Symmetry Fields of Geodesic Flows

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A symmetry field of the dynamical system defined by a vector field X is a vector field commuting with X. We study symmetry fields for systems on 3-dimensional manifolds with a smooth invariant measure, and for geodesic flows on surfaces. It turns out that the existence of a symmetry field implies the existence of a multivalued integral. Geodesic flows with nonobvious symmetry fields form a very special class of integrable flows.

§1. SYSTEMS WITH INVARIANT MEASURE

Consider the dynamical system defined by a smooth vector field X on a connected 3-dimensional manifold Σ:

$$\dot{x} = X(x), \qquad x \in \Sigma. \tag{1.1}$$

We assume that $X \neq 0$ on Σ and that the phase flow of system (1.1) preserves the smooth measure defined by a nondegenerate volume 3-form ω on Σ, i.e., $L_X\omega = 0$, where L_X is the Lie derivative.

Example 1.1. Energy level. Let N be a 4-dimensional symplectic manifold (the phase space of a Hamiltonian system with two degrees of freedom), Ω a symplectic 2-form on N, and H a Hamilton function. The Hamiltonian vector field X_H on N is defined by the equation $i_{X_H}\Omega = \Omega(X_H, \cdot) = -dH$. For the Hamiltonian system, H is an integral and the 4-form $\Omega \wedge \Omega$ is invariant. Let $\Sigma = \Sigma_h = \{z \in N : H(z) = h\}$ be a regular level of H. Then the restriction X of the field X_H to Σ defines a dynamical system with an invariant 3-form ω defined by the relation $\Omega \wedge \Omega/2 = dH \wedge \omega$. We are most interested in the following:

Example 1.2. Geodesic flow. Let M be a 2-dimensional Riemannian manifold. The *geodesic flow* is a Hamiltonian system in $N = T^*M \setminus M$ endowed with the standard symplectic structure $\Omega = dy \wedge dx$ and with the Hamiltonian $H(x, y) = \|y\|^2/2$, where the norm of $y \in T^*_x M$ is defined by the Riemannian metric. We restrict the geodesic flow to $\Sigma = T^*_1 M$. The volume form ω on Σ equals $\lambda \wedge \Omega$, where $\lambda = y\,dx$.

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Recall that by the Maupertuis principle [1], a Hamiltonian system on T^*M with $H = \|y\|^2/2 + V(x)$ is reduced to a geodesic flow on an energy level Σ_h.

Example 1.3. Contact dynamical system. A generalization of a geodesic flow is a contact system on a contact manifold (Σ, λ). The contact field X satisfies the equations $\lambda(X) = 1$ and $i_X d\lambda = 0$ [1, 9]. Any contact vector field can be obtained as a Hamiltonian field on an energy level of a homogeneous Hamiltonian [1].

According to the classical theorem of Euler and Jacobi, if we know a nonconstant integral $f\colon \Sigma \to \mathbb{R}$ of system (1.1) with a smooth invariant measure, then Eqs. (1.1) can be integrated in quadratures on the set of regular points of f. The phase flow is described by the following theorem of A. N. Kolmogorov [12]:

Theorem 1.1. *Let c be a regular value of the integral f. Then every compact connected component Λ of the level surface $\{f = c\}$ is diffeomorphic to the torus $\mathbb{T}^2$, and in some coordinates $\psi_1, \psi_2 \bmod 2\pi$, and I in a neighborhood of Λ, Eqs.* (1.1) *take the form*

$$\dot{\psi}_1 = \omega_1(I)/F, \quad \dot{\psi}_2 = \omega_2(I)/F, \quad \dot{I} = 0, \tag{1.2}$$

where $F(\psi, I)$ is a smooth positive function. The volume form ω is given by

$$\omega = F\, dI \wedge d\psi_1 \wedge d\psi_2. \tag{1.3}$$

If we normalize F by the averaging condition $\langle F\rangle_\psi \equiv 1$, then ω_1 and ω_2 are the *frequencies* of the system. For a Hamiltonian system, $F \equiv 1$.

Definition 1.1. A closed 1-form ϕ on Σ is a *multi-valued integral* if $\phi(X) \equiv 0$.

Then $\phi = df$, where f is a local integral of system (1.1). Since the Euler–Jacobi theorem is local, it holds when system (1.1) has a nonzero multi-valued integral.

Remark. Sometimes a multi-valued integral is defined as an invariant closed 1-form ϕ such that $L_X\phi = 0$ [9]. By the homotopy formula, $L_X\phi = i_X d\phi + d i_X\phi = d i_X\phi = 0$. Hence $\phi(X) \equiv c$ and thus $\dot{f} = df(X) = c$ if $\phi = df$. In our definition of a multi-valued integral, $c = 0$. If ϕ is exact and Σ is compact, or, more generally, if there exists a Lagrange stable trajectory, then these two definitions are equivalent. Recall that a trajectory is called *Lagrange stable* [4] if it is defined for all $t \geqslant 0$ or $t \leqslant 0$ and the closure of the corresponding semitrajectory is compact.

Definition 1.2 (S. Lie). A vector field U on Σ is a *symmetry field* of system (1.1) if $[U, X] = 0$.

Equivalently, the phase flow $g_U\colon \Sigma \times \mathbb{R} \to \Sigma$ of U is a symmetry group: g_U takes trajectories of (1.1) to trajectories of the same system. Symmetry fields were studied in [15–17].

Example 1.4. Hamiltonian symmetry fields. For Hamiltonian systems, it makes sense to study symmetry fields both in the whole phase space N and on an energy level Σ. Let $U = X_F$ be a Hamiltonian symmetry field on N with Hamiltonian $F\colon N \to \mathbb{R}$. Then $[U, X] = -X_{\{F,H\}} = 0$, where $\{\cdot\,,\cdot\}$ is the Poisson bracket. Hence $\dot{F} = \{F, H\} \equiv c$. If there exists a Lagrange stable trajectory, then $c = 0$ and F is an integral. Thus $UH = 0$, so that U is tangent to any energy level Σ and defines a *Hamiltonian symmetry field* of system (1.1) on Σ. More generally, a closed invariant 1-form on N defines a locally Hamiltonian symmetry field.

We call a symmetry field $U = fX$ *trivial.* Then f is an integral of system (1.1). If U is nontrivial and analytic, then $U \neq fX$ on an open dense set in Σ. In general, for a noncompact Σ, the existence of a nontrivial symmetry field does not imply integrability of the system. For compact Σ, the situation is different.

Theorem 1.2. *Let Σ be compact and let system (1.1) have a nontrivial symmetry field U. If U is not measure-preserving, system (1.1) has a nonconstant integral. If U is measure-preserving, system (1.1) has a nonzero multi-valued integral. If $H^1(\Sigma, \mathbb{R}) = 0$, the integral is single-valued.*

Proof. The Lie derivative $L_U\omega$ is a smooth 3-form on Σ. Since $\omega \neq 0$, we have

$$L_U\omega = g\omega, \tag{1.4}$$

where g is a smooth function. Since $L_X\omega = 0$ and $[L_U, L_X] = 0$, we obtain

$$0 = L_U L_X \omega = L_X L_U \omega = L_X(g\omega) = \dot{g}\omega.$$

Thus g is an integral of system (1.1). If $g \not\equiv \text{const}$, the proof is complete.

Now suppose that $g \equiv c$. Then (1.4) and the homotopy formula yield

$$L_U\omega = i_U d\omega + d i_U \omega = d(i_U\omega) = c\omega.$$

Integrating over the compact manifold Σ and using the Stokes theorem, we obtain $c = 0$. Therefore, the form ω is U-invariant. Put

$$\phi = i_X i_U \omega. \tag{1.5}$$

The 1-form ϕ is closed. Indeed,

$$d\phi = -di_U(i_X\omega) = -L_U(i_X\omega) + i_U d(i_X\omega), \tag{1.6}$$

where

$$L_U(i_X\omega) = i_X L_U\omega - i_{[U,X]}\omega = 0 \quad \text{and} \quad d(i_X\omega) = L_X\omega - i_X d\omega = 0.$$

Since $\phi(X) = \omega(X, U, X) = 0$, by Definition 1.1, ϕ is a multi-valued integral of system (1.1). Since the fields U and X are not everywhere collinear, $\phi \not\equiv 0$. The proof is complete.

Remarks. 1. If U is a measure-preserving symmetry field, then the local integral f that corresponds to the multi-valued integral (1.5) is constant not only on the trajectories of system (1.1), but also on the orbits of the symmetry group g_U. This is an analog of the Bernoulli theorem in hydrodynamics, where the field U is the vorticity field. If $H^1(\Sigma, \mathbb{R}) = 0$, then Σ is foliated by 2-dimensional tori with quasiperiodic motion.

2.If Σ is an energy level of a Hamiltonian system and $U = X_F$ is a Hamiltonian symmetry field, then $\phi = -i_U\Omega = dF\big|_\Sigma$ is the multi-valued integral (1.5).

3. If X is a contact vector field on a contact manifold (Σ, λ), then $\phi = -i_U d\lambda$. By (1.6), Theorem 1.2 holds also for noncompact Σ if we define a multi-valued integral as a closed invariant 1-form and replace the form (1.5) by $\phi + c\lambda$.

Non-measure-preserving symmetry fields exist only for *degenerate* systems.

Theorem 1.3. *Suppose Σ is compact and the symmetry field U is not measure-preserving. Let C be the set of critical values of the integral g in (1.4). Then:*

(1) *the ratio of frequencies $\rho = \omega_1/\omega_2$ is constant on every connected component D of the set of regular points $\Sigma \setminus g^{-1}(C)$;*
(2) *if $\rho|_D$ is irrational, then ω_1 and ω_2 are constant on D.*

Theorem 1.3 holds also for noncompact Σ if the connected components of regular levels of g are compact. If Σ is an energy level of a Hamiltonian system, then the frequencies are constant for a rational ρ, too.

Proof of Theorem 1.3. Using Theorem 1.1, we introduce the coordinates $\psi_1, \psi_2 \bmod 2\pi$, and I in a neighborhood of a component Λ of a regular level of g. We write

$$Y = \omega_1 \frac{\partial}{\partial \psi_1} + \omega_2 \frac{\partial}{\partial \psi_2}, \quad Z = -\omega_2 \frac{\partial}{\partial \psi_1} + \omega_1 \frac{\partial}{\partial \psi_2}. \tag{1.7}$$

By (1.2), $X = fY$, where $f = F^{-1}$. Let

$$U = a\frac{\partial}{\partial I} + bY + cZ. \tag{1.8}$$

Then

$$[X, U] = fYa\frac{\partial}{\partial I} + (fYb - Uf - af\mu)Y + f(Yc - a\nu)Z \equiv 0, \tag{1.9}$$

where

$$\mu(I) = \frac{\omega_1\omega_1' + \omega_2\omega_2'}{\omega_1^2 + \omega_2^2}, \quad \nu(I) = \frac{\omega_1\omega_2' - \omega_2\omega_1'}{\omega_1^2 + \omega_2^2}.$$

Suppose that system (1.1) is nondegenerate: $\nu(I) \neq 0$ for I in some interval. Then $\rho(I)$ is irrational for almost all I. Expanding the equation $Ya \equiv 0$ as a Fourier series in $\psi \in \mathbb{T}^2$, we obtain that $a = a(I)$. Averaging the coefficient $Yc - a\nu$ in (1.9) over $\mathbb{T}^2$, we get $a \equiv 0$ and hence $c = c(I)$. Thus (1.8) yields $U = bY + cZ$. Since $Yc = 0$, we rewrite the coefficient of Y in (1.9) as

$$fYb - (bY + cZ)f = f^2(Y(Fb) + Z(Fc)) \equiv 0.$$

By (1.7), $\operatorname{div} Y = \operatorname{div} Z = 0$. Hence $\operatorname{div}(FU) = Y(Fb) + Z(Fc) = 0$. Thus U preserves the form (1.3) and $g = f \operatorname{div}(FU) \equiv 0$. This is a contradiction.

The proof of (2) is similar. If $\nu(I) \equiv 0$ and ρ is irrational, then $a = a(I)$ and $c = c(I)$. Equations (1.9) and $\langle F \rangle_\psi \equiv 1$ imply $a\mu \equiv 0$. If U is not measure-preserving, then $a \not\equiv \text{const}$, and, thus $\mu \equiv 0$. □

A symmetry field of a Hamiltonian system is said to be ***obvious*** if it is a sum of trivial and Hamiltonian symmetry fields. Theorems 1.2–1.3 do not describe the class of systems with nonobvious symmetry fields. In §§3–5, we describe symmetry fields of geodesic flows on surfaces. It turns out that measure-preserving symmetry fields are usually obvious. The existence of fields that are not measure-preserving imposes strong conditions on the metric. The main results are Theorems 4.1, 5.1, and 5.2. They can be generalized to contact dynamical systems. In §6, we study surfaces with nonobvious polynomial symmetry fields.

§2. REPRESENTATION OF SYMMETRY FIELDS OF GEODESIC FLOWS

Let M be a Riemannian manifold (possibly with boundary) with metric $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$. The *geodesic flow* is the Hamiltonian system $X = X_H$ on the phase space $N = T^*M \setminus M$ endowed with the symplectic structure $\Omega = dy \wedge dx$ and the Hamiltonian $H = \|y\|^2/2$. To use the standard notation of Riemannian geometry, we identify TM and T^*M by setting $v = H_y$. Then $N = TM \setminus M = \{(x, v) \in TM : v \neq 0\}$, $H = \|v\|^2/2$, and the equations of the geodesic flow take the form

$$\dot{x} = v, \quad \frac{Dv}{dt} = 0, \tag{2.1}$$

where D/dt is the covariant derivative.

Remark. The problem of symmetry fields for the geodesic flow on N is more general than for the geodesic flow on the energy level $\Sigma = T_1M \subset N$ (Example 1.2). Symmetry fields of $X|_\Sigma$ correspond to symmetry fields U of X such that $UH \equiv 0$. Sometimes, the geodesic flow is defined as the extension $\widetilde{X}$ of X to TM. Taylor series of smooth symmetry fields for $\widetilde{X}$ yield a sequence of homogeneous polynomial symmetry fields [15, 16]. Thus, if we admit the latter definition of the geodesic flow, then we study polynomial symmetry fields.

The Riemann connection yields the decomposition $T_{(x,v)}N = T_xM \oplus T_xM$ [11]. A vector $U \in T_{(x,v)}N$ is represented as $U = (u, u') \in T_xM \oplus T_xM$, where $u = d\pi(U)$ and $\pi\colon TM \to M$ is the projection. If $f\colon TM \to \mathbb{R}$ is a polynomial in velocity v, then

$$Uf = \nabla_u f + \langle f_v, u' \rangle, \tag{2.2}$$

where $\nabla_u f$ is the covariant derivative of the symmetric tensor field f, and $f_v \in T_xM$ is the gradient with respect to $v \in T_xM$. The standard 1-form $\lambda = y\,dx$ on $T^*M = TM$ takes the form $\lambda(U) = \langle v, u \rangle$, and the symplectic form Ω is given by

$$\Omega(U, W) = \langle u', w \rangle - \langle u, w' \rangle, \qquad U = (u, u'), \quad W = (w, w').$$

Any vector field $U = (u, u')$ on N is represented by the functions $u, u'\colon N \to TM$. For example, relations (2.1)–(2.2) imply $X = (v, 0)$.

Definition 2.1. Let $\tau\colon N \to N$ be the time-reversing involution $\tau(x, v) = (x, -v)$. We call a vector field U on N *even* (*odd*) if $d\tau(U) = \mp U \circ \tau$, respectively.

Equivalently, $u(x, -v) \equiv \mp u(x, v)$ and $u'(x, -v) \equiv \pm u'(x, v)$. If $F\colon N \to \mathbb{R}$ is even (odd), then the Hamiltonian field X_F is even (odd). Any symmetry field U of the geodesic flow is a sum of even and odd components: $U = U_+ + U_-$. Since X is even, $U_\pm$ are symmetry fields. Therefore, we may study even and odd symmetry fields separately.

Definition 2.2. Let $h_\lambda\colon N \to N$ be the homothety $h_\lambda(x, v) = (x, \lambda v)$. A field U on N is *homogeneous* of order $\deg U = n$ if $\lambda^n dh_\lambda(U) = U \circ h_\lambda$ for all $\lambda > 0$.

Equivalently, $u(x, \lambda v) \equiv \lambda^{n-1} u(x, v)$ and $u'(x, \lambda v) \equiv \lambda^n u(x, v)$. The field U is a *polynomial field* if u and u' are polynomials in v. Every polynomial symmetry field is the sum of homogeneous polynomial symmetry fields [16].

Lemma 2.1. *A field $U = (u, u')$ is a symmetry field of the geodesic flow on N if and only if*

$$\frac{D}{dt}u = u', \quad \frac{D}{dt}u' = -R(u, v)v, \tag{2.3}$$

where R is the curvature tensor of the Riemannian metric [11].

Indeed, (2.3) are the variational equations of the geodesic flow [11]. Hence, a symmetry field is defined by a solution $u\colon N \to TM$ of the Jacobi equation

$$\frac{D^2u}{dt^2} = -R(u, v)v. \tag{2.4}$$

Corollary 2.1. *Let $F = \lambda(U) = \langle u, v \rangle$. Then $\dot{F} = UH$ is an integral of the geodesic flow. Hence $\dot{F} = 0$ on Lagrange stable trajectories.*

This follows from (2.1)–(2.3): $\dot{F} = \langle u', v \rangle$ and $\ddot{F} = -\langle R(u, v)v, v \rangle \equiv 0$ [11].

Corollary 2.2. *If Lagrange stable trajectories of the geodesic flow are dense (for example, if M is compact), then $UH \equiv 0$ for any symmetry field U on N. Thus, $U|_\Sigma$ is a symmetry field of the geodesic flow on each energy level Σ.*

For analytic fields, it is sufficient that Lagrange stable trajectories form a uniqueness set for analytic functions on Σ. Since for dynamically nontrivial flows most of the trajectories are Lagrange stable, below we study symmetry fields on the fixed energy level $\Sigma = T_1 M$. Any symmetry field on Σ can be uniquely extended to a homogeneous symmetry field U on N of given order n such that $UH \equiv 0$ (in general, U is not polynomial). The *order* $n = \deg U$ of a polynomial field U on Σ is the lowest possible order of its homogeneous polynomial extension to N. Note that $\deg F \leqslant \deg U$.

Definition 2.3. A symmetry field $U = (u, u')$ is said to be *reduced* if $F = \langle u, v\rangle \equiv 0$.

For a Hamiltonian vector field $U = X_f$, we have $u = f_v$ and hence $F = \langle f_v, v\rangle$. If f is homogeneous of order n, then $F = nf$. If $U = fX$ is trivial, then $F = f$. Hence nonzero reduced fields are nontrivial and not Hamiltonian.

Lemma 2.2. *Every symmetry field U on Σ can be represented in the form*

$$U = FX + U_1 = X_F/n + U_2, \tag{2.5}$$

where U_1 and U_2 are reduced symmetry fields on Σ and X_F is the Hamiltonian field with homogeneous Hamiltonian of given order n coinciding with F on Σ.

The first representation of (2.5), which is standard for Jacobi fields, separates the trivial component of U, while the other representation separates the Hamiltonian component. If U is a polynomial field of order n, then U_2 is a polynomial field and $\deg U_2 \leqslant n$. For polynomial fields, the second representation is more convenient (since in general we have $\deg U_1 = n+2$). Putting aside trivial and Hamiltonian fields, from now on we study reduced symmetry fields.

Corollary 2.3. *Suppose that Lagrange stable trajectories are dense and $U = (u, u')$ is a nontrivial symmetry field on Σ such that the sectional curvature $K_{u,v}$ is nonpositive. Then u is parallel and $K_{u,v} \equiv 0$.*

Proof. By (2.3), we have

$$\frac{d^2}{dt^2}\left(\frac{\|u\|^2}{2}\right) = \frac{d}{dt}\langle u, u'\rangle = \langle u', u'\rangle + \langle R(u, v)v, u\rangle \geqslant 0, \tag{2.6}$$

where

$$K_{uv} = -\langle R(u, v)v, u\rangle/\|u\|^2 \leqslant 0. \tag{2.7}$$

If $\langle u, u'\rangle\big|_{t=0} = c > 0$ for some Lagrange stable trajectory, then (2.6) implies that $\|u\|^2/2 \geqslant ct$ for $t \geqslant 0$. Hence, on Lagrange stable trajectories, $\langle u, u'\rangle \equiv 0$. By (2.6), $u' \equiv 0$ and $K_{u,v} \equiv 0$. □

For compact manifolds of negative sectional curvature, there are no nontrivial symmetry fields. Of course, this follows from general properties of Anosov systems.

Corollary 2.4. *If $U = (u, u')$ is a polynomial symmetry field of first order on Σ, then U is a Hamiltonian field and u is a Killing field on M.*

By (2.5), it is sufficient to show that any reduced polynomial field $U = (u, u')$ of order one is zero. For such a field, $u = u(x)$ and $\langle u(x), v\rangle = 0$ for all $v \in T_x M$. Hence $u \equiv 0$. □

No assumptions on M are needed here. Corollaries 2.2 and 2.4 imply that polynomial symmetry fields of first order on N are Hamiltonian if Lagrange stable trajectories are dense (for example, if M is compact). This was established in [17] for $M = \mathbb{T}^2$. Since affine vector fields on M define symmetry fields of first order on N, we obtain the well-known fact that on compact Riemannian manifolds, affine vector fields are Killing fields [11].

§3. GEODESIC FLOWS ON SURFACES

Let M be a 2-dimensional oriented Riemannian manifold and let $J: T_xM \to T_xM$ be the counter-clockwise rotation through the angle $\pi/2$. For nonorientable M, we pass to the oriented covering $\widetilde{M} \to M$. If $U = (u, u')$ is a reduced symmetry field on Σ, we have

$$u = fJv, \quad u' = \dot{f}Jv, \tag{3.1}$$

where f is a function on Σ. Thus, the field $U = U_f$ is defined by $f \in E = C^\infty(\Sigma)$. If U is a polynomial field of order n, then u is a polynomial field of order $n-1$ and, thus, $f \in E_{n-2}$, where $E_{n-2} \subset E$ is the set of restrictions to Σ of polynomials of order $\leqslant n-2$.

The Gaussian curvature $K = K_{v,Jv}$ is defined by (2.7). Then (2.3) yields:

Lemma 3.1. *The field U_f is a reduced symmetry field on Σ if and only if*

$$\ddot{f} + Kf = (X^2 + K)f \equiv 0, \qquad f \in E. \tag{3.2}$$

Thus, the classification of reduced smooth symmetry fields (or polynomial symmetry fields of order n) on Σ is equivalent to the classification of solutions of the Jacobi Eq. (3.2) in E (or E_{n-2}, respectively). We introduce basis vector fields X, Y, Z on Σ:

$$X = (v, 0), \quad Y = (0, -Jv), \quad Z = (Jv, 0). \tag{3.3}$$

Note that Y and Z are not Hamiltonian, but measure-preserving. By (3.1),

$$U_f = fZ - \dot{f}Y. \tag{3.4}$$

Lemma 3.2.

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = KY. \tag{3.5}$$

Proof. It is sufficient to calculate the operators in (3.5) on the polynomials of first order in velocity. For example, we verify the latter equality for a linear function $l = \langle w(x), v\rangle$. Using (2.2) and (3.3), we obtain

$$Xl = \langle \nabla_v w, v\rangle, \quad Yl = \langle Jw, v\rangle, \quad Zl = \langle \nabla_{Jv} w, v\rangle. \tag{3.6}$$

Further,

$$[Z, X]l = \langle \nabla_{Jv}\nabla_v w, v\rangle - \langle \nabla_v \nabla_{Jv} w, v\rangle = \langle R(Jv, v)w, v\rangle.$$

Put $w = av + bJv$. The second equality of (3.6) implies $b = -Yl$. Thus,

$$[Z, X]l = b\langle R(Jv, v)Jv, v\rangle = -bK = KYl.$$

The proof of the remaining equalities in (3.5) is just the same. □

Remark. Let ξ, η, ζ be the basis 1-forms on Σ dual to the basis fields X, Y, Z. Then ξ and ζ are the standard 1-forms on the $SO(2)$-bundle $\Sigma = T_1M \to M$, and $-\eta$ is the connection form [11]. Thus, Eqs. (3.5) are equivalent to the fundamental equations of the Riemannian connection [11]:

$$d\xi = \zeta \wedge \eta, \quad d\zeta = \eta \wedge \xi, \quad d\eta = K\xi \wedge \eta. \tag{3.7}$$

Proposition 3.1. *If U_f is a reduced symmetry field on Σ, then $g = (YX - Z)f$ is an integral of the geodesic flow. It coincides with the integral g in (1.4).*

Proof. Since $K: M \to \mathbb{R}$, we have $YK = 0$. From (3.5) it follows that

$$Y(X^2 + K) = X(YX - Z), \qquad (X^2 + K)Y = (XY + Z)X. \tag{3.8}$$

By the first identity of (3.8), if $(X^2 + K)f = 0$, then $Xg = X(YX - Z)f = 0$. To prove the second statement, note that the 3-vector $\tilde{\omega} = X \wedge Y \wedge Z$ is dual to the volume form $\omega = \xi \wedge \eta \wedge \zeta$ on Σ. Using (3.4) and (3.5), we obtain $L_U\tilde{\omega} = (YX - Z)f\tilde{\omega}$. Thus, $g = (YX - Z)f$ is the integral in (1.4). □

The field Y on Σ is the generator of the group of rotations $(x, v) \to (x, e^{-J\theta}v)$, θ mod 2π. In other words, Y is the operator of differentiation with respect to the angle variable θ in T_xM. Every function $f \in E$ can be expressed by the Fourier series

$$f = \sum_{k\in\mathbb{Z}} f_k = f_0 + \sum_{k=1}^{\infty} f_{(k)}, \qquad f_{(k)} = f_k + \overline{f_k}, \tag{3.9}$$

where $Yf_k = ikf_k$ and $Y^2 f_{(k)} = -k^2 f_{(k)}$. Here $f_0 = \langle f\rangle$ is the angle average of f, and $f_{(k)}$ is a harmonic homogeneous polynomial of order k in velocity. We obtain:

Lemma 3.3. $\operatorname{Ker} Y = E_0 = \{f \in E : f = f_0\}$, *and any $f \in E$ is represented as*

$$f = Y\hat{f} + f_0, \qquad \langle\hat{f}\rangle = 0. \tag{3.10}$$

$\operatorname{Ker}(Y^2 + 1) = \{f \in E : f = f_{(1)}\}$, *and $Y^2 + 1$ is invertible on $\{f \in E : f_{(1)} = 0\}$.*

Lemma 3.4. *The integral g can be represented as*

$$g = (Y^2 + 1)X\hat{f} - 2Zf_0. \tag{3.11}$$

Proof. Since $Yf_0 = 0$, relations (3.5) and (3.10) yield

$$\begin{aligned} g &= (YX - Z)(Y\hat{f} + f_0) = Y([X,Y] + YX)\hat{f} - ZY\hat{f} + (-[X,Y] + XY)f_0 - Zf_0 \\ &= (Y^2 + 1)X\hat{f} - 2Zf_0. \quad \square \end{aligned}$$

Proposition 3.2. *The field U_f is a symmetry field if and only if g is an integral and*

$$\langle X^2 f_{(2)}\rangle + (\Delta/2 + K)f_0 = 0. \tag{3.12}$$

Here $\Delta = \operatorname{div}\operatorname{grad}$ is the Laplace–Beltrami operator, and the divergence of a field u is defined by $\operatorname{div} u = \operatorname{Tr}\nabla u$ [11].

Proof. Let $Xg = 0$. The first identity of (3.8) yields $Y(X^2 + K)f = 0$. By Lemma 3.3, $(X^2 + K)f \in E_0$. Thus, U_f is a symmetry field provided $\langle X^2 f\rangle + Kf_0 = 0$. Since, in the Fourier series (3.9) of $Xf_{(k)}$, only monomials of order $k \pm 1$ are nonzero, we have $\langle X^2 f\rangle = \langle X^2 f_{(2)} + X^2 f_0\rangle$. Here $X^2 f_0 = \langle\nabla_v \operatorname{grad} f_0, v\rangle$. Hence,

$$2\langle X^2 f_0\rangle = \operatorname{Tr}\nabla\operatorname{grad} f_0 = \Delta f_0. \quad \square$$

§4. ODD SYMMETRY FIELDS

Recall that each symmetry field is a sum of even and odd components. First we study odd fields. Let $E_\pm \subset E$ be the set of even (odd) functions on Σ.

Proposition 4.1. *A field U_f is an odd reduced symmetry field on Σ if and only if $g = (YX - Z)f$ is an even integral. If g is constant and the geodesic flow has a Lagrange stable trajectory, U is measure-preserving and $\hat{f}$ is an odd integral.*

Proof. If U_f is an odd reduced symmetry field, then $f \in E_-$. Thus, $f_0 = f_{(2)} \equiv 0$ and, therefore, Eq. (3.12) holds. By (3.11), $g = (Y^2+1)X\hat{f} \in E_+$. If $g \equiv c$, then Lemma 3.3 yields

$$X\hat{f} = (Y^2+1)^{-1}g = (Y^2+1)^{-1}c = c.$$

The existence of a Lagrange stable trajectory implies $c = 0$. □

Corollary 4.1. *Any odd reduced measure-preserving symmetry field U on Σ is the sum of trivial and Hamiltonian symmetry fields:*

$$U = U_{Yh} = Yh \cdot Z - Zh \cdot Y = nhX - X_h, \tag{4.1}$$

where $h \in E_-$ is an odd integral and X_h is the Hamiltonian field on Σ with homogeneous Hamiltonian of order n coinciding with h on Σ.

Proposition 4.1 yields the exact sequence

$$0 \longrightarrow I_-\{h\} \xrightarrow{Y} S_-\{f\} \xrightarrow{YX-Z} I_+\{g\},$$

where $I_\pm \subset E_\pm$ is the set of even (odd) integrals and $\{U_f : f \in S_\pm\}$ the set of even (odd) reduced symmetry fields. The subspace $G_- = (YX - Z)S_- \subset I_+$ represents non-measure-preserving odd symmetry fields. By Theorem 1.3, $G_- = 0$ for nondegenerate flows. In §5 we give an example where G_- is the set of all even integrals with zero average over Σ.

For polynomial fields of odd order n, we obtain the exact sequence

$$0 \longrightarrow I_{n-2}\{h\} \longrightarrow S_n\{f\} \longrightarrow I_{n-1}\{g\},$$

where $I_k = I \cap E_k$. For compact M without boundary, all the vector spaces are finite-dimensional, since the symbol of the differential operator $X : E_n \longrightarrow E_{n+1}$ on polynomials on TM is elliptic [18]: $\sigma(X)(\xi)f = \langle \xi, v \rangle f$ for $\xi \in T^*M$.

Theorem 4.1. *Suppose that the geodesic flow has a Lagrange stable trajectory. Any odd symmetry field on Σ is a sum of the following symmetry fields:*

(1) *a Hamiltonian field X_F, where F is an odd integral;*
(2) *a measure-preserving reduced field U_{Yh}, where h is an odd integral;*
(3) *a reduced field $U_{Yh'}$ that is not measure-preserving, where $g = (Y^2+1)\dot{h}'$ is an even integral.*

By (4.1), for odd reduced measure-preserving fields, $\phi = dh$ is the multi-valued integral of the form (1.5). Hence, if a nontrivial odd symmetry field on Σ exists, then there is a nonconstant single-valued integral (independent of the topology on M; compare with Theorem 1.2). Thus, the geodesic flow is Liouville integrable on regular levels of the integral. In general, the representation in Theorem 4.1 is not unique, since the sum of fields of types (2) and (3) is a field of type (3). Theorem 1.3 yields:

Corollary 4.2. *If the conditions of the Liouville-Arnold theorem hold (for example, if M is a compact manifold without boundary) and the flow is nondegenerate on a dense set, then any odd symmetry field on Σ is uniquely represented as a sum of fields* (1) *and* (2).

By (4.1), such a field is obvious. If the Liouville-Arnold theorem holds but the system is degenerate, the representation is unique if $E_- = I_- \oplus I_-^\perp$: any function in E_- is a sum $h + h'$, where h is an odd integral and the time average $\langle h' \rangle_t$ is zero almost everywhere. Equivalently, $\int h' f \omega = 0$ for any odd integral f with compact support.

Corollary 4.3. *Theorem* 4.1 *holds for nonorientable M.*

Proof. We pass to a two-sheet oriented covering $\widetilde{M} \to M$. Let $\sigma\colon \widetilde{M} \to \widetilde{M}$ be the orientation-reversing involution such that $M = \widetilde{M}/\sigma$. Then $\sigma' \circ J = -J \circ \sigma'$, where $\sigma'\colon T\widetilde{M} \to T\widetilde{M}$ is the derivative of σ. Hence, a reduced symmetry field U on Σ is defined by a function f on $T_1\widetilde{M}$ such that $f \circ \sigma' = -f$ and (3.2) holds. Since $d\sigma'(Y) = -Y \circ \sigma'$ and $d\sigma'(Z) = -Z \circ \sigma'$, we have $\hat{f} \circ \sigma' = \hat{f}$ and $g \circ \sigma' = g$. Thus, $h = \hat{f}$ and g are well defined functions on Σ. □

Corollary 4.4. *If U is a polynomial symmetry field of order n, then $F \in I_n$, $g \in I_{n-1}$ and $h \in I_{n-2}$. The representation of Theorem* 4.1 *is unique if M is a compact manifold without boundary and $h' \perp I_{n-2}$.*

Indeed, $E_{n-2} = I_{n-2} \oplus I_{n-2}^\perp$ since $X : E_n \to E_{n+1}$ is elliptic.

For example, let m be the lowest possible order of polynomial integrals of the geodesic flow and suppose that the flow is nondegenerate. Then odd polynomial symmetry fields on Σ exist only for odd m, and they are sums of Hamiltonian fields of order km and measure-preserving reduced fields of order $km + 2$, where k is odd.

§5. EVEN SYMMETRY FIELDS

Let dS be the area 2-form on M defined by $dS(v, Jv) = 1$ for $\|v\| = 1$.

Proposition 5.1. *Let U_f be an even reduced symmetry field such that $g = \text{const}$. Then $g \equiv 0$, $f_0 \equiv c$, and the function $X\hat{f} = l = \langle w(x), v \rangle$ is linear in velocity. If we regard l as a differential 1-form on M, then*

$$dl = cKdS \quad \Longleftrightarrow \quad \operatorname{div}(Jw) = -cK. \tag{5.1}$$

Proof. We have $f, \hat{f} \in E_+$ and $g \in E_-$. Hence $g \equiv 0$. By (3.11),

$$(Y^2 + 1)X\hat{f} = 2Zf_0 = 2\langle \operatorname{grad} f_0, Jv \rangle \in \operatorname{Ker}(Y^2 + 1).$$

By Lemma 3.3, $X\hat{f} = l = \langle w(x), v \rangle$ and $Zf_0 = 0$. Thus $f_0 \equiv c$. Then (3.2) yields

$$(X^2 + K)f = (X^2 + K)Y\hat{f} + Kc = 0.$$

Using the second identity (3.8), we obtain

$$(XY + Z)l + Kc = 0. \tag{5.2}$$

Using (3.7) and the expression for dl in terms of ∇w [11], we get

$$(XY + Z)l = -\langle \nabla_v w, Jv \rangle + \langle \nabla_{Jv} w, v \rangle = -dl(v, Jv). \tag{5.3}$$

Now (5.1) follows from (5.2) and (5.3). □

Corollary 5.1. *Any even reduced measure-preserving symmetry field U on Σ can be represented as*

$$U = U_{Yh+c} = nhX - X_h + cZ - (0, w), \tag{5.4}$$

where $h \in E_+$, $h_0 = 0$, $\dot{h} = \langle w, v\rangle$, the vector field w on M satisfies (5.2), *and X_h is the Hamiltonian field on Σ with homogeneous Hamiltonian of order n coinciding with h on Σ.* □

The 1-form

$$\phi = dh - \pi^* l + c\eta \tag{5.5}$$

is an even multi-valued integral and $Yh + c = \phi(Y)$. Indeed, (3.7) yields $d\eta = K\pi^*\, dS$. By (5.1) and (5.5), ϕ is closed and $\phi(X) = 0$. For reduced measure-preserving fields, ϕ is the multi-valued integral (1.5).

Theorem 5.1. *Every even symmetry field on Σ is a sum of:*

(1) *a Hamiltonian field X_F, where F is an even integral;*
(2) *a measure-preserving reduced field U_{Yh+c} in* (5.4)*;*
(3) *a reduced field U_f that is not measure-preserving, where $g = (YX - Z)f$ is an odd integral and* (3.12) *holds.*

In particular, we have an exact sequence

$$0 \longrightarrow \tilde{I}_+\{\phi\} \xrightarrow{Y} S_+\{f\} \xrightarrow{YX-Z} I_-\{g\},$$

where $\tilde{I}_+$ is the set of multi-valued even integrals. The representation in Theorem 5.1 is unique if there are no odd single-valued integrals. If such an integral exists, then the Liouville-Arnold theorem holds, and the flow is nondegenerate on a dense set, moreover, the representation is unique, since by Theorem 1.3, all symmetry fields are measure-preserving. If U is polynomial of order n, then $F \in I_n$, $g \in I_{n-1}$, and $\phi \in \tilde{I}_{n-2}$.

For example, let m be the lowest possible order of polynomial integrals of the geodesic flow. For simplicity, let us assume that if m is even, the integral is single-valued. If the flow is nondegenerate, every polynomial symmetry field on Σ is a sum of Hamiltonian fields of order km and measure-preserving reduced fields of order $km + 2$, where k is an integer.

Proposition 5.2. *Let M be a compact manifold with boundary Γ, consisting of simple closed geodesics (Γ may be empty) and U_f an even reduced measure-preserving symmetry field. If $\chi(M) \neq 0$, then the multi-valued integral* (5.5) *is exact: there is a nonconstant single-valued integral h on Σ such that $f = Yh$ and $\phi = dh$.*

Proof. Integrating (5.1) over M and using the Gauss-Bonnet formula, we obtain

$$0 = \oint_\Gamma X\hat{f} = \oint_\Gamma l = c \iint_M K\, dS = c2\pi\chi(M),$$

since the geodesic curvature of Γ vanishes. Thus $c = 0$ and $dl = 0$ by (5.1). Let us prove that the cohomology class $[l] \in H^1(M, \mathbb{R})$ is zero. Indeed, every homotopy class in M can be represented by a closed geodesic γ, and on γ we have $l = X\hat{f}$. Hence, $\oint_\gamma l = 0$. Thus, there is a function h_0 on M such that $l = -dh_0$. We can put $h = \hat{f} + h_0$. □

For example, we may apply Proposition 5.2 if there is a contractible closed geodesic without self-intersections. Then $c = 0$ for any topology of M.

Corollary 5.2. *Let M be a compact manifold with geodesically convex boundary ∂M and $\chi(M) < 0$. If the metric is analytic, all analytic symmetry fields on Σ are trivial. For a smooth metric, polynomial fields with C^1 coefficients are trivial.*

Indeed, this case is reduced to that of Proposition 5.2 [7]. Then Corollary 5.2 follows from Theorems 4.1, 5.1, Proposition 5.2, and the results on topological obstacles to the existence of analytic and polynomial integrals [5, 13, 14]. In fact, there exist transversal separatrices [7]. Another proof is based on the positivity of the topological entropy of the geodesic flow [10]. Corollary 5.2 was proved in [16] for polynomial symmetry fields under the assumption that M is a compact manifold without boundary. In this case, polynomial symmetry fields are analytic if the metric is analytic. Since $X : E_n \to E_{n+1}$ is elliptic [18], this can be deduced from Proposition 3.2.

Note that under the assumptions of Corollary 5.2, there may exist smooth nonconstant integrals and nontrivial symmetry fields.

Corollary 5.3. *For nonorientable M, Theorem* 5.1 *holds with $c = 0$. If M is a compact manifold with geodesically convex boundary, then the integral* (5.5) *is exact.*

Proof. Since $f_0 \circ \sigma = -f_0$, we have $c = 0$ and $\underset{\sim}{dl} = 0$ on $\widetilde{M}$. As in the proof of Proposition 5.2, using the fact that any homotopy class in $\widetilde{M}$ is represented by a closed geodesic, we see that l is exact. □

Finally, we collect some results for compact manifolds M without boundary.

Theorem 5.2. *Any smooth symmetry field on Σ is a sum of:*

(1) *a Hamiltonian symmetry field X_F;*
(2) *a measure-preserving reduced symmetry field U_{Yh}, where h is an integral except for $M = \mathbb{T}^2$, when it may be necessary to replace h by the multi-valued integral* (5.5) *of the cohomology class $[\phi] = c[d\theta] \in H^1(\Sigma, \mathbb{R})$, where θ is the angle in the tangent plane;*
(3) *a non-measure-preserving field U_f, where $g = (YX - Z)f$ is an integral,* (3.12) *holds, and the frequencies are constant on connected components of the set of regular points of g.* □

Possibly except for $M = \mathbb{T}^2$, the geodesic flow with a nontrivial symmetry field has a nonconstant single-valued integral (compare with Theorem 1.2). There is an exact sequence

$$0 \longrightarrow I \xrightarrow{Y} S \xrightarrow{YX-Z} I, \tag{5.6}$$

where I is the set of integrals with zero average over Σ, and S the set of reduced symmetry fields. We give an example for which $G = (YX - Z)S = I$, so that sequence (5.6) is exact in the last term.

Example 5.1. C-manifolds. Suppose that all geodesics in M are closed and have the same length [3]. Then M is diffeomorphic to the sphere S^2. Let us show that $G_\pm = I_\mp$ and $S_\pm \cong I_+ \oplus I_-$, so that $S \cong I \oplus I$.

Indeed, let $\Gamma = \Sigma/S^1$ be the set of oriented geodesics. Then Γ is diffeomorphic to S^2 and has a natural Riemannian metric [3] and a complex structure J. The area form on Γ defines a reduced symplectic structure [1]. The antipodal involution $\sigma \colon \Gamma \to \Gamma$ (corresponding to time reversion) is an isometry. The projection $\pi \colon \Sigma \to \Gamma$ maps any reduced symmetry field U on Σ to a vector field ξ on Γ, and U is even (odd) if and only if $d\sigma(\xi) = \mp\xi \circ \sigma$. The integrals g and h on Σ are projected into smooth functions ϕ and ψ on Γ such that

$$\iint_\Gamma \phi \, dS = \iint_\Gamma \psi \, dS = 0.$$

If U is even (odd), then $\phi \circ \sigma = \mp\phi$ and $\psi \circ \sigma = \pm\psi$. Definition (1.4) of g yields $\operatorname{div}\xi = \phi$. The representation of the reduced field U as the sum of fields (2) and (3) is equivalent to the representation $\xi = J\operatorname{grad}\psi + \eta$, where $\operatorname{div}\eta = \phi$, and $J\operatorname{grad}\psi$ is a Hamiltonian field on Γ. The representation is unique if we take $\eta = \operatorname{grad}\chi$, where χ is the unique solution of the equation $\Delta\chi = \phi$ with zero average. Thus, in this example, I is the set of smooth functions on Γ with zero average, and the decomposition of the set of symmetry fields of the geodesic flow in Theorem 5.1 takes the form $I \oplus I \oplus I$.

Let M be compact and let the metric and a nontrivial symmetry field be analytic. By Corollary 5.2, $M = S^2$, $\mathbb{R}P^2$, $\mathbb{T}^2$, or a Klein bottle $\mathbb{K}^2$.

Proposition 5.3. *For $M = \mathbb{T}^2$ or $\mathbb{K}^2$, analytic symmetry fields are measure-preserving.*

Thus, analytic symmetry fields for $\mathbb{K}^2$ and odd analytic fields for $\mathbb{T}^2$ are *obvious* (sums of trivial and Hamiltonian fields).

Conjecture 5.1. *For a smooth metric, Proposition 5.3 holds for C^1 polynomial fields.*

If the metric is analytic, Conjecture 5.1 is a consequence of Proposition 5.3. Proposition 5.3 follows from Theorem 1.3 and:

Lemma 5.1. *A geodesic flow on $\mathbb{T}^2$ with a nonconstant analytic integral cannot be everywhere degenerate.*

Proof. We use a method of [14]. Let C be the set of critical values of g. Then C is finite and $\Lambda = \Sigma \setminus g^{-1}(C)$ consists of a finite number of connected components. If the system is degenerate everywhere on Λ, then Λ contains closed geodesics from at most a finite number of homotopy classes Γ_i. On geodesics from other homotopy classes, g takes values in C. Choose a circle $S = \Sigma \cap T_xM$. For any homotopy class $\Gamma \in \mathbb{Z}^2$ of closed curves in $M = \mathbb{T}^2$, there is a trajectory of the geodesic flow starting with initial velocity $v_\Gamma \in S$ and asymptotically close to a closed geodesic from the class Γ [8]. Hence, g takes values from a finite set C on an infinite number of points $v_\Gamma \in S$, where $\Gamma \neq \Gamma_i$. Since g is analytic, $g|_S \equiv \text{const}$. □

Conjecture 5.2. *If $M = \mathbb{T}^2$ and $K \not\equiv 0$, every analytic (for an analytic metric) or polynomial multi-valued integral of the geodesic flow is exact (single-valued).*

Recall that odd integrals are always single valued. By Proposition 5.3, Conjecture 5.2 implies that all analytic and polynomial symmetry fields on a nonflat torus are obvious. On a flat torus, Z is a nonobvious symmetry field [17]. In §6, we study nonobvious polynomial symmetry fields. They are either non-measure-preserving of order $n \geqslant 2$, or measure-preserving of even order $n \geqslant 4$ and correspond to multi-valued integrals. In particular, we prove Conjectures 5.1 and 5.2 for polynomial fields of order $n < 11$ and $n < 6$, respectively.

§6. NONOBVIOUS POLYNOMIAL SYMMETRY FIELDS

Representation in conformal coordinates

For local calculations, it is convenient to introduce a local conformal coordinate $z \in \mathbf{C}$ on M such that the Riemannian metric is $\lambda\, dz d\bar{z}$. Setting $p = \lambda\dot{\bar{z}}$, we obtain complex canonical coordinates z, $\bar{z}$, p, $\bar{p}$ on N. Then $\Sigma = \{p\bar{p} = \lambda\}$. The fields (3.3) on Σ can be represented in the form

$$X = \xi + \bar{\xi}, \quad Y = i(\eta - \bar{\eta}), \quad Z = i(\xi - \bar{\xi}), \tag{6.1}$$

where ξ and η are complex vector fields:

$$\xi = p^{-1}\partial + \partial \ln \lambda \frac{\partial}{\partial p}, \quad \eta = p\frac{\partial}{\partial p}.$$

For simplicity, we write $\partial = \partial/\partial z$. The decomposition (6.1) is standard for vector fields on complex manifolds [11]. The Fourier series (3.9) of a function f on Σ takes the form

$$f = \sum_{k\geqslant 1} f_k p^k + \sum_{k\geqslant 1} \overline{f_k p^k} + f_0, \qquad p\overline{p} = \lambda. \tag{6.2}$$

By (6.1)–(6.2), the Fourier coefficients of $\dot{f} = Xf$ have the following form [6]:

$$(\dot{f})_k = \overline{\partial} f_{k-1}/\lambda + \partial f_{k+1} + (k+1) f_{k+1}\partial \ln\lambda, \quad (\dot{f})_0 = 2\,\mathrm{Re}(\partial f_1 + f_1 \partial \ln \lambda). \tag{6.3}$$

Symmetry fields of order 2

The main problem in the classification of polynomial symmetry fields is the absence of the classification of integrals of order greater than 2. Thus, a complete description is possible only for measure-preserving reduced fields of order $n \leqslant 4$, and non-measure-preserving fields of order $n \leqslant 3$. We already know that all symmetry fields of first order are Hamiltonian.

Since there are no nonconstant integrals (and no nonzero multi-valued integrals) of order 0, from Corollary 5.1 it follows that any measure-preserving reduced symmetry field of order 2 equals cZ; such a field exists only for $K \equiv 0$. It turns out that non-measure-preserving fields of order 2 exist only if the curvature is constant.

Proposition 6.1. *If there is a non-Hamiltonian symmetry field of order* 2 *on* Σ, *then* $K \equiv \text{const}$.

The result is local: no assumptions on M are needed. Corollary 2.2 implies that Proposition 6.1 holds for symmetry fields on N if Lagrange stable trajectories are dense (or, in the analytic case, form a uniqueness set). For $M = \mathbb{T}^2$, Proposition 6.1 was proved in [17], and for $M = S^2$ by P. Anikeev.

Proof. Let U_f be a reduced symmetry field of order 2. Since $\deg f = 0$, we have $f = f_0$. By Proposition 3.2, $g = -2Zf$ is a nonconstant linear integral and

$$\Delta f + 2Kf = 0. \tag{6.4}$$

The representation (6.2) yields $g = g_1 p + \overline{g_1 p}$. Since g is an integral, g_1 is holomorphic [4]. Following Birkhoff [4], in a neighborhood of any point of M where $g_1 \neq 0$ we choose a local conformal coordinate $z \in \mathbf{C}$ such that $g_1 \equiv 1$. Then the conformal multiplier λ depends only on $y = \mathrm{Im}\, z$ [4]. Using (6.1), we obtain

$$g = -2Zf = 2i(\overline{\partial} f p - \partial f \overline{p})/\lambda = p + \overline{p}.$$

Hence $\overline{\partial} f = -i\lambda/2$. Thus, $f = f(y)$ and $\lambda = -f'$. Since

$$K = -(\ln\lambda)''/(2\lambda) \quad \text{and} \quad \Delta f = f''/\lambda, \tag{6.5}$$

(6.4) yields $f'' = (\ln\lambda)'' f$. Since $f' = -\lambda$, the variables can be separated:

$$f'/f = (\ln\lambda)''/(\ln\lambda)'.$$

Thus, $(\ln\lambda)' = cf$ and $(\ln\lambda)'' = -c\lambda$. By (6.5), $K = c/2 = \text{const}$. □

Symmetry fields for constant K

We describe reduced symmetry fields of order 2 for the standard sphere ($K = 1$) and for the Lobachevsky plane ($K = -1$). Both manifolds can be realized as surfaces in $\mathbb{R}^3$:

$$M = \{\mathbf{r} \in \mathbb{R}^3 : \langle A\mathbf{r}, \mathbf{r}\rangle = \pm 1\}, \qquad A = \operatorname{diag}(1, 1, \pm 1),$$

with the metric $\|\dot{\mathbf{r}}\|^2 = \langle A\dot{\mathbf{r}}, \dot{\mathbf{r}}\rangle$ for $\langle A\mathbf{r}, \dot{\mathbf{r}}\rangle = 0$. Equations (2.1) of the geodesic flow on Σ take the form $\ddot{\mathbf{r}} \pm \mathbf{r} = 0$, and (3.2) becomes $\ddot{f} \pm f = 0$, where f is a function on M. Any homogeneous linear function f on $\mathbb{R}^3$ satisfies this equation, and, therefore, U_f is a reduced symmetry field of order 2. There are no other complete nonflat surfaces with non-Hamiltonian symmetry fields of order 2.

Now let $K \equiv 0$. If M is the plane $\mathbb{R}^2$, then (3.2) becomes $\ddot{f} = 0$, and the solutions are linear functions f on $\mathbb{R}^2$. For $M = \mathbb{T}^2$, $f \equiv c$, and so any reduced symmetry field cZ is measure-preserving. For $M = \mathbb{K}^2$, there are no non-Hamiltonian symmetry fields of order 2.

Thus, for a compact M, the existence of a non-measure-preserving symmetry field of order 2 implies that M is the standard sphere. We conjecture that if the metric is smooth, this holds for polynomial fields of any order. For analytic fields, this is not the case (Example 5.1).

Symmetry fields of order 3

All measure-preserving symmetry fields of order 3 are obvious. Geodesic flows possessing non-measure-preserving symmetry fields of order 3 have a quadratic integral. Hence by [4], the corresponding metrics form a special class of Liouville metrics (in particular cases, metrics of revolution). If M is compact, then $M = S^2$ or $\mathbb{R}P^2$ by Proposition 5.1 for an analytic metric, and by Conjecture 5.1 for a smooth metric (we prove this at the end of this section). All Liouville metrics on S^2 are classified in [13], and so it is possible to describe non-measure-preserving symmetry fields of order 3.

Symmetry fields of order 4

Since the classification of integrals of order 3 is unknown, we give a local description only for measure-preserving reduced fields U_f of order 4, or, equivalently, for fields with $g \equiv \text{const}$.

Proposition 6.2. *If there is a measure-preserving reduced symmetry field of order 4, then in a neighborhood of every point of M, except for a discrete set of points, there exists a local conformal coordinate $z = x + iy$ such that λ satisfies the equation*

$$2\lambda_{xy} = c\Delta \ln \lambda, \tag{6.6}$$

where Δ is the Euclidean Laplace operator.

Proof. We analyze the conditions of Corollary 5.1. Since $\deg U = 4$, we have $\deg h = 2$ in (5.4). Hence, $h = h_2 p^2 + \overline{h_2 p^2}$. By (6.3), the equation $\dot{h} = l$ implies that h_2 is holomorphic. In a neighborhood of any point of M where $h_2 \neq 0$, there is a local conformal coordinate z such that $h_2 \equiv 1/4$. Then (6.3) implies

$$l = \dot{h} = (p\partial \ln \lambda + \overline{p\partial} \ln \lambda)/2 = (\partial\lambda\dot{\overline{z}} + \overline{\partial}\lambda\dot{z})/2 = \operatorname{Re}(\overline{\partial}\lambda\, \dot{z}).$$

For the corresponding 1-form $l = \operatorname{Re}(\overline{\partial}\lambda\, dz)$, we get $dl = \operatorname{Re}(\partial^2\lambda\, dz \wedge d\overline{z})$. Since $K\, dS = i\partial\overline{\partial} \ln \lambda\, dz \wedge d\overline{z}$, the equation $dl = cK\, dS$ implies $\operatorname{Im} \partial^2\lambda = c\partial\overline{\partial} \ln \lambda$. Since $\lambda_{xy} = 2 \operatorname{Im} \partial^2 \lambda$ and $\Delta = 4\partial\overline{\partial}$, the proof is complete. □

For $c = 0$, Eq. (6.6) means that the metric is locally Liouville. The proof of Proposition 6.2 implies that if λ satisfies (6.6), then locally in M there exists a multi-valued quadratic integral of the cohomology class $c[d\theta]$. Obviously, (6.6) holds for constant λ. By the Cauchy–Kovalevskaya theorem, Eq. (6.6) has many local analytic solutions for any c. If $c \neq 0$ and $\lambda \not\equiv$ const, there are no single-valued quadratic integrals of the geodesic flow.

Now consider the global problem. We know that $c = 0$ if $M \neq \mathbb{T}^2$ is compact.

Proposition 6.3. *If M is a nonflat torus, then $c = 0$. Thus all multi-valued integrals of order 2 are exact, and measure-preserving non-Hamiltonian symmetry fields of order 4 exist only for globally Liouville metrics on $\mathbb{T}^2$.*

There exists a global conformal coordinate z on M such that Eq. (6.6) holds on $\mathbf{C}$ and λ is a doubly periodic function. Proposition 6.3 follows from:

Lemma 6.1. *For $c \neq 0$, all doubly periodic solutions of (6.6) are constant.*

Sketch of the proof. Equation (6.6) is elliptic for $0 < \lambda < |c|$, and hyperbolic for $\lambda > |c|$. If for given doubly periodic $\lambda > 0$ the elliptic region is nonempty, Lemma 6.1 follows from the Hopf maximum principle. In the hyperbolic region, Lemma 6.1 can be proved by the Lax method based on the strong nonlinearity of Eq. (6.6). This method was used in [3] for a different equation in order to prove the nonexistence of integrals of order 3 for a forced oscillation problem with one degree of freedom. We omit the details since the analysis in our case is similar. □

Proposition 6.3 implies Corollary 5.2 for $n < 6$. It seems that the same method may work for $n \geqslant 6$, but the calculations become complicated. It is also probable that single-valued polynomial integrals of a geodesic flow on $\mathbb{T}^2$ exist only for Liouville metrics, but this is not proved even for integrals of order 3, except for in the following particular case.

Trigonometric metrics on $\mathbb{T}^2$

Let $M = \mathbb{T}^2$ and the doubly periodic function λ be a trigonometric polynomial. In [17] it was proved that if there is a nonconstant polynomial integral of the geodesic flow, the metric is of the Liouville type. A similar method yields this result for multi-valued polynomial integrals. All polynomial analytic symmetry fields are analytic and, hence, measure-preserving by Proposition 5.3. Thus, for a nonflat trigonometric metric on $\mathbb{T}^2$, polynomial symmetry fields are obvious and exist only for Liouville metrics. For analytic fields, this question remains open.

Non-measure-preserving fields on $\mathbb{T}^2$

Let U_f be a reduced non-measure-preserving symmetry field of order n of the geodesic flow on a torus. Then $h = \hat{f}$ has order $n-2$ and $g = (Y^2+1)Xh - 2Zf_0$ is a *nonconstant* integral of order $\leqslant n-1$. For polynomial fields of order $n \leqslant 10$, Conjecture 5.1 follows from the next assertion.

Lemma 6.2. *If g is a nonconstant integral, then $2 \leqslant m = \deg g \leqslant n-9$.*

We can assume that U is either even or odd. Let us represent h and g in the form (6.2) with coefficients h_k, where $k = n-2, n-4, \dots$, and g_k, where $k = n-1, n-3, \dots$ The first nonzero coefficient g_m is a holomorphic function on $\mathbb{T}^2$ and thus $g_m = \text{const} \neq 0$. Now Lemma 6.2 follows from:

Lemma 6.3. *For $m \geqslant n-7$ or $m = 1$, we have*

$$\iint_M \lambda g_m \, dz \wedge d\bar{z} = 0. \tag{6.7}$$

Sketch of the proof. Using (6.3), we obtain $\lambda g_1 = 2i\bar{\partial} f_0$ and

$$\lambda g_k = (1 - k^2)\big(\bar{\partial} h_{k-1} + \lambda \partial h_{k+1} + (k+1) h_{k+1} \partial \lambda\big) \tag{6.8}$$

for $k > 1$. Thus (6.7) holds for $m = 1$. Now let $m > 1$. For $k = n - 1$, we obtain

$$\lambda g_{n-1} = (1 - (n-1)^2)\bar{\partial} h_{n-2},$$

which yields (6.7) for $m = n - 1$. If $g_{n-1} \equiv 0$, then $h_{n-2} = \text{const}$. By (6.8),

$$\lambda g_{n-3} = (1 - (n-3)^2)\big(\bar{\partial} h_{n-4} + (n-2) h_{n-2} \partial \lambda\big), \tag{6.9}$$

which yields (6.7) for $m = n - 3$.

Let $m = n - 5$. Multiplying the metric by a constant, we can represent the conformal multiplier as $\lambda = 1 + \partial\bar{\partial}\phi$. Putting $g_{n-3} = 0$ in (6.9) and using (6.8), after algebraic manipulations we obtain

$$\lambda g_{n-5} = \partial(\lambda h_{n-4}) + \bar{\partial}\big(h_{n-6} + (n-5) h_{n-4} \partial^2 \phi + (n-5)(n-2) h_{n-2} (\partial^2 \phi)^2/2\big).$$

Hence, (6.7) is proven for $m = n - 5$. The case $m = n - 7$ is similar, and so we omit the details. Probably, Lemma 6.3 holds for all $m \leqslant n - 1$, but for $m \leqslant n - 9$ the computations become difficult.

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