Mathematical Notes, Vol. 58, No. 6, 1995

On Solutions with Generalized Power Asymptotics to Systems of Differential Equations

V. V. Kozlov and S. D. Furta

UDC 517.9

ABSTRACT. In the paper we study methods for constructing particular solutions with nonexponential asymptotic behavior to a system of ordinary differential equations with infinitely differentiable right-hand sides. We construct the corresponding formal solutions in the form of generalized power series whose first terms are particular solutions to the so-called truncated system. We prove that these series are asymptotic expansions of real solutions to the complete system. We discuss the complex nature of the functions that are represented by these series in the analytic case.

§1. Introduction

Consider the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

whose right-hand sides are representable as formal power series

$$f^j = \sum_{i_0, i_1, \dots, i_n} f^j_{i_0, i_1, \dots, i_n} t^{i_0} (x^1)^{i_1} \dots (x^n)^{i_n}, \tag{2}$$

where the sum is taken over positive integers $i_1, \dots, i_n$ and integers i_0. For such systems we find sufficient conditions that guarantee the existence of particular solutions with generalized power asymptotics of the form $\sim t^{-\mathbf{G}}$, $t \to \pm 0$ or $t \to \pm\infty$, where $\mathbf{G}$ is a real matrix.

For simplicity, we assume that the right-hand sides of (1) are infinitely differentiable functions of $\mathbf{x}, t$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. In specific applications, we mainly deal with polynomial systems, for which this requirement is always satisfied.

To construct such solutions, we must first truncate system (1) to a model system that has explicitly constructible solutions with the cited asymptotics. Usually, these model systems are taken to be quasihomogeneous systems, which are singled out by using Newton polytopes of the exponents in the expansions (2) [1]. However, this approach permits one to find only solutions with the classical power asymptotics (the matrix $\mathbf{G}$ is diagonal), whereas in specific applications one often deals with solutions whose leading terms of asymptotic expansions involve expressions of the form $t^\alpha \sin(\delta \ln t)$ or $t^\alpha \cos(\delta \ln t)$. A similar situation occurs, for instance, for autonomous systems, say, for autonomous multidimensional Hamiltonian systems with even-order resonance of frequencies [2]. Thus, we must slightly modify the conventional approach to the construction of truncated systems.

§2. Quasihomogeneous and semiquasihomogeneous systems

Consider an $(n+1)$-dimensional Fuchsian system of differential equations of the form

$$\mu \frac{d\mathbf{x}}{d\mu} = \mathbf{G}\mathbf{x}, \qquad \mu \frac{dt}{d\mu} = -t. \tag{3}$$

The flow of system (3) will be denoted by

$$t \mapsto \mu^{-1} t, \qquad \mathbf{x} \mapsto \mu^{\mathbf{G}} \mathbf{x}. \tag{4}$$

Translated from *Matematicheskie Zametki*, Vol. 58, No. 6, pp. 851–861, December, 1995.
Original article submitted March 20, 1995.

 0001-4346/95/5856-1286 $12.50

Definition 1. System (1) is said to be *quasihomogeneous* with respect to the structure generated by the matrix $\mathbf{G}$ if (1) is invariant with respect to the phase flow (4) of system (3).

In the following, we equip the right-hand sides of quasihomogeneous systems with a subscript q (the *quasihomogeneity index*). Sometimes this subscript will be used to denote some generalized "degree" of quasihomogeneity.

Thus, the truncated systems will be quasihomogeneous systems

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}, t). \tag{5}$$

It is natural to seek solutions to (5) in the form of "quasihomogeneous rays"

$$\mathbf{x}^{\gamma}(t) = (\gamma t)^{-\mathbf{G}} \mathbf{x}_0^{\gamma}, \tag{6}$$

where $\gamma = \pm 1$ and $\mathbf{x}_0^{\gamma}$ is a constant real vector. If we are interested in the asymptotics for positive t, then we set $\gamma = +1$, otherwise we must take $\gamma = -1$.

In what follows, we assume that the truncated system (5) has a real solution of the form (6) and we will try to complete this solution so as to obtain a solution to the complete system (1).

Let us now define a class of systems for which quasihomogeneous systems of the type (5) can serve as truncated ones.

Definition 2. System (1) is said to be *semiquasihomogeneous* if the flow (4) transforms its right-hand side to the form

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}, t) + \mathbf{f}^*(\mathbf{x}, t, \mu), \tag{7}$$

where $\mathbf{f}_q(\mathbf{x}, t)$ is a quasihomogeneous vector field and $\mathbf{f}^*(\mathbf{x}, t, \mu)$ is a formal series in powers of μ^{β}, the exponent β being a nonnegative real number. For $\beta > 0$, system (1) is said to be *positive semiquasihomogeneous*, and for negative β, it is said to be *negative semiquasihomogeneous*.

If in (7) we set $\mu = 0$ for a positive semiquasihomogeneous system or $\mu = \infty$ for a negative semiquasihomogeneous system, then we obtain the truncated system (5).

There are numerous examples of semiquasihomogeneous systems of differential equations. A classical example of a nonautonomous semiquasihomogeneous system is given by the first and the second Painlevé equations [3] reduced to the form of second-order systems.

Looking ahead, we note that if a system is positive semiquasihomogeneous, then we must seek particular solutions with generalized power asymptotics as $t \to \pm\infty$; otherwise, we must study the asymptotics of solutions as $t \to \pm 0$.

§3. Main theorem

Theorem 1. *Let system* (1) *be semiquasihomogeneous and let the corresponding truncated system* (5) *have a particular solution of type* (6). *Then* (1) *has a particular solution for which* (6) *is the main term of the asymptotic expansion as* $t^{\chi} \to \gamma \times \infty$, *where* $\chi = \operatorname{sign} \beta$ *is the "semiquasihomogeneity sign."*

Proof. First, we formally construct the asymptotic solution to system (1). We seek this solution in the form of a series

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}} \sum_{k=0}^{\infty} \mathbf{x}_k(\ln(\gamma t))(\gamma t)^{-k\beta}, \tag{8}$$

where the coefficients $\mathbf{x}_k$ are polynomial vector functions of $\ln(\gamma t)$.

It is interesting to note that the series (8) has the same form as the series applied in the integration of systems of linear differential equations with a Fuchsian singularity by means of the Frobenius method [4].

Obviously, the right-hand sides of (1) admit formal expansions into series in "quasihomogeneous forms":

$$\mathbf{f}(\mathbf{x}, t) = \sum_{m=0} \mathbf{f}_{q+\chi m}(\mathbf{x}, t).$$

The following identities hold:

$$\mathbf{f}_{q+\chi m}(\mu^{\mathbf{G}}\mathbf{x}, \mu^{-1}t) = \mu^{\mathbf{G}+(1+\beta m)\mathbf{E}}\mathbf{f}_{q+\chi m}(\mathbf{x}, t). \tag{9}$$

By using identities (9), we see that after the change of the dependent and the independent variables given by

$$\mathbf{x}(t) = (\gamma t)^{-\mathbf{G}}\mathbf{y}(\gamma t), \qquad s = (\gamma t)^{-\beta},$$

system (1) acquires the form

$$-\gamma\beta s\mathbf{y}' = \gamma\mathbf{G}\mathbf{y} + \sum_{m=0} s^m \mathbf{f}_{q+\chi m}(\mathbf{y}, \gamma), \tag{10}$$

where the prime denotes the derivative with respect to the new independent variable s.

In the new variables, the series (8) become

$$\mathbf{y}(s) = \sum_{k=0}^{\infty} \mathbf{x}_k\Big(-\frac{1}{\beta}\ln s\Big)s^k. \tag{11}$$

Let us substitute (11) into (10) and match the coefficients of like powers of s. Assume that the zeroth coefficient in (8) is constant; then for the coefficients of s^0 we have the system of algebraic equations

$$-\gamma\mathbf{G}\mathbf{x}_0 = \mathbf{f}_q(\mathbf{x}_0, \gamma). \tag{12}$$

Clearly, the solvability of (12) for $\mathbf{x}_0 = \mathbf{x}_0^{\gamma}$ is equivalent to the existence of a particular solution (6) to the truncated system (5).

For the coefficients of higher powers of s, we have the systems

$$\gamma\frac{d\mathbf{x}_k}{d\tau} - \mathbf{K}_k\mathbf{x}_k = \mathbf{\Phi}_k(\mathbf{x}_0, \dots, \mathbf{x}_{k-1}), \tag{13}$$

where $\mathbf{\Phi}_k$ is a polynomial vector function of $\mathbf{x}_0, \dots, \mathbf{x}_{k-1}$, $\tau = -\beta^{-1}\ln s = \ln(\gamma t)$, $\mathbf{K}_k = k\gamma\beta\mathbf{E} + \mathbf{K}$, and

$$\mathbf{K} = \gamma\mathbf{G} + d\mathbf{f}_q(\mathbf{x}_0^{\gamma}, \gamma) \tag{14}$$

is the so-called Kowalewski matrix [5].

If we assume that all coefficients, up to the kth one, are determined, then the $\mathbf{\Phi}_k$ become known polynomials in τ. The resulting system (13) can be viewed as a system of ordinary differential equations with constant coefficients and polynomial right-hand side; such a system always has a polynomial particular solution $\mathbf{x}_k(\tau)$. Thus, we can successively find all coefficients of the series (11). The formal construction of a particular asymptotic solution to (1) in the form (8) is thus complete.

The presence of logarithms in the expansion (8) means in general that we are considering the generic case. For example, consider the simple situation in which the truncated system (5) is autonomous. By a slight modification of the argument in [5], we can show that $-\gamma$ is always an eigenvalue of the Kowalewski matrix (14) with the eigenvector $\mathbf{f}_q(\mathbf{x}_0^{\gamma})$. Since, as a rule, in specific applications we have $|\beta| = 1/(q-1)$, where $q \geq 2$ is an integer, we see that, in the positive semiquasihomogeneous case, at least one of the matrices $\mathbf{K}_k$ is degenerate, and, in general, the logarithms are inherent in expansions (8).

In the next step of the proof of Theorem 1, we must show that either the series (8) converge uniformly together with their derivatives on the given interval of t, or the series (8) are asymptotic expansions of a real solution to (1).

In the classical situation (for which the right-hand sides of (1) are autonomous, system (1) is positive semiquasihomogeneous, the matrix G is diagonal, and the elements of G are nonnegative integers), the proof of the latter assertion can be obtained with the help of a well-known theorem from [6]. However, since the case under consideration is formally not contained in the framework of the theorem in [6], we prove our assertion independently.

Without loss of generality, throughout the following, we assume that $\gamma = +1$ and, for brevity, omit the subscript γ. The case $\gamma = -1$ can be reduced by time inversion to the case considered. $\square$

§4. Infinitely differentiable case

Suppose that the right-hand sides of system (10) are infinitely differentiable functions on the Cartesian product of a small neighborhood of $\mathbf{y} = \mathbf{x}_0$ and a small neighborhood of $s = 0$. In system (10) we change the "time scale"

$$s = \varepsilon\xi, \qquad 0 < \varepsilon \ll 1,$$

and rewrite the system in the form

$$-\beta\xi\frac{d\mathbf{y}}{d\xi} = \mathbf{G}\mathbf{y} + \sum_{m=0} \varepsilon^m \xi^m \mathbf{f}_{q+\chi m}(\mathbf{y}, \gamma). \tag{15}$$

For $\varepsilon = 0$, this system becomes

$$-\beta\xi\frac{d\mathbf{y}}{d\xi} = \mathbf{G}\mathbf{y} + \mathbf{f}_q(\mathbf{y}, \gamma),$$

which has a particular equilibrium solution $\mathbf{y}_0(\xi) = \mathbf{x}_0$.

After the transformation described above, the Kth partial sum of the series (8) acquires the form

$$\mathbf{y}_K^\varepsilon(\xi) = \sum_{k=0}^{K} \varepsilon^k \mathbf{x}_k\Big(-\frac{1}{\beta}\ln(\varepsilon\xi)\Big)\xi^k,$$

whence it follows that for $\varepsilon \to +0$ this sum converges to $\mathbf{y}_0(\xi) = \mathbf{x}_0$ uniformly on $[0, 1]$.

Let $K \in \mathbb{N}$ be so large that $-\beta K < \operatorname{Re}\rho_i$, $i = 1, \dots, n$, where ρ_i are the eigenvalues of the Kowalewski matrix $\mathbf{K}$.

We seek a particular solution to system (15) for sufficiently small $\varepsilon > 0$ on the interval $[0, 1]$ in the form

$$\mathbf{y}(\xi) = \mathbf{y}_K^\varepsilon(\xi) + \mathbf{z}(\xi)$$

with the initial condition $\mathbf{y}(+0) = \mathbf{0}$, where $\mathbf{z}(\xi)$ has the asymptotics $\mathbf{z}(\xi) = O(\xi^{K+\delta})$ as $\xi \to +0$; we assume that $\delta > 0$ is chosen sufficiently small.

Let us write (15) as an equation in a Banach space:

$$\mathcal{G}(\varepsilon, \mathbf{z}) = \mathbf{0},$$
$$\mathcal{G}(\varepsilon, \mathbf{z}) = \beta\xi\frac{d}{d\xi}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \mathbf{G}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \sum_{m=0} \varepsilon^m \xi^m \mathbf{f}_{q+\chi m}(\mathbf{y}_K^\varepsilon + \mathbf{z}, \gamma). \tag{16}$$

We regard $\mathcal{G}(\varepsilon, \mathbf{z})$ as a mapping

$$\mathcal{G}\colon (0, \varepsilon_0) \times \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta},$$

where $\mathfrak{B}_{1,\Delta}$ is the Banach space of vector functions $\mathbf{z}\colon [0, 1] \to \mathbb{R}^n$ that are continuous on $[0, 1]$ together with the first derivatives and have finite norm

$$\|\mathbf{z}\|_{1,\Delta} = \sup_{[0,1]} \xi^{-\Delta}\big(\|\mathbf{z}(\xi)\| + \xi\|\mathbf{z}'(\xi)\|\big)$$

(here the prime denotes the derivative with respect to ξ), and $\mathfrak{B}_{0,\Delta}$ is the Banach space of vector functions $\mathbf{u}\colon [0, 1] \to \mathbb{R}^n$ that are continuous on $[0, 1]$ and have finite norm

$$\|\mathbf{u}\|_{0,\Delta} = \sup_{[0,1]} \xi^{-\Delta}\|\mathbf{u}(\xi)\|, \qquad \text{where} \quad \Delta = K + \delta.$$

We note some properties of the mapping $\mathcal{G}$:

a) $\mathcal{G}(0, \mathbf{0}) = \beta\xi\frac{d}{d\xi}\mathbf{y}_0(\xi) + \mathbf{G}\mathbf{y}_0(\xi) + \mathbf{f}_q\big(\mathbf{y}_0(\xi), \gamma\big) = \mathbf{0}$;

b) $\mathcal{G}$ is continuous with respect to $(\varepsilon, \mathbf{z})$ on $(0, \varepsilon_0) \times \mathfrak{U}_{1,\Delta}$, where $\mathfrak{U}_{1,\Delta}$ is a neighborhood of the origin in $\mathfrak{B}_{1,\Delta}$;

c) $\mathcal{G}$ is strongly differentiable with respect to $\mathbf{z}$ on $(0, \varepsilon_0) \times \mathfrak{U}_{1,\Delta}$ and the Frechét differential $\nabla_{\mathbf{z}}\mathcal{G}(\varepsilon, \mathbf{z})$, given by

$$\nabla_{\mathbf{z}}\mathcal{G}(\varepsilon, \mathbf{z})\mathbf{h} = \beta\xi\frac{d}{d\xi}\mathbf{h} + \mathbf{G}\mathbf{h} + \sum_{m=0} \varepsilon^m\xi^m d\mathbf{f}_{q+\chi m}(\mathbf{y}_K^\varepsilon + \mathbf{z}, \gamma)\mathbf{h}, \qquad \mathbf{h} \in \mathfrak{B}_{1,\Delta},$$

is a bounded operator continuously depending on $\varepsilon, \mathbf{z}$;

d) the operator $\nabla_{\mathbf{z}}\mathcal{G}(0, 0)\colon \mathfrak{B}_{1,\Delta} \to \mathfrak{B}_{0,\Delta}$,

$$\nabla_{\mathbf{z}}\mathcal{G}(0, 0) = \beta\xi\frac{d}{d\xi} + \mathbf{K},$$

is continuously invertible.

Assertions a), b), and c) are quite obvious. The proof of an assertion similar to d) can be found in [7]. Thus, all assumptions of the implicit function theorem [8] are satisfied; therefore, if $\varepsilon_0 > 0$ is sufficiently small, then for every $\varepsilon \in (0, \varepsilon_0)$ the functional equation (16) has a solution in $\mathfrak{B}_{1,\Delta}$ that continuously depends on ε. On proceeding to the variables $(\mathbf{y}, s)$, we see that the differential equation (10) has a particular solution $\mathbf{y}(s) \in C^1[0, \varepsilon]$ with the asymptotics

$$\mathbf{y}(s) = \sum_{k=0}^{K} \mathbf{x}_k\Big(-\frac{1}{\beta}\ln s\Big)s^k + o(s^K).$$

Since the right-hand side of (10) is a smooth vector function, we actually have $\mathbf{y} \in C^\infty[0, \varepsilon]$. Returning to the original variables $\mathbf{x}, t$, we obtain the required assertion concerning the existence of a smooth (on $[T, +\infty)$ or on $(0, T^{-1}]$, $T = \varepsilon_0^{-1/\beta}$, depending on the sign of the semiquasihomogeneity) particular solution to the original system with the given leading term of the asymptotics. Theorem 1 is thereby proved.

§5. Analytic case

The above procedure does not provide any information about the convergence of the series (8). Moreover, if the right-hand sides of (10) are not analytic functions, the series (8) may well diverge. Let us study the convergence of the series (8) for the case in which the right-hand sides of (10) are complex analytic functions in a small neighborhood of the point $\mathbf{y} = \mathbf{x}_0$, $s = 0$.

We even pose a somewhat broader problem and try to clarify the "complex nature" of the solutions to (1) representable in the form (8). Obviously, the presence of logarithms in the expansions (8) is the main obstacle to solving this problem by using the abstract implicit function theorem. The point is that the infinite-sheeted Riemann surface of the logarithm is noncompact; therefore, it is impossible to construct a reasonable complete normed space of analytic functions that contains the vector function $t^{\mathbf{G}}\mathbf{x}(t)$. If, for some reason, the logarithms in expansion (8) were absent, we would readily show that $t^{\mathbf{G}}\mathbf{x}(t)$ is holomorphic on a "piece" of the Riemann surface of the function $s = t^{-\beta}$. The idea of the proof will be clarified below. However, as was mentioned above, for positive semiquasihomogeneous systems, the occurrence of logarithms in the expansion (8) is the generic case. Nevertheless, later on we shall see how to overcome this obstacle in a rather typical situation.

Theorem 2. *Let system* (1) *be positive semiquasihomogeneous, let the truncated system* (5) *be autonomous, let* $\beta^{-1} = q - 1$ *be a positive integer, and let all assumptions of Theorem* 1 *be satisfied with* $\gamma = 1$. *If the number* -1 *is a unique solution of the form* $\rho = -k\beta$, $k \in \mathbb{N}$, *to the characteristic equation* $\det(\mathbf{K} - \rho\mathbf{E}) = 0$, *then there exists a particular solution* $\mathbf{x}(t)$ *to system* (1) *with asymptotic expansion* (8) *such that the function* $t^{\mathbf{G}}\mathbf{x}(t)$ *is holomorphic on the covering of the domain* $|t| \ge T$, *for* $T > 0$ *sufficiently large, by the Riemann surface of the solution to the differential equation*

$$\dot{s} = -\beta\frac{s^q}{(1 + as^{q-1})} \tag{17}$$

with the "initial" condition $s(+\infty) = 0$, *where* a *is a real parameter.*

Proof. We apply the technical trick proposed in [9]. Let us make the change of the dependent variable $\mathbf{x} = t^{-\mathbf{G}}\mathbf{y}$ and the "uniformizing" change (17) of the time variable; after these changes, system (1) takes the form

$$-\beta s\mathbf{y}' = \mathbf{G}\mathbf{y} + (1 + as^{q-1})\sum_{m=0} s^m \mathbf{f}_{q+\chi m}(\mathbf{y}, \gamma). \tag{18}$$

For $a = 0$, this system is reduced to (10). The similarity of systems (10) and (18) is also related to the fact that the solution to (17) with the initial condition $s(+\infty) = 0$ has the asymptotics $s(t) \sim t^{-\beta}$.

Let us show that under the above assumptions, the parameter a can be chosen in such a way that the formal solution to system (18) contains no powers of logarithms. We seek a particular solution of (18) in the form of an ordinary Taylor series

$$\mathbf{y}(s) = \sum_{k=0}^{\infty} \mathbf{y}_k s^k. \tag{19}$$

Let us substitute (19) into (18) and match the coefficients of like powers of s. For the coefficient of s^0, we obtain

$$-\mathbf{G}\mathbf{y}_0 = \mathbf{f}_q(\mathbf{y}_0),$$

whence it follows that $\mathbf{y}_0 = \mathbf{x}_0$.

For the coefficients of the kth powers of s, $k < q - 1$, we have the equations

$$\mathbf{K}_k\mathbf{y}_k = \mathbf{\Phi}_k(\mathbf{y}_0, \dots, \mathbf{y}_{k-1}), \qquad \mathbf{K}_k = k\beta\mathbf{E} + \mathbf{K}, \tag{20}$$

where the $\mathbf{\Phi}_k$ depend polynomially on their arguments and are not yet dependent of the parameter a, which is to be defined later.

Since for $k \neq q-1$ the matrices $\mathbf{K}_k$ are nondegenerate, the coefficients $\mathbf{y}_k$ can be uniquely determined from (20) by the formulas

$$\mathbf{y}_k = \mathbf{K}_k^{-1}\mathbf{\Phi}_k(\mathbf{y}_0, \dots, \mathbf{y}_{k-1}).$$

For $k = q - 1$ we have

$$\mathbf{K}_{q-1}\mathbf{y}_{q-1} = a\mathbf{f}_q(\mathbf{y}_0) + \mathbf{\Phi}_{q-1}(\mathbf{y}_0, \dots, \mathbf{y}_{q-2}), \qquad \mathbf{K}_{q-1} = \mathbf{K} + \mathbf{E}. \tag{21}$$

Note that $\mathbf{f}_q(\mathbf{y}_0) = \mathbf{p}$, where $\mathbf{p}$ is an eigenvector of the Kowalewski matrix $\mathbf{K}$ with the eigenvalue $\rho = -1$.

Let us decompose $\mathbf{y}_{q-1}$ and $\mathbf{\Phi}_{q-1}$ into sums of components belonging to the eigenspace of the matrix $\mathbf{K}$ generated by the vector $\mathbf{p}$ and to its orthogonal complement, respectively:

$$\mathbf{y}_{q-1} = y_{q-1}\mathbf{p} + \mathbf{y}_{q-1}^{\perp}, \qquad \mathbf{\Phi}_{q-1} = \varphi_{q-1}\mathbf{p} + \mathbf{\Phi}_{q-1}^{\perp}.$$

Since the operator $\mathbf{K}_{q-1}$ is nondegenerate on the invariant subspace orthogonal to the vector $\mathbf{p}$, we have

$$\mathbf{y}_{q-1}^{\perp} = \mathbf{K}_{q-1}^{-1}\mathbf{\Phi}_{q-1}^{\perp}.$$

By setting $a = -\varphi_{q-1}$, we completely satisfy relation (21). The number y_{q-1} can now be chosen arbitrarily.

For $k > q - 1$, relations for $\mathbf{y}_k$ also have the form (20), where the $\mathbf{\Phi}_k$ in addition depend on the parameter a. As above, these equations can readily be solved for the $\mathbf{y}_k$, since the matrices $\mathbf{K}_k$ are nondegenerate.

Thus, we have shown that Eq. (18) has a particular formal solution that can be represented by the formal Taylor series (19). Let us now prove that the series (19) is the Taylor series of a function holomorphic in the disk $|s| < \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small.

The proof almost literally reproduces that in the preceding section. After the change $s = \varepsilon\xi$, $0 < \varepsilon \ll 1$, system (18) acquires the form

$$-\beta\xi\frac{d\mathbf{y}}{d\xi} = \mathbf{G}\mathbf{y} + (1 + a\varepsilon^{q-1}\xi^{q-1})\sum_{m=0} \varepsilon^m\xi^m \mathbf{f}_{q+\chi m}(\mathbf{y}, \gamma). \tag{22}$$

By $\mathbf{y}_K^\varepsilon$ denote the finite sum

$$\mathbf{y}_K^\varepsilon(\xi) = \sum_{k=0}^{K} \varepsilon^k \mathbf{y}_k \xi^k,$$

where $-\beta K < \operatorname{Re}\rho_i$, $i = 1, \dots, n$. We seek a particular solution to (22) in the form

$$\mathbf{y}(\xi) = \mathbf{y}_K^\varepsilon(\xi) + \mathbf{z}(\xi),$$

where $\mathbf{z}(\xi)$ is a holomorphic function in the disk $|\xi| < 1$ and has a zero of order $K+1$ at the point $\xi = 0$.

To prove the existence of such a solution, it suffices to apply the abstract implicit function theorem [8] to the equation

$$\mathcal{G}(\varepsilon, \mathbf{z}) = \mathbf{0}, \qquad \mathcal{G}\colon (0, \varepsilon_0) \times \mathfrak{E}_{1,K} \to \mathfrak{E}_{0,K},$$

in the corresponding Banach spaces, where

$$\mathcal{G}(\varepsilon, \mathbf{z}) = \beta\xi \frac{d}{d\xi}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + \mathbf{G}(\mathbf{y}_K^\varepsilon + \mathbf{z}) + (1 + a\varepsilon^{q-1}\xi^{q-1}) \sum_{m=0} \varepsilon^m \xi^m \mathbf{f}_{q+\chi m}(\mathbf{y}_K^\varepsilon + \mathbf{z}, \gamma).$$

Here $\mathfrak{E}_{1,K}$ is the Banach space of vector functions $\mathbf{z}\colon \mathcal{K}_1 \to \mathbb{C}^n$ that are holomorphic inside the unit disk $\mathcal{K}_1 = \{\xi \in \mathbb{C},\ |\xi| < 1\}$, continuous up to the boundary $\partial\mathcal{K}_1$ together with the first derivatives, take real values on the real axis ($\mathbf{z}(\bar{\xi}) = \bar{\mathbf{z}}(\xi)$), and have a zero of order $K+1$ at the center $\xi = 0$ of the disk; the norm on $\mathfrak{E}_{1,K}$ is defined by the expression

$$\|\mathbf{z}\|_{1,K} = \sup_{|\xi| \le 1} \xi^{-(K+1)} \big(\|\mathbf{z}(\xi)\| + |\xi| \|\mathbf{z}'(\xi)\|\big)$$

(where the prime denotes the derivative with respect to the variable ξ); furthermore, $\mathfrak{E}_{0,K}$ is the Banach space of vector functions $\mathbf{u}\colon \mathcal{K}_1 \to \mathbb{C}^n$ that are holomorphic inside the unit disc $\mathcal{K}_1$, continuous up to the boundary, take real values on the real axis ($\mathbf{z}(\bar{\xi}) = \bar{\mathbf{z}}(\xi)$), and have a zero of order $K+1$ at the center $\xi = 0$ of the disk. The norm of $\mathfrak{E}_{0,K}$ is defined by

$$\|\mathbf{u}\|_{1,K} = \sup_{|\xi| \le 1} \xi^{-(K+1)} \|\mathbf{u}(\xi)\|.$$

The assumptions of the implicit function theorem are verified directly.

Equation (17) has a particular solution $s(t)$, $s(+\infty) = 0$, given by the function inverse to

$$t(s) = s^{1-q} - a\beta^{-1} \ln s.$$

Let $\mathcal{R}$ be the Riemann surface of the functions s and t. Returning to the original variables, we see that system (1) has a particular solution $\mathbf{x}(t)$ such that the series (8) is the asymptotic expansion of this solution and for which the vector function $t^{\mathbf{G}}\mathbf{x}(t)$ is holomorphic on the covering of the domain $|t| > \varepsilon_0^{-1/\beta}$ by the Riemann surface $\mathcal{R}$.

In closing, we note that if, for some reason, the original expansion (8) does not contain logarithms, then we must set the parameter a in (17) equal to zero. The theorem is thus proved. □

§6. Conclusion

The problems of finding particular solutions to systems of ordinary differential equations with nonexponential asymptotics have been studied for a rather long time. Attention was mainly paid to the effective construction of truncated systems and to finding their particular solutions [1]. However, the problem of extending the obtained "quasihomogeneous ray" to a solution of the complete system has not been studied well enough until now. In particular, there are no general theorems concerning the complex structure of extendable solutions in the analytic case. Investigations of this kind were carried out only for specific model systems of mathematical physics [10, 11]. Obviously, the theorem proved in the preceding section does not solve the posed problem completely. However, the assumptions of this theorem are satisfied in a number of important problems. For example, Theorem 2 can be applied to prove the convergence of the asymptotic series constructed in the paper (12), where these series were applied to obtain a new case of inversion of the Lagrange–Dirichlet theorem concerning the stability of the equilibrium.

References

1. A. D. Bryuno, *A Local Method of Nonlinear Analysis for Differential Equations* [in Russian], Nauka, Moscow (1979).
2. A. P. Markeev, "Resonances and asymptotic trajectories in Hamiltonian systems," *Prikl. Matem. Mekh.* [*J. Appl. Math. Mech.*], **54**, No. 2, 207–212 (1990).
3. V. I. Gromak, "Solutions of the second Painlevé equation," *Differentsial'nye Uravneniya* [*Differential Equations*], **18**, No. 5, 753–763 (1982).
4. W. Wazow, *Asymptotic Expansions for Solutions of Ordinary Differential Equations*, Interscience Publishers [John Wiley & Sons, Inc.], New York–London–Sydney (1965).
5. H. Yoshida, "Necessary condition for the existence of algebraic first integrals. I, II," *Celestial Mech.*, **31**, 363–379, 381–399 (1983).
6. A. N. Kuznetsov, "On the existence of solutions entering a singular point of an autonomous system that has a formal solution," *Funktsional. Anal. i Prilozhen.* [*Functional Anal. Appl.*], **23**, No. 4, 63–74 (1989).
7. S. D. Furta, "Asymptotic solutions of systems of differential equations with deviating argument in some critical cases," *Mat. Sb. (N. S.)* [*Math. USSR-Sb.*], **184**, No. 2, 43–56 (1993).
8. L. V. Kantorovich and G. P. Akilov, *Functional Analysis* [in Russian], 3rd ed., Nauka, Moscow (1984).
9. S. D. Taliaferro, "Instability of an equilibrium in a potential field," *Arch. Rat. Mech. and Anal.*, **109**, No. 2, 183–194 (1990).
10. M. Tabor and J. Weiss, "Analytic structure of the Lorenz system," *Phys. Rev. Ser. A*, **24**, No. 4, 2157–2167 (1981).
11. J. D. Fournier, G. Levine, and M. Tabor, "Singularity clustering in the Duffing oscillator," *J. Phys. Ser. A. Math., Gen.*, **21**, 33–54 (1988).
12. V. V. Kozlov and V. P. Palamodov, "Asymptotic solutions of equations of classical mechanics," *Dokl. Akad. Nauk SSSR* [*Soviet Math. Dokl.*], **163**, No. 2, 285–289 (1982).

(V. V. Kozlov) Moscow State University

(S. D. Furta) Moscow Aviation Institute

Translated by A. I. Shtern