigid body with fixed point, a dynamical system of the form (1.4) arises. The following propositions hold:

a) The vortex fields w on the group $SO(3)$ commuting with the field of velocities v generate the rotations of a rigid body with angular velocity $\omega = \mu\gamma$, $\mu = \text{const}$. In particular, the vortex fields are right-invariant and all the vortex curves are closed. The fibration of the group $SO(3)$ by the vortex curves coincides with the well-known Hopf fibration.

b) These fields are defined by the relations

$$i_w \Omega = 0, \quad i_w \omega = \text{const}.$$

c) The Hamiltonian system on $T^*SO(3)$ with the Hamiltonian $H' = |\vec{K}|^2/2$ has the same invariant surfaces $\vec{K} = k\vec{\gamma}$. The projection of the corresponding Hamiltonian vector field on $SO(3)$ is a vortex field commuting with the field v .

d) The metric on $SO(3)$ defined by the kinetic energy of the body allows us to calculate curl v . This field turns out to be a vortex field.

e) The phase flow of system (1.4) on $SO(3)$ conserves the bilaterally invariant Haar measure. Recall that on each Lie group there exists a unique (up to a constant multiplier) measure invariant with respect to all left (right) group shifts. It is called the *left-sided (right-sided) Haar measure*. For unimodular Lie groups these two measures coincide. All the compact groups are unimodular.

f) The "Bernoulli integral" f equals $(I^{-1}\gamma, \gamma)/2$, where I is the inertia operator of the rigid body. If the operator I is not spherical, then the flow on the group $SO(3)$ is vortex. The critical points of the function f are the orbits of the constant rotations of a body about the principal axes of the inertia ellipsoid (with a fixed value of angular momentum \vec{K}) and the critical values coincide with the values of the energy on these rotations. If c is not a critical value of the function f , then the "Bernoulli surface"

$$B_c = \{f = c\}$$

is the two-dimensional torus with a conditionally-periodic motion. The tori B_c are the natural projections of the two-dimensional Liouville tori from the phase space $T^*SO(3)$ to the group $SO(3)$.

Statements a)–f) constitute the vortex theory of the Euler top. Some of them admit generalizations to arbitrary Lie groups. For example, property e) holds for unimodular groups. In the nonunimodular case, the phase flow of system (1.4) preserves the right Haar measure. This result was obtained recently by the author and Yaroshchuk.

It is useful to note that the Hamiltonian H' is a Casimir function on the algebra $so(3)$ of the rotations group: this function commutes with all functions on $so(3)$ with respect to the corresponding Lie–Poisson bracket. It is evident that $H' \equiv \text{const}$ are the invariant surfaces.

§3. Three lemmas on Poisson brackets

This section is auxiliary. Let f_1, \dots, f_n be functions on the phase space M^{2n} , and

$$\frac{\partial(f_1, \dots, f_n)}{\partial(y_1, \dots, y_n)} \neq 0.$$

Then in accordance with the implicit function theorem the system of equations

$$(3.1) \quad f_1(x, y) = c_1, \dots, f_n(x, y) = c_n$$

can be solved (at least locally) with respect to momenta y :

$$y_1 = u_1(x, c), \dots, y_n = u_n(x, c), \quad c = (c_1, \dots, c_n).$$

Let us introduce the Poisson bracket matrix $A = (\{f_i, f_j\})$. The following simple lemma holds

LEMMA 1. *We have* $\text{rank } A = \text{rank}(\text{curl } u)$.

As a consequence, we obtain the well-known statement which is usually used in the proof of the Liouville theorem on the complete integrability of Hamiltonian systems: if the functions f_1, \dots, f_n are (pairwise) in involution, then the 1-form $u_1(x, c) dx_1 + \dots + u_n(x, c) dx_n$ is closed for all values of c .

For the proof we note that the functions

$$(3.2) \quad F_k(x, c) = f_k(x, u(x, c)), \quad 1 \leq k \leq n,$$

are identically equal to c_k . Consequently,

$$\frac{\partial F_k}{\partial x_i} = \frac{\partial f_k}{\partial x_i} + \sum \frac{\partial f_k}{\partial y_j} \frac{\partial u_j}{\partial x_i} = 0, \quad \frac{\partial F_s}{\partial x_i} = \frac{\partial f_s}{\partial x_i} + \sum \frac{\partial f_s}{\partial y_j} \frac{\partial u_j}{\partial x_i} = 0.$$

We multiply the first equality by $\partial f_s / \partial y_i$ and the second one by $-\partial f_k / \partial y_i$ and sum them over i . As a result we obtain the following relations:

$$(3.3) \quad \{f_k, f_s\} + \sum_{i,j} \frac{\partial f_k}{\partial y_j} \frac{\partial f_s}{\partial y_i} \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) = 0, \quad 1 \leq k, s \leq n.$$

We set $B = \text{curl } u$ and represent these relations in the form

$$\left(B \frac{\partial f_k}{\partial y}, \frac{\partial f_s}{\partial y} \right) = a_{ks},$$

where a_{ks} are the elements of the matrix A formed by the Poisson brackets of the functions f_1, \dots, f_n .

In accordance with our assumption the vectors

$$\frac{\partial f_1}{\partial y}, \dots, \frac{\partial f_n}{\partial y}$$

are linearly independent. Consequently, the matrices A and B are connected by some relation of the form $CBD = A$, where C and D are nondegenerate matrices. It is well known from linear algebra that in this case the ranks of the matrices A and B coincide. \square

Let $\Phi(x, y)$ be one more function on M^{2n} commuting with all the functions f_1, \dots, f_n . The Hamiltonian vector field v_Φ generated by the Hamiltonian function Φ is then tangent to the n -dimensional surface Σ_c defined by equations (3.1). Let w be the projection of the field v_Φ to the tangent to the coordinate space $\{x\}$ plane. It is evident that

$$w = \frac{\partial \Phi}{\partial y} \Big|_{y = u(x, c)}$$

LEMMA 2. If Φ is a function of f_1, \dots, f_n then w is a vortex vector:

$$(3.4) \quad (\text{curl } u)w = 0.$$

PROOF. Since each function F_k (see (3.2)) is identically equal to c_k , the following relations hold:

$$(3.5) \quad \frac{\partial F_i}{\partial c_j} = \delta_{ij} = \sum_{s=1}^n \frac{\partial u_i}{\partial c_s} \frac{\partial f_s}{\partial y_j}.$$

Here δ_{ij} is the Kronecker symbol.

Let us multiply equalities (3.3) by the derivatives $\partial u_p / \partial c_k, \partial u_q / \partial c_s$ and then sum them over k and s . As the result we get:

$$\sum_{k,s} \frac{\partial u_p}{\partial c_k} \frac{\partial u_q}{\partial c_s} \{f_k, f_s\} + \frac{\partial u_p}{\partial x_q} - \frac{\partial u_q}{\partial x_p} = 0.$$

To verify equality (3.4), we multiply these relations by

$$\frac{\partial \Phi}{\partial y_p} \Big|_{y=u(x,c)}$$

and sum them over p from 1 to n . We are interested in the values of

$$\sum_{k,s} \sum_p \frac{\partial u_p}{\partial c_k} \frac{\partial \Phi}{\partial y_p} \Big|_{y=u} \frac{\partial u_q}{\partial c_s} \{f_k, f_s\}.$$

In accordance with our assumption, the function $\varphi(x, c) = \Phi(x, u(x, c))$ depends only on $c_1 = f_1, \dots, c_n = f_n$. Consequently,

$$\sum_p \frac{\partial \Phi}{\partial y_p} \Big|_{y=u} \frac{\partial u_p}{\partial c_k} = \frac{\partial \varphi}{\partial c_k}.$$

The sum

$$\sum_k \frac{\partial \varphi}{\partial c_k} \{f_k, f_s\}$$

is equal to the Poisson bracket $\{\Phi, f_s\}$, which vanishes by our assumption. The lemma is proved.

The collection of functions f_1, \dots, f_n is called *closed* if all the Poisson brackets $\{f_i, f_j\}$ are functions of f_1, \dots, f_n . Let us find a simple condition for being closed by using properties of the collection of the covector fields $u(x, c)$.

The covectors

$$a_1 = \frac{\partial u}{\partial c_1}, \dots, a_n = \frac{\partial u}{\partial c_n}$$

are linearly independent. The dual collection of n linearly independent vectors b_1, \dots, b_n such that $(a_i, b_j) = \delta_{ij}$, $1 \leq i, j \leq n$, corresponds uniquely to them. In accordance with (3.5), $b_j = \partial f_j / \partial y$, $1 \leq j \leq n$.

LEMMA 3. The collection of functions f_1, \dots, f_n is closed if and only if for all $i, j = 1, \dots, n$ the functions (Bb_i, b_j) do not depend on x .

Here, as above, $B = \text{curl } u$. Thus the verification of the fact that the collection is closed requires only differentiations and algebraic operations, including the operation of inversion of functions.

Lemma 3 is a simple consequence of relations (3.3) and (3.5).

§4. Noncommutative integration

Let us suppose that a Hamiltonian system with n degrees of freedom has a collection of m independent integrals F_1, \dots, F_m closed with respect to the Poisson brackets

$$\{F_i, F_j\} = f_{ij}(F_1, \dots, F_n), \quad 1 \leq i, j \leq n.$$

The matrix $(f_{i,j})$ is skew-symmetric, therefore its rank is even. Assume that $\text{rank}(f_{i,j}) = 2k$.

The basic assumption of the *noncommutative theory* of Hamiltonian equations integration is as follows:

$$(4.1) \quad m = n + k.$$

For $k = 0$ we obtain the classical Liouville condition of complete integrability.

One can formulate a geometric consequence of condition (4.1): if the surface of the joint levels of the integrals F_1, \dots, F_m is regular and compact, then each of its connected component is a $(2n - m) = (n - k)$ -dimensional torus; moreover, the phase flow of the Hamiltonian system in question is a “winding” on these tori. An analytical consequence of (4.1) is the integrability in quadratures of Hamiltonian equations.

The theory of noncommutative integration of Hamiltonian systems was developed in the papers of N. N. Nekhoroshev, A. S. Mishchenko, A. T. Fomenko, Ya. V. Tatarinov, A. V. Strel'tsov, A. V. Brailov, and others. It is set forth in detail in the book [3]; the necessary references can be found there as well.

According to the classical Lie–Cartan theorem (see [4, p. 126]) there exist $m - 2k$ functions $\Phi_1, \dots, \Phi_{m-2k}$ of F_1, \dots, F_m , that commute with all of these m functions, and $2k$ functions Ψ_1, \dots, Ψ_{2k} of F_1, \dots, F_m can be chosen in such a way that their Poisson bracket matrix is nondegenerate. Let us consider the $2(n - k)$ -dimensional invariant surface

$$M_\alpha = \{\Psi_1 = \alpha_1, \dots, \Psi_{2k} = \alpha_{2k}\}.$$

It is a symplectic manifold: the restriction of the symplectic structure (2-form) $dx \wedge dy$ on M_α is a closed nondegenerate 2-form. The restriction of $n - k$ functions $\Phi_1, \dots, \Phi_{m-2k}$ on M_α are independent and commute (with respect to the symplectic structure on M_α). Since, in accordance with (4.1), we have

$$m - 2k = n - k = \dim M_\alpha / 2,$$

we can apply the conventional Liouville theorem on involutive integrals. In this way one can, in fact, prove the theorem on the fibration of the original $2n$ -dimensional phase space by $(n - k)$ -dimensional tori with conditionally-periodic motions.

It is essential that the Lie–Cartan theorem cannot be used to prove the integrability quadratures, since this theorem does not give a constructive way to obtain the functions Φ and Ψ . A. V. Brailov proved the exact integrability of Hamiltonian equations with condition (4.1) by using other arguments.

Condition (4.1) has a clear meaning: one cannot add other independent functions to the functions F_1, \dots, F_m keeping the rank of their Poisson bracket matrix fixed.

The basic examples of the application of noncommutative integration theory are connected with the problem of inertial motion on a Lie group with left-invariant kinetic energy (see §2). Let us discuss these examples in more detail.

Let G be a Lie group, g its algebra, $n = \dim G$. The phase space T^*G can be represented as the direct product $g^* \times G$, where g^* is the dual to g . Lie algebra g is the space of velocities $\omega = (\omega_1, \dots, \omega_n)$, and g^* is the space of momenta $k = (k_1, \dots, k_n)$. These objects are connected by the linear relations

$$(4.2) \quad k_s = \sum_{p=1}^n I_{sp} \omega_p,$$

where $I = (I_{sp})$ is the constant inertia tensor of the system.

Hamiltonian equations on T^*G include n dynamical *Euler–Poincaré equations*

$$(4.3) \quad \dot{k}_i = \sum c_{ip}^j k_j \omega_p, \quad 1 \leq i \leq n,$$

and n geometric relations; their exact form is irrelevant here. In (4.3), the coefficients c_{ip}^j denote structure constants of the algebra g . System (4.3) can be considered either as a closed system on the algebra g (then relations (4.2) must be substituted for k) or as a system on the coalgebra g^* (then (4.2) must be solved with respect to ω). For the Lie algebra $so(3)$, equations (4.3) coincide with the famous Euler dynamical equations of a rotating top.

The Lie–Poisson bracket

$$\{H, F\} = \sum c_{is}^j k_j \frac{\partial H}{\partial k_i} \frac{\partial F}{\partial k_s}$$

is closely connected to equations (4.3). To each ordered pair of functions H, F on g^* there corresponds the bracket $\{H, F\}$, which is a function on g^* too. The Lie–Poisson bracket is degenerate: there exist nonconstant functions on g^* commuting with all basic functions k_1, \dots, k_n . These nonconstant functions are called the *Casimir functions*. They can also be characterized in the following way: they are exactly the functions of n Noether integrals that do not depend on the position of a system. For example, for the algebra $so(3)$ the square of the length of the angular momentum vector $k_1^2 + k_2^2 + k_3^2$ (cf. with §3) is a Casimir function.

Authors, who studied the problem of exact integration of Hamilton equations with left-invariant metric on Lie groups, considered, as a rule, only the Euler–Poincaré equations (4.3) (one can find a survey of the corresponding results in [3]). Let us suppose that equations (4.3) admit l independent integrals which are not Casimir functions; let $2p$ be the maximum rank of their Poisson bracket matrix and $2k$ the maximum rank of the Noether integrals Poisson bracket matrix. If

$$(4.4) \quad l = k + p,$$

then equations (4.3) are integrable in quadratures and almost the entire g^* is fibered by $(k - p)$ -dimensional tori with conditionally periodic trajectories.

The derivation of condition (4.4) is based on the following arguments. The joint level of Casimir functions in the typical situation are $2k$ -dimensional invariant

symplectic manifolds with l independent integrals. Thus condition (4.4) is the noncommutative integrability condition (4.1).

It is easy to understand that condition (4.4) guarantees the integrability of a Hamiltonian system on the whole phase space T^*G . To see that, we rewrite (4.4) in the following way:

$$n + l = n + (k + p).$$

This is the integrability condition (4.1). Actually, the system of Hamiltonian equations admits $n + l$ independent integrals (l integrals of the Euler–Poincaré equations and n Noether integrals). Moreover, the rank of their Poisson bracket matrix equals $2(k + p)$ (since the integrals of equations (4.3) commute with the Noether integrals). Hence almost all the phase space $T^*G = g^* \times G$ is foliated by $2n - l - n = (n - k - p)$ -dimensional tori with conditionally periodic motions.

Concluding this section, we consider Hamiltonian systems from the hydrodynamical point of view. Let $(c_1, \dots, c_n) = c$ be constants of the Noether integrals (2.1). Consider the typical case when for the given value of c the rank of the Poisson bracket matrix of these integrals is maximal (equal to $2k$). It was noted in §2 that a dynamical system on G arises after projecting the n -dimensional invariant surface $\Sigma_c \subset T^*G$ on the group G .

LEMMA 4. *Functions of k_1, \dots, k_n commute with the Noether integrals.*

PROOF. Let Φ be a function on g^* . It can be regarded as a function on $T^*G = g^* \times G$ invariant with respect to left shifts. Consequently, according to the Noether theorem, the Hamiltonian vector field v_Φ admits n Noether integrals. \square

Let Φ be a function on g^* . In accordance with Lemma 4, the Hamiltonian vector field v_Φ is tangent to Σ_c . Thus the projection onto G of the restriction of v_Φ to Σ_c is well defined.

Let $w_1, \dots, w_{n-2k} (v_1, \dots, v_l)$ be tangent fields on G that are the projections of the Hamiltonian fields generated by the Casimir functions (along the remaining integrals of the Euler–Poincaré equations). These fields are independent and we have $[w_i, v_j] = 0$ for all $i = 1, \dots, n - 2k, j = 1, \dots, l$.

We denote by F_1, \dots, F_l independent integrals of equations (4.3) which are not Casimir functions. Let f_1, \dots, f_l be functions on the group G defined uniquely by the commutative diagram

$$\begin{array}{ccc} \Sigma_c & \xrightarrow{\text{pr}} & G \\ F|_{\Sigma_c} \searrow & & \swarrow f \\ & \mathbb{R} & \end{array}$$

PROPOSITION. *The fields w_1, \dots, w_{n-2k} are vortex and f_1, \dots, f_l are Bernoulli integrals (they are constant on the flow lines and on the vortex manifolds).*

This proposition is a simple consequence of Lemma 2 (see §3).

In the typical situation, all the vortex vectors are linear combinations of the vectors w_1, \dots, w_{n-2k} . However, when the rank of the Poisson bracket matrix drops, new vortex vectors arise.

Regular “Bernoulli surfaces”

$$B_\alpha = \{f_1 = \alpha_1, \dots, f_l = \alpha_l\}$$

are $n - l = (n - k - p)$ -dimensional tori with conditionally-periodic motions. They are projections onto the group G of Liouville tori foliating $2n$ -dimensional phase space in accordance with the noncommutative integration theorem.

These observations extend conclusion c) (see §2) associated with the Euler top in three-dimensional space to the multi-dimensional case. It is easy to understand that the vortex fields w_1, \dots, w_{n-2k} are right invariant and the vortex manifolds are closed (this is the analog of conclusion a)). We show that $i_{w_s}\omega = \text{const}$, $1 \leq s \leq n$. Indeed,

$$(4.5) \quad i_{w_s}\omega = \left(u, \frac{\partial \Phi}{\partial y} \Big|_{y=u} \right),$$

where Φ is the corresponding Casimir function. Since these functions are homogeneous polynomials of the Noether integrals, by the Euler theorem, the right-hand side of (4.5) is a constant (depends only on c_1, \dots, c_n). These arguments prove conclusion b). An analog of conclusion f) was formulated above. It would be interesting to obtain a natural multi-dimensional extension of property d).

§5. The vortex integration method

We shall follow the integration method of the equations for geodesic lines on Lie groups and set forth an integration method of Hamiltonian equations (1.3) given in the phase space $M^{2n} = T^*N$. Let $H(x, y)$ be a Hamiltonian function. We rewrite the Lamb equation (1.6) in a more explicit form:

$$(5.1) \quad (\text{curl } u) \left(\frac{\partial H}{\partial y} \Big|_{y=u} \right) = - \frac{\partial H(x, u)}{\partial x}.$$

DEFINITION 1. A solution $u(x, c_1, \dots, c_n)$ of system (5.1) is called *complete* if

$$\det \frac{\partial u}{\partial c} \neq 0.$$

DEFINITION 2. A solution $u(x, c_1, \dots, c_n)$ is called *closed* if the conditions of Lemma 3 (see §3) hold.

Before the formulation of our result, we need the following remark. To each function $\Phi(x, y)$ defined on M^{2n} there corresponds the equation

$$(5.2) \quad (\text{curl } u) \left(\frac{\partial \Phi}{\partial y} \Big|_{y=u} \right) = - \frac{\partial \Phi(x, u)}{\partial x}$$

of the form (1.5). Every solution $u(x)$ of (5.2) forms an n -dimensional surface $y = u(x)$ which is tangent to the Hamiltonian vector field v_Φ .

THEOREM. *Suppose a complete and closed n -parametric solution $u(x, c)$ of equation (5.1) is found and*

$$\text{rank} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) = 2k.$$

Moreover, let a closed collection of l integrals Φ_1, \dots, Φ_l of Hamiltonian equations (1.3) exist and also suppose that

1) $\text{rank} (\{\Phi_i, \Phi_j\}) = 2p,$

2) *the functions $u(x, c)$ satisfy all equations obtained from (5.2), by replacing the function Φ with the integrals Φ_1, \dots, Φ_l .*

3) $\Phi_1(x, u(x, c)), \dots, \Phi_k(x, u(x, c))$ *are independent as functions of x .*

If $l = k + p$, then the Hamiltonian equations (1.3) can be integrated in quadratures.

This result contains as a special case the Jacobi theorem on the complete integrals: Indeed, let

$$u = \frac{\partial S(x, c)}{\partial x}.$$

Then $\text{curl } u \equiv 0$, consequently $k = 0$ and the function $S(x, c)$ satisfies the Hamilton–Jacobi equation. It remains to set $l = p = 0$.

One of the most interesting cases is contained in the following

COROLLARY. *Suppose a complete and closed family of solutions $u(x, c)$ of equation (5.1) is found and*

1) $\text{rank} (\text{curl } u) = 2,$

2) $d_x H(x, u(x, c)) \neq 0.$

Then the Hamiltonian equations with the Hamiltonian H are integrable in quadratures.

In this case $l = 1, p = 0$, and the Hamiltonian function is an additional integral. This statement is more effective for $n = 3$ because $k \leq 1$ in this case.

PROOF OF THE THEOREM. By the completeness of the solution, the equations

$$y_1 = u_1(x, c_1, \dots, c_n), \dots, y_n = u_n(x, c_1, \dots, c_n)$$

are locally solvable with respect to c_1, \dots, c_n :

$$c_1 = F_1(x, y), \dots, c_n = F_n(x, y).$$

The functions F_1, \dots, F_n form a collection of integrals of equations (1.3). Moreover,

1) the Poisson brackets $\{F_i, F_j\}$ depend only on F_1, \dots, F_n (Lemma 3),

2) $\text{rank} (\{f_i, F_j\}) = 2k$ (Lemma 1).

By the nondegeneracy condition of the solution $u(x, c)$ and assumption 3 of the theorem, the functions

$$(5.3) \quad F_1, \dots, F_n, \Phi_1, \dots, \Phi_l$$

are independent. Let v_1, \dots, v_l be the Hamiltonian vector fields corresponding to the Hamiltonians Φ_1, \dots, Φ_l . Due to assumption 2 of the theorem, the fields

v_i are tangent to the invariant surfaces $\Sigma_c = \{x, y : y = u(x, c)\}$ for all values of c . Since F_1, \dots, F_n are independent, these functions are integrals of the fields v_1, \dots, v_l . Consequently, all the Poisson brackets $\{F_i, \Phi_j\}$ vanish. This in turn implies that the rank of the Poisson bracket matrix of integrals (5.3) equals $k + p$. Hence (due to the assumption), $n + l = n + (k + p)$ and condition (4.1) of the noncommutative theory holds. This guarantees the exact integrability in quadratures of the Hamiltonian equations. The theorem is proved.

The method for finding a complete integral based on the separation of variables is known for the Hamilton–Jacobi equation. We know of nothing similar for the more general equation (5.1).

In conclusion, we consider again the rotation of the Euler top, which is described by the Euler dynamical equations

$$(5.4) \quad \dot{k} = k \times \omega,$$

and the geometrical relations for the fixed unit vectors α, β, γ :

$$(5.5) \quad \dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega, \quad \dot{\gamma} = \gamma \times \omega.$$

Here k is the angular momentum of the rigid body; it is equal to $I\omega$, where I is the constant inertia operator. It is clear that equations (5.4)–(5.5) are equivalent to the Hamiltonian equations of the Euler problem.

We see three-dimensional invariant surfaces that are projected one-to-one onto the group $SO(3)$. This means that the momentum k is a function of α, β, γ . From (5.4), we obtain the partial equation in vector form

$$(5.6) \quad \frac{\partial k}{\partial \alpha} (\alpha \times \omega) + \frac{\partial k}{\partial \beta} (\beta \times \omega) + \frac{\partial k}{\partial \gamma} (\gamma \times \omega) = k \times \omega.$$

Here instead of ω one must set $I^{-1}k$. Obviously, this equation is equivalent to equation (5.1) presented in noncanonical variables. The passage from (5.1) to (5.6) is completely similar to the passage from the Lagrangian (Hamiltonian) equations to the Euler–Poincaré (Chetaev) equations on Lie algebras.

It is evident that the function

$$(5.7) \quad k = c_1 \alpha + c_2 \beta + c_3 \gamma$$

is one of the complete solutions of (5.6), where c_1, c_2, c_3 are arbitrary constants. This is a consequence of the invariance of the vector k in the fixed space. The fact that the solution (5.7) is closed is evident.

If $c \neq 0$ and the inertial tensor is not spherical, then assumptions 1 and 2 of the corollary hold.

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