# Integral Invariants of the Hamilton Equations

V. V. Kozlov UDC 517.933

ABSTRACT. Conditions are found for the existence of integral invariants of Hamiltonian systems. For two-degrees-of-freedom systems these conditions are intimately related to the existence of nontrivial symmetry fields and multivalued integrals. Any integral invariant of a geodesic flow on an analytic surface of genus greater than 1 is shown to be a constant multiple of the Poincaré–Cartan invariant. Poincaré's conjecture that there are no additional integral invariants in the restricted three-body problem is proved.

## §1. Introduction

The general theory of integral invariants was developed by H. Poincaré and presented in his work [1]. Several important complements are due to E. Cartan [2]. Let us recall the basic definitions.

Let

$$\dot{x} = v(x), \qquad x \in M^n, \tag{1.1}$$

be a smooth dynamical system on a manifold $M$. The Lie derivative along the vector field $v$ will be denoted by $\mathcal{L}_v$. By the homotopy formula,

$$\mathcal{L}_v = di_v + i_v d.$$

Let $\varphi$ be a $k$-form, $\gamma$ a $k$-chain, and $g_v^t$ the phase flow of system (1.1). Then [3, Chap. VII]

$$\left.\frac{d}{dt}\right|_{t=0} \int_{g^t(\gamma)} \varphi = \int_\gamma \mathcal{L}_v \varphi.$$

Thus, if

$$\mathcal{L}_v \varphi = 0, \tag{1.2}$$

then the integral

$$I[\gamma] = \int_\gamma \varphi \tag{1.3}$$

is an *absolute integral invariant* of system (1.1):

$$I\left[g^t(\gamma)\right] = I[\gamma] \qquad \forall t \in \mathbb{R}. \tag{1.4}$$

If

$$\mathcal{L}_v \varphi = d\psi, \tag{1.5}$$

where $\psi$ is a $(k-1)$-form, then Eq. (1.4) holds for any $k$-cycle $\gamma$, $\partial\gamma = 0$. In this case the integral (1.3) is called a *relative integral invariant.*

Some cases of interest are beyond the scope of Poincaré's classification of integral invariants into absolute and relative ones. For example, if

$$\mathcal{L}_v \varphi = \psi, \qquad d\psi = 0, \tag{1.6}$$

where the $k$-form $\psi$ is not exact, then Eq. (1.4) holds for any $k$-cycle homologous to zero. The corresponding integral invariant will be referred to as *conditional.*

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Let us give a simple example of integral invariant that is conditional but not relative ($k = 1$ in our example). Let

$$M^2 = \mathbb{T} \times \mathbb{R} = \{q \bmod 2\pi, p\}, \qquad \dot{q} = 0, \quad \dot{p} = 1; \qquad \varphi = p\,dq.$$

Then

$$\mathcal{L}_v \varphi = i_v d\varphi = dq.$$

The form $\psi = dq$ is closed but not exact, and so $\dot{I}[g^t(\gamma)] = 2\pi$ for any closed contour $\gamma$ that goes around the cylinder $M$ (say, $\gamma = \{0 \le q < 2\pi,\ p = 0\}$).

Let a $k$-form $\varphi$ generate a conditional or a relative integral invariant. Then the $(k+1)$-form $d\varphi$ obviously generates an absolute invariant.

Indeed,

$$\mathcal{L}_v d\varphi = d\mathcal{L}_v \varphi = d\psi = 0.$$

This observation is actually due to Poincaré [1, item 238].

Now let $M^{2n} = T^*N^n$ be the phase space of a Hamiltonian system with configuration space $N^n = \{x\}$. We introduce the canonical momenta $y \in T_x^*N$ and the 1-form

$$\varphi = y\,dx = \sum_1^n y_k dx_k.$$

According to Poincaré [1, item 255], the Hamilton equations

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \quad \dot{y}_k = -\frac{\partial H}{\partial x_k}; \qquad 1 \le k \le n, \tag{1.7}$$

admit only the linear relative invariant

$$\int_\gamma \sum y_k\, dx_k, \qquad \partial\gamma = 0. \tag{1.8}$$

It is interesting that the invariant (1.8) is independent of the Hamiltonian $H$ in Eqs. (1.7). That is why the integral (1.8) is sometimes referred to as the universal integral invariant. Hwa-Chung-Lee [4] proved that each linear integral invariant of the Hamilton equations is a constant multiple of the Poincaré invariant (1.8). However, this result is formal; the proof is based on studying the invariance of the integral of the same 1-form $\varphi$ with respect to the phase flows of Hamiltonian systems with various specific Hamiltonians.

We should point out that Hwa-Chung-Lee's theorem is proved for the case $M = \mathbb{R}^{2n}$ and is no longer valid if the first Betti number of $M$ is not zero. In the latter case, one can add a closed nonexact 1-form to $\varphi$, thus modifying the value of the integral (1.8) on cycles nonhomologous to zero by some additive constants. In general, Hwa-Chung-Lee's theorem is only valid for relative integral invariants.

Poincaré posed the problem whether the equations of dynamics (in particular, in the three-body problem) have other integral invariants. In [1, item 257] he wrote: "One could ask if there exist algebraic integral invariants other than those constructed in the preceding ... We might apply either Broonce's method or the one I used in Chaps. IV and V ... ."

Poincaré realized that this problem is closely related to the integrability conditions for the Hamilton equations. That is why he referred to Chap. V, where he proved a theorem saying that there are no single-valued analytic integrals for a generic perturbation of the Hamiltonian. Let us show that a completely integrable system actually admits several distinct integral invariants in the vicinity of an invariant torus. In the action-angle variables ($J$, $\varphi \bmod 2\pi$) the Hamilton equations read

$$\dot{J}_1 = \cdots = \dot{J}_n = 0, \qquad \dot{\varphi}_1 = \omega_1, \ldots, \dot{\varphi}_n = \omega_n, \tag{1.9}$$

where each $\omega_k$ is a function of $J$. Let us consider the nondegenerate case, in which

$$\frac{\partial(\omega_1, \ldots, \omega_n)}{\partial(J_1, \ldots, J_n)} \ne 0.$$

Then system (1.9) can be represented in various nonequivalent Hamiltonian forms following [5]. One takes the symplectic structure

$$\omega = d\varphi, \qquad \varphi = \sum_1^n \frac{\partial K}{\partial \omega_k} d\varphi_k,$$

and the Hamiltonian function

$$H = \sum_1^n \omega_k \frac{\partial K}{\partial \omega_k} - K,$$

where $K$ is a nondegenerate function of the frequencies $\omega_1, \ldots, \omega_n$,

$$\det\left\| \frac{\partial^2 K}{\partial \omega_i \partial \omega_j} \right\| \neq 0.$$

The various Hamiltonian representations of system (1.9) are numbered by the functions $K(\omega)$. Hence, by the Poincaré theorem, system (1.9) admits the integral invariants

$$\oint \varphi = \oint \sum_1^n \frac{\partial K}{\partial \omega_k} d\varphi_k.$$

Poincaré tried to find a relationship between the new integral invariants and the properties of multipliers of periodic solutions to the Hamilton equations. He showed [1, item 259] that if there are $p$ distinct integral invariants (with the corresponding 1-forms $\varphi$ linearly independent) and if the coefficients of the forms $\varphi$ are linear in the canonical variables (as is the case in Eq. (1.8)), then $p$ multipliers are equal to $1$. Unfortunately, Poincaré did not obtain definitive results in the general case. He wrote: "The three-body problem is likely to admit no invariant algebraic relations other than those already known. However, I am not able to prove this yet" [1, item 258].

Our aim is to prove *Poincaré's conjecture* for some simplified versions of the three-body problem.

## §2. Perturbation theory and integral invariants

Poincaré's idea that the problem about integral invariants is related to the problem of small denominators [1, item 257] was developed in [6], where the following system with a small parameter $\varepsilon$ was considered:

$$\dot{x} = u_0 + \varepsilon u_1 + \cdots, \quad \dot{y} = v_0 + \varepsilon v_1 + \cdots, \quad \dot{z} = \varepsilon w_1 + \cdots. \tag{2.1}$$

The right-hand sides of these equations are power series in $\varepsilon$ whose coefficients are analytic in $x$, $y$, and $z$ and $2\pi$-periodic in $x$ and $y$. One can assume that the coefficients are defined and analytic on the direct product $\Delta \times \mathbb{T}^2$, where $\Delta$ is an interval in $\mathbb{R} = \{z\}$ and $\mathbb{T}^2 = \{x, y \bmod 2\pi\}$. It is assumed that the functions $u_0$ and $v_0$ depend only on $z$. Then for $\varepsilon = 0$ the system is completely integrable; the level surfaces of the integral $z = \text{const}$ are two-dimensional tori with conditionally periodic trajectories. Systems of the form (2.1) occur very frequently in the theory of nonlinear vibrations (e.g., see [7]).

The author [6] studied the existence of a relative integral invariant

$$\oint \varphi_\varepsilon \tag{2.2}$$

of system (2.1). Here the coefficients of the 1-form $\varphi_\varepsilon$ are single-valued analytic functions on $\Delta \times \mathbb{T}^2$ and depend on $\varepsilon$ analytically. Needless to say, the trivial case

$$d\varphi_\varepsilon = 0, \tag{2.3}$$

in which the integral (2.2) vanishes identically by the Stokes theorem, must be excluded from our considerations.

Let us expand the function $w_1$ in the double Fourier series

$$w_1 = \sum W_{mn}(z)\exp\big[i(mx+ny)\big].$$

Consider the set $P \subset \Delta$ of points $z$ such that

(1) $mu_0(z) + nv_0(z) = 0$ for some integers $m$ and $n$ that are not zero simultaneously;

(2) $W_{mn}(z) \neq 0$.

Such sets were originally considered by Poincaré in connection with the integrability problem for the Hamilton equations [1, Chap. V].

**Theorem 1** [6]. *Suppose that*

(A) *the set $P$ has an accumulation point $z_*$ in $\Delta$;*

(B) $u_0'v_0 - u_0v_0'|_{z_*} \neq 0$;

(C) $W_{00}(z) \not\equiv 0$.

*Then system* (2.1) *does not have any nontrivial integral invariants of the form* (2.2).

Condition (B) means that the nonperturbed system ($\varepsilon = 0$) is nondegenerate, that is, the frequency ratio $u_0/v_0$ is not constant. Furthermore, it follows from (B) that for $z = z_*$ and $\varepsilon = 0$ the right-hand sides in system (2.1) are different from zero. Conditions (A) and (B) together guarantee that there are no nonconstant analytic integrals and nontrivial symmetry fields analytic in $\varepsilon$ [6].

One can try to apply Theorem 1 to Hamiltonian systems close to completely integrable systems. Specifically, we speak of two-degrees-of-freedom systems whose order is reduced by one by means of the energy integral. The application of the Whittaker method to the reduced system results in a nonautonomous Hamiltonian system with time-periodic Hamiltonian [8, Chap. 1].

Thus, let us consider the Hamilton equations

$$\dot{x} = 1, \qquad \dot{y} = \frac{\partial H}{\partial z}, \qquad \dot{z} = -\frac{\partial H}{\partial y}, \qquad H_\varepsilon = H_0(z) + \varepsilon H_1(x, y, z) + \cdots . \tag{2.4}$$

Here ($y \bmod 2\pi$, $z$) are the canonical action–angle variables of the nonperturbed system and the function $H$ is $2\pi$-periodic in "time" $x = t$.

For system (2.4) we have

$$u_0 = 1, \qquad v_0 = \frac{\partial H_0}{\partial z}, \qquad w_1 = -\frac{\partial H_1}{\partial y}. \tag{2.5}$$

Consequently, condition (B) is equivalent to the nondegeneracy condition

$$\frac{d^2 H_0}{dz^2} \neq 0 \tag{2.6}$$

for the nonperturbed Hamiltonian. The set $P$ obviously coincides with the set

$$\left\{ z \in \Delta : \frac{dH_0}{dz} = -\frac{n}{m},\ H_{mn} \neq 0 \right\}, \tag{2.7}$$

where $H_{mn}$ are the Fourier coefficients of the perturbation $H_1$. It follows from (2.5) that condition (C) never holds for Hamiltonian systems ($W_{00} \equiv 0$), which is by no means surprising since system (2.4) has the Poincaré–Cartan integral invariant

$$\oint z\,dy - H_\varepsilon\,dx. \tag{2.8}$$

Clearly, this invariant is nontrivial (the degeneracy condition (2.5) is violated).

Let us indicate conditions sufficient for the nonexistence of a second integral invariant. To this end, we need the following lemma.

**Lemma 1** [6]. *Let conditions* (A) *and* (B) *of Theorem* 1 *be satisfied. Then there exists a function*

$$\lambda_\varepsilon = \lambda_0(z) + \varepsilon\lambda_1(z) + \cdots,$$

*such that*

$$d\varphi_\varepsilon = i_v(\lambda_\varepsilon\Omega), \tag{2.9}$$

*where $v_\varepsilon$ is the vector field* (2.1) *and $\Omega = dx \wedge dy \wedge dz$.*

Let us show how the conclusion of Theorem 1 can be derived from this lemma. To this end, we integrate the 2-forms on both sides of Eq. (2.9) over the two-dimensional torus $z = \text{const}$. By the Stokes theorem, the integral of $d\varphi$ is zero, whereas the integral on the right-hand side is equal to $\lambda_\varepsilon W_{00} + o(\varepsilon)$. From condition (C) we obtain $\lambda_\varepsilon = 0$. Hence, Eq. (2.9) coincides with the degeneracy condition (2.3).

**Lemma 2.** *If Eq.* (2.9) *holds, then the 3-form $\lambda\Omega$ generates an absolute integral invariant of system* (2.1).

Indeed,

$$0 = dd\varphi = di_v(\lambda\Omega) = di_v(\lambda\Omega) + i_v d(\lambda\Omega) = \mathcal{L}_v(\lambda\Omega). \quad \square$$

**Lemma 3.** *Suppose that system* (2.1) *has one more absolute invariant generated by a 3-form $\lambda'\Omega$ with $\lambda' \neq 0$. Then the ratio $\lambda/\lambda'$ is an integral of system* (2.1).

Set $\mathcal{L}_v\Omega = \mu\Omega$. Then, by the Leibniz rule,

$$\mathcal{L}_v(\lambda\Omega) = (\mathcal{L}_v\lambda)\Omega + \lambda\mathcal{L}_v\Omega = (\dot\lambda + \lambda\mu)\Omega = 0.$$

Since $\Omega \neq 0$, it follows that

$$\dot\lambda + \mu\lambda = 0. \tag{2.10}$$

Similarly,

$$\dot\lambda' + \mu\lambda' = 0. \tag{2.11}$$

Consequently,

$$\dot{\left(\frac{\lambda}{\lambda'}\right)} = 0$$

by virtue of (2.10) and (2.11), as desired. $\square$

The phase flow of the Hamilton equations (2.4) is known to preserve the "standard" volume 3-form $\Omega$. Furthermore, the energy-momentum 1-form in (2.8) satisfies (2.9) with $\lambda_\varepsilon = 1$.

**Theorem 2.** *Let condition* (2.5) *be satisfied, and let the set* (2.7) *have an accumulation point in the interval $\Delta$. Then any conditional integral invariant* (2.2) *of the Hamiltonian system* (2.4) *is the product of the Poincaré–Cartan integral invariant* (2.8) *by a constant factor $c_\varepsilon$.*

**Proof.** Suppose that we are given an integral invariant of the form (2.2) of system (2.4). Since conditions (A) and (B) of Theorem 1 are satisfied, it follows that Eq. (2.9) is valid. Note that $\mathcal{L}_v\Omega = 0$; then, by Lemmas 2 and 3, the factor $\lambda_\varepsilon$ in (2.9) is an integral of system (2.4). However, under the conditions of Theorem 2, $\lambda_\varepsilon = c_\varepsilon = \text{const}$ [8, Chap. 1]. Hence,

$$d\varphi_\varepsilon = c_\varepsilon d(zdy - H_\varepsilon dx).$$

It follows that the values taken by the integrals (2.2) and (2.8) on cocycles homologous to zero differ by the factor $c_\varepsilon$. $\square$

**Remark.** Suppose that

1) $u_0'v_0 - u_0v_0' \neq 0$,
2) $P$ is everywhere dense in $\Delta$,
3) system (2.1) admits a nontrivial invariant (2.2).

Then it can be shown that any other conditional integral invariant of system (2.1) is a constant multiple of (2.2) and that the proportionality factor is analytic in $\varepsilon$.

Theorem 2 can be applied to the planar circular restricted three-body problem. Here the small parameter $\varepsilon$ is the ratio of the mass of Jupiter to that of the Sun. In the rotating frame of reference in which the Sun and Jupiter are at rest, the dynamics of a third body (asteroid) of negligibly small mass is described by the Hamilton equations [9]

$$\begin{gathered} \dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}; \qquad k = 1, 2, \\ H = H_0 + \varepsilon H_1 + \cdots, \qquad H_0 = -\frac{1}{2p_1^2} - p_2. \end{gathered} \tag{2.12}$$

The double Fourier series expansion of the perturbing function, found by Leverrier, has the form

$$H_1 = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h_{uv} \cos\left[uq_1 - v(q_1 + q_2)\right],$$

where the coefficients $h_{uv}$ depend on $p_1$ and $p_2$ and are nonzero in general.

By taking the angular variable $q_2$ as the new "time" and by applying Whittaker's order-reducing procedure, we arrive at Hamilton equations of the form (2.4) with

$$H_0(z) = -\frac{1}{z^2}.$$

Thus, condition (2.6) is automatically satisfied. It can be shown that the set $P$ is necessarily everywhere dense on the semiaxis $z > 0$. We see that the reduced Hamilton equations for the restricted three-body problem do not have new relative integral invariants analytic in the parameter $\varepsilon$ and independent of the Poincaré–Cartan integral invariant.

## §3. Integral invariants and symmetries

Let us return to system (1.1), now assuming that $M$ is a three-dimensional manifold and that $v$ is a smooth vector field without critical points. Moreover, assume that system (1.1) admits an invariant volume form $\Omega$:

$$\mathcal{L}_v\Omega = 0.$$

The volume form determines the canonical orientation of $M$. If $M$ is compact, then we can assume that

$$\int_M \Omega > 0.$$

In particular, $\Omega$ determines a smooth measure invariant with respect to system (1.1).

The most important example is given by Hamiltonian systems with two degrees of freedom. Here $M^3$ is a connected component of a nonsingular level surface of the Hamiltonian, $v$ is the restriction of the Hamiltonian field to $M$, and the volume form is Liouville's invariant 4-form (e.g., see [10] for details).

**Lemma 4** (Cartan [2, item 91]). *Under the cited assumptions, the 2-form*

$$\Phi = i_v \Omega \tag{3.1}$$

*is closed and generates an absolute integral invariant of system* (1.1).

Indeed,

$$d\Phi = di_v\Omega = \mathcal{L}_v\Omega - i_v d\Omega = 0,$$
$$\mathcal{L}_v\Phi = \mathcal{L}_v i_v\Omega = i_v\mathcal{L}_v\Omega = 0.$$

Since the form (3.1) is closed, locally we have $\Phi = d\varphi$. But $i_v\Phi = 0$, and so

$$\mathcal{L}_v\varphi = i_v d\varphi + di_v\varphi = d(i_v\varphi).$$

Consequently, the 1-form $\varphi$ gives rise to a "local" relative integral invariant.

If the cohomology class of the 2-form $\Phi$ is zero, then the 1-form $\varphi$ is globally well defined. This is necessarily the case if

$$H^2(M, \mathbb{R}) = 0. \tag{3.2}$$

This argument is in fact contained in [2, item 91], but only for the special case $M = \mathbb{R}^3$.

Throughout the sequel we assume that the theorem about the partition of unity is valid for $M^3$. This is always the case if $M^3$ is a compact manifold.

**Lemma 5.** *Let* $\Psi$ *be a smooth* 2*-form on* $M$. *Then there exists a vector field* $x \mapsto u(x)$ *such that*

$$\Psi = i_u\Omega. \tag{3.3}$$

Indeed, let $\{\lambda_\alpha(x)\}$ be a partition of unity subordinate to some open cover of $M$. We assume that coordinates can be introduced globally in each domain $\operatorname{supp}\lambda_\alpha$. Obviously, in the domain $\operatorname{supp}\lambda_\alpha$ the algebraic equation (3.3) with $\Psi$ replaced by $\lambda_\alpha\Psi$ has a unique smooth solution $u_\alpha$ such that

$$\operatorname{supp} u_\alpha \subset \operatorname{supp}\lambda_\alpha.$$

It remains to set

$$u(x) = \sum_\alpha u_\alpha(x).$$

**Remark.** In the analytic case the field $u$ is obviously analytic.

**Lemma 6.** *Suppose that system* (1.1) *has a conditional integral invariant* $\oint \varphi$. *Let*

$$d\varphi = i_u\Omega. \tag{3.4}$$

*Then* $u$ *is a symmetry field of system* (1.1), *that is,* $[u, v] = 0$.

**Proof.** By the definition of a conditional invariant, we have $\mathcal{L}_v\varphi = \psi$, with $d\psi = 0$ Consequently,

$$0 = d\mathcal{L}_v\varphi = \mathcal{L}_v d\varphi = \mathcal{L}_v i_u\Omega = (\mathcal{L}_v i_u - i_u\mathcal{L}_v)\Omega = i_{[v,u]}\Omega.$$

Since the volume form is nondegenerate, it follows that the fields $u$ and $v$ commute. □

**Remark.** Lemma 6 remains valid if we replace the form $d\varphi$ in Eq. (3.4) by any closed 2-form. In [11] we obtained conditions for the existence of nontrivial symmetry fields (the vectors $u(x)$ and $v(x)$ are independent almost everywhere) of the Hamilton equations.

Lemma 6 has important applications in Hamiltonian mechanics. By way of example, let us consider the geodesic flow on a closed two-dimensional surface $\Sigma$. The flow is determined by specifying a Riemannian metric. The geodesics on $\Sigma$ are described by the Hamilton equations with Hamiltonian $H$ that is the Riemannian metric expressed via the canonical coordinates on $T^*\Sigma$. It is well known that for positive values of the total energy $h$ the Hamiltonian systems on the three-dimensional isoenergetic surfaces

$$\{x \in T^*\Sigma : H(x) = h\} \tag{3.5}$$

are isomorphic. One usually sets $h = 1$; the corresponding dynamical system is called the *geodesic* flow on $\Sigma$. Obviously, the geodesic flow has a relative Poincaré–Cartan integral invariant.

**Theorem 3.** *Let $\Sigma$ be an analytic surface of genus $> 1$ equipped with an analytic Riemannian metric. If a conditional invariant of the geodesic flow on $\Sigma$ is determined by an analytic* 1*-form on the surface* (3.5), *then this invariant is proportional to the Poincaré–Cartan invariant.*

**Proof.** Let $\Omega$ be an invariant analytic volume 3-form on the surface (3.5). If the geodesic flow has a conditional integral invariant determined by an analytic 1-form $\varphi$, then (by Lemma 6) one can find an analytic symmetry field $u$. However, the geodesic flow on an analytic surface does not have nontrivial symmetries [12]:

$$u = cv, \qquad c = \text{const}.$$

Consequently, by (3.4),

$$d\varphi = ci_v\Omega.$$

Hence, the conditional integral invariant in question is the Poincaré–Cartan invariant times the constant factor $c$. □

In closing this section, let us indicate yet another application of the cited results to one of the restricted versions of the three-body problem. Let two heavy bodies of the same mass rotate about their center of mass in elliptic orbits with nonzero eccentricity, and let the third body of negligibly small mass permanently move along a line orthogonal to the plane of motion of the first two bodies (see [13] for details). This problem was suggested by A. N. Kolmogorov as a tool for verifying the possibility of combinations of final three-body motions in the Chasy classification.

The dynamics of the third body is described by a nonautonomous Hamiltonian system of the form (2.4) with a periodic Hamiltonian. The extended phase space is the Cartesian product

$$\mathbb{T} \times \mathbb{R}^2 = \{x \bmod 2\pi, y, z\}.$$

Obviously, this system has the Poincaré–Cartan integral invariant (2.8).

Kolmogorov's problem is not integrable: it does not admit nonconstant analytic integrals [13]. This is due to the quasistochastic behavior of the trajectories. In particular, there are infinitely many nondegenerate long-period trajectories. As was shown in [11], nontrivial analytic symmetry fields are missing in this case: $u = cv$, $c = \text{const}$. By Lemma 6, the equations of this problem do not admit new conditional integral invariants. Similarly, one can prove that new analytic integral invariants are missing on fixed energy manifolds with large negative energy in the planar circular restricted three-body problem. The necessary preliminary results concerning the structure of the set of long-periodic nondegenerate trajectories were established in [14] by methods of symbolic dynamics.

## §4. Higher-order invariants

In §§ 2 and 3 we dealt with the existence of linear integral invariants. Let us now study second-order conditional invariants

$$\int_D \Phi, \tag{4.1}$$

where $D$ is a two-dimensional cycle in $M^3$ and $\Phi$ is a 2-form. The invariance condition for the integral (4.1) has the form

$$\mathcal{L}_v\Phi = \Psi, \qquad d\Psi = 0. \tag{4.2}$$

For relative invariants, $\Psi$ is exact, whereas $\Psi = 0$ for absolute invariants.

Since the invariant volume 3-form $\Omega$ is nondegenerate, it follows that

$$d\Phi = f\Omega, \qquad f\colon M^3 \to \mathbb{R}. \tag{4.3}$$

**Lemma 7.** *The function $f$ is an integral of system (1.1) on $M^3$.*

Indeed, from Eqs. (4.2) and (4.3) we obtain

$$0 = d\Psi = d\mathcal{L}_v\Phi = \mathcal{L}_v d\Phi = \mathcal{L}_v(f\Omega) = (\mathcal{L}_v f)\Omega + f\mathcal{L}_v\Omega = \dot{f}\Omega.$$

Consequently, $\dot{f} = 0$. □

By Lemma 4, system (1.1) has the absolute invariant $i_v\Omega$. Thus, we can speak of the existence of yet another integral invariant.

In the following we use the notion of multivalued integral of system (1.1). This is a closed 1-form $\vartheta$ such that

$$i_v\vartheta = 0. \tag{4.4}$$

Locally we have $\vartheta = dg$, where

$$\dot{g} = i_v dg = 0$$

by (4.4). Thus, locally $g$ is a usual integral of system (1.1). If

$$H^1(M, \mathbb{R}) = 0, \tag{4.5}$$

then $g$ is well defined globally, and the multivalued integral is an ordinary integral of system (1.1). Since $\dim M = 3$, conditions (3.2) and (4.5) are equivalent by the Poincaré duality theorem.

Throughout the following $M$, $v$, $\Omega$, and $\Phi$ are assumed to be analytic.

**Theorem 4.** *Let $M^3$ be compact, and let system (1.1) admit a conditional integral invariant (4.1) with*

$$\Phi \neq c i_v\Omega, \qquad c = \text{const}. \tag{4.6}$$

*Then system (1.1) has a nontrivial multivalued integral $\vartheta \neq 0$.*

**Proof.** By Lemma 7, the function $f$ in Eq. (4.3) is an integral of system (1.1). If $f \neq \text{const}$, then the proof is complete. Let $f = \alpha = \text{const}$. By integrating both sides of the identity

$$d\Phi = \alpha\Omega \tag{4.7}$$

over the compact manifold $M$ and by applying the Stokes theorem, we obtain

$$\alpha \int_M \Omega = 0.$$

Since the 3-form $\Omega$ is a volume form, it follows that $\alpha = 0$. Hence, the form $\Phi$ is closed by (4.6).

Set (Lemma 5)

$$\Phi = i_u\Omega.$$

Since the 2-form $\Phi$ is closed, it follows from Lemma 6 that $u$ commutes with $v$. Two cases are possible: 1) the vectors $u(x)$ and $v(x)$ are linearly dependent at all points $x \in M$; 2) these vectors are independent almost everywhere. Since $v \neq 0$, we see that in the first case

$$u(x) = \lambda(x)v(x), \qquad \lambda\colon M \to \mathbb{R}.$$

Since $u$ is a symmetry field, it follows that $\lambda$ is an integral of system (1.1) [11]. If $\lambda \neq \text{const}$, then the proof is complete. The case $\lambda = \text{const}$ is impossible in view of condition (4.6). In the second case, as shown in [15], the existence of a nontrivial analytic symmetry field implies the existence of a multivalued analytic integral $\vartheta \neq 0$. The proof uses the fact that $M$ is three-dimensional and has an invariant volume 3-form. □

**Corollary.** *Under the assumptions of Theorem 4, Eq.* (1.1) *can be integrated explicitly by using finitely many algebraic operations, differentiations, and quadratures.*

The additional differentiations are needed to find the multivalued integral (see also [15]).

**Remark.** Theorem 4 remains valid if there is a linear integral invariant

$$\oint \varphi$$

provided that the 2-form $\Phi = d\varphi$ satisfies condition (4.6).

Since the differential equations of the various versions of the three-body problem cited above do not admit nontrivial symmetry fields and multivalued integrals, it follows that any conditional integral invariant (4.1) of these equations must be a constant multiple of the invariant

$$\int_D dz \wedge dy - dH \wedge dx.$$

Since $\dim M = 3$, it makes sense to consider only third-order absolute integral invariants. The corresponding 3-form is $f\Omega$, and by Lemma 3, $f$ is an integral of (1.1). For the cited equations of dynamics, $f = \text{const}$.

Conditions for the existence of integral invariants in Hamiltonian systems with many degrees of freedom demand further investigation.

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## References

1. H. Poincaré, *Les Méthodes nouvelles de la mechanique céleste*, Vol. 3, Gauthier-Villars, Paris (1892).
2. E. Cartan, *Leçons sur les invariants intégraux*, Hermann, Paris (1922).
3. C. Godbillon, *Geometrie différentielle et mechanique analytique*, Hermann, Paris (1969).
4. Hwa-Chung-Lee, "The universal integral invariants of Hamiltonian systems and application to the theory of canonical transformations," *Proc. Royal Soc. of Edinburgh. Sect. A.*, **LXII**, No. 3, 237–246 (1946–48).
5. V. V. Kozlov, "The Liouville property of invariant measures in completely integrable systems and the Monge–Ampere equation," *Mat. Zametki* [*Math. Notes*], **53**, No. 4, 45–52 (1993).
6. V. V. Kozlov, "Dynamical systems determined by the Navier-Stokes equations," *Rus. J. Math. Phys.*, **1**, No. 1, 57–69 (1993).
7. N. N. Bogolyubov and Yu. L. Mitropol'skii, *Asymptotic Methods in the Theory of Nonlinear Vibrations* [in Russian], Nauka, Moscow (1974).
8. V. V. Kozlov, *Qualitative Methods in Dynamics of Solids* [in Russian], Izdat. Moskov. Univ., Moscow (1980).
9. C. L. Charlier, *Die Mechanik des Himmels*, W. de Gruyter, Berlin–Leipzig (1927).
10. D. V. Anosov and Ya. G. Sinai, "Some smooth ergodic systems," *Uspekhi Mat. Nauk* [*Russian Math. Surveys*], **22**, No. 5, 107–172 (1967).
11. V. V. Kozlov, "Symmetry Groups of Dynamical Systems," *Prikl. Mat. Mekh.* [*J. Appl. Math. Mech.*], **52**, No. 4, 531–541 (1988).
12. V. V. Kozlov, "Symmetry groups of geodesic flows on closed surfaces," *Mat. Zametki* [*Math. Notes*], **48**, No. 5, 62–67 (1990).
13. V. M. Alekseev, "Final motions in the three-body problem and symbolic dynamics," *Uspekhi Mat. Nauk* [*Russian Math. Surveys*], **36**, No. 4, 161–176 (1981).
14. J. Llibre and C. Simo, "Oscillatory solutions in the planar restricted three-body problem," *Math. Ann.* **248**, No. 2, 153–184 (1980).
15. S. V. Bolotin and V. V. Kozlov, "Symmetry fields of geodesic flows," *Russ. J. Math. Phys.* (to appear).

Moscow State University

Translated by V. E. Nazaikinskii