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SYMMETRIES AND THE TOPOLOGY OF DYNAMICAL SYSTEMS WITH TWO DEGREES OF FREEDOM

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ABSTRACT. The problem of geodesic curves on a closed two-dimensional surface and some of its generalizations related with the addition of gyroscopic forces are considered. The authors study one-parameter groups of symmetries in the four-dimensional phase space that are generated by vector fields commuting with the original Hamiltonian vector field. If the genus of the surface is greater than one, then there are no nontrivial symmetries. For a surface of genus one (a two-dimensional torus) it is established that if there is an additional integral polynomial in the velocities, even or odd with respect to each component of the velocity, then there is a polynomial integral of degree one or two. For a surface of genus zero examples of nontrivial integrals of degree three and four are given. Fields of symmetries of first and second degree are studied. The presence of such symmetries is related to the existence of ignorable cyclic coordinates and separated variables. The influence of gyroscopic forces on the existence of fields of symmetries with polynomial components is studied.

Bibliography: 9 titles.

1. INTRODUCTION

Let M be a closed two-dimensional surface that is a *configuration space* of a dynamical system with two degrees of freedom. Local coordinates on M will be denoted q_1, q_2. In mechanics they are usually called *generalized coordinates*. Let p_1, p_2 be the *canonical momenta* conjugate to q_1, q_2. Thus,

$$x = (q_1, q_2, p_1, p_2)$$

are local coordinates in the *phase space* T^*M (the total space of the cotangent bundle of M). An essential role in the geometry of phase space is played by a *symplectic structure*, a closed nondegenerate 2-form

$$\omega = \sum dp_s \wedge dq_s .$$

Let H be a real function on T^*M. It is uniquely associated to a vector field v_H according to the rule

$$\omega(v, \cdot) = -dH. \tag{1.1}$$

The field v is termed *Hamiltonian*. It generates a Hamiltonian dynamical system on T^*M

$$\dot{x} = v(x). \tag{1.2}$$

In the local coordinates q, p these equations take the usual form of *Hamilton's canonical equations*

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}, \qquad s = 1, 2. \tag{1.3}$$

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As a Hamiltonian H we take some Riemannian metric on M, a positive definite quadratic form in p_1, p_2 whose coefficients are uniquely determined functions on M. In mechanics the equations (1.3) with such a Hamiltonian describe the inertial motion, and in Riemannian geometry they are the equations of geodesics of the Riemannian metric under consideration. Equations (1.3) possess the following simple property: together with a solution $q(t), p(t)$ they admit the solution $q(-t), -p(-t)$. Following Birkhoff, such dynamical systems are said to be *reversible.*

As generalized coordinates we can take isothermal (conformal) coordinates of the Riemannian metric H. Then the Hamiltonian will have the following form:

$$H = \frac{\Lambda}{2}(p_1^2 + p_2^2), \tag{1.4}$$

where Λ is a positive function of q_1, q_2.

We are interested in the problem of symmetries of the dynamical system (1.2) that are generated by vector fields u on T^*M which commute with the original Hamiltonian field v:

$$v: [u, v] = 0.$$

Such fields will be called *fields of symmetries.* It turns out that the presence and shape of fields of symmetries depend in an essential way on the topology of the configuration space M. All the objects occurring below will be assumed to be C^∞.

The operator of differentiation along the vector field v has the following form:

$$L_v = \Lambda p_1 \frac{\partial}{\partial q_1} + \Lambda p_2 \frac{\partial}{\partial q_2} - \frac{1}{2}\frac{\partial \Lambda}{\partial q_1}(p_1^2 + p_2^2)\frac{\partial}{\partial p_1} - \frac{1}{2}\frac{\partial \Lambda}{\partial q_2}(p_1^2 + p_2^2)\frac{\partial}{\partial p_2}.$$

Let u be a vector field with differentiation operator

$$L_u = Q_1 \frac{\partial}{\partial q_1} + Q_2 \frac{\partial}{\partial q_2} + P_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}.$$

If u is a field of symmetries, then the operators L_v and L_u of course commute.

Definition. If Q_1 and Q_2 are polynomials of degree $n-1$ in the momenta, and P_1, P_2 are polynomials of degree n, then the field u is called a *homogeneous field of degree* n.

The degree of a homogeneous field u will be denoted $\deg u$. In particular, $\deg v = 2$. A field u of symmetries can be expanded in a formal series in homogeneous fields:

$$u = u_0 + u_1 + u_2 + \cdots, \qquad \deg u_k = k, \ \ k \geq 1.$$

The operator of differentiation along the vector field u_0 has the form

$$\alpha_1 \frac{\partial}{\partial p_1} + \alpha_2 \frac{\partial}{\partial p_2}$$

where α_1, α_2 are functions on M. We have a simple

Proposition. *Each homogeneous piece of the vector field* u *is itself a field of symmetries.*

The field u_0 is obviously equal to zero. Otherwise $\Lambda = 0$. Therefore we can restrict consideration to homogeneous fields of symmetries of degree ≥ 1.

A field of symmetries u is said to be *Hamiltonian* if

$$\omega(u, \cdot) = -dF,$$

where F is a uniquely determined function in phase space. If F is a homogeneous polynomial of degree m in the momenta, then $\deg u = m$. We set $u = v_F$. Then, as is well known,

$$[v_H, v_F] = v_{\{H,F\}},$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. If u and v commute, then

$$\{H, F\} = \text{const}.$$

Since the Hamiltonian is quadratic, this constant is equal to zero. Hence, the Hamiltonian F is an integral of the dynamical system (1.2). Integrals that are linear in the momenta p_1, p_2 generate Hamiltonian symmetries, which were studied by Emmy Noether.

A field of symmetries u will be called *locally Hamiltonian* if the 1-form $\omega(u, \cdot)$ is closed but not exact. In this case the equations (1.2) admit the closed 1-form as an invariant, which can be called a many-valued integral.

It should not be assumed that fields of symmetries of (1.2) are always Hamiltonian (or locally Hamiltonian). Here is simple counterexample: if $\Lambda = \text{const}$, then the quadratic vector field with differentiation operator

$$p_2\frac{\partial}{\partial q_1} - p_1\frac{\partial}{\partial q_2} \tag{1.5}$$

is a field of symmetries. However, it is not even locally Hamiltonian relative to the standard symplectic structure ω.

2. Main results

First we discuss the problem of the structure of Hamiltonian fields of symmetries. It is obvious that if the equations (1.2) admit a smooth integral

$$F: T^*M \to \mathbb{R},$$

when any homogeneous form of its Maclaurin series expansion in the momenta is also an integral.

In [1] the case is considered when the Euler characteristic χ of the surface M is negative, and it is proved that equations (1.2) do not admit an integral independent of the energy integral H. According to the Gauss-Bonnet formula, the condition $\chi(M) < 0$ is equivalent to the assumption of the negativity of the mean Gauss curvature of the Riemannian metric on M. Later, Kolokol′tsov [2] gave another proof of the result of [1], based on the introduction of a conformal structure in M. This method goes back to Birkhoff [3], who studied the problem of the existence of local integrals of first and second degree in the momenta. Birkhoff considered conditional integrals, whose derivative vanishes for some value of the total energy $H = h$. He showed that the existence of a conditional linear integral is connected with the existence of an ignorable *cyclic coordinate* (on which the Hamiltonian function does not depend), and the existence of a conditional quadratic integral is connected with the existence of ignorable *separated variables.* Global versions of these results of Birkhoff, concerning the existence of conditional linear and quadratic integrals of dynamical systems with a configuration space in the form of a two-dimensional torus, were established in [4].

Recently Bolotin proved [5] that under the condition $\chi(M) < 0$ there are unstable closed trajectories with transversally intersecting separatrices on the energy surfaces $H = \text{const} > 0$. From this we obtain the presence of domains with stochastic behavior and, as a consequence, the absence of an additional integral.

For $\chi \geq 0$ the surface M is homeomorphic to a sphere or a torus. A description of the metrics on the two-dimensional sphere and torus that admit nontrivial linear and quadratic integrals is given in Kolokol'tsov's paper [2] in the spirit of Birkhoff's theory.

We discuss the case $\chi = 0$ in more detail. Here

$$M = \mathbb{T}^2 = \{q_1, q_2 \bmod 2\pi\}.$$

It is known that a Riemannian metric on the torus can always be reduced to the form (1.4) in the large. The question is whether there exist polynomial integrals of degree ≥ 3 that are independent of H and do not reduce to polynomials of degree ≤ 2. Bolotin conjectured that there are no such integrals for the case of the torus. This conjecture has not yet been completely proved. However, we do have

Theorem 1. *Assume that equations* (1.3) *with the Hamiltonian* (1.4) *admit an integral* F *homogeneous in the momenta and independent of the energy integral, and that one of the following additional conditions holds*:

a) F *is an even function of* p_1 *and* p_2;

b) F *is an even function of* p_1 *(resp.,* p_2*) and an odd function of* p_2 *(resp.,* p_1*).*

Then the equations (1.3) *admit an additional integral of degree* ≤ 2.

In case M is homeomorphic to a two-dimensional sphere, the situation is different. There are examples of metrics on the sphere for which the equations of geodesics admit integrals of degree 3 and 4 in the momenta that do not reduce to polynomials of lower degree.

We shall indicate the scheme for constructing such examples. With this goal we consider the problem of rotation of a heavy rigid body with a fixed point. This system admits the rotation group around the vertical. Fixing the zeroth constant of the corresponding Noether integral (which in mechanics is usually called the area integral) and factoring by the orbits of the action of the symmetry group, we reduce this problem to a system with two degrees of freedom on the sphere S^2. The Lagrangian has the form $T - V$, where T is some Riemannian metric on $M = S^2$ (the kinetic energy of the reduced system) and $V: S^2 \to \mathbb{R}$ is the potential energy (see [6]). If the Goryachev-Chaplygin or Kovalevskaya conditions hold, then the equations with Lagrangian $T - V$ admit an additional integral which is of degree 3 or 4, respectively, in the velocities. For example, in the Kovalevskaya case it has the form

$$F = F_4 + F_2 + F_0, \tag{2.1}$$

where F_k is a homogeneous polynomial of degree k in the velocities.

We now use the Jacobi least action principle. For this we fix the energy constant

$$h = T + V > \max_M V.$$

According to Jacobi's principle, the trajectories on M with energy h are geodesic curves of the Riemannian metric

$$(h - V)T. \tag{2.2}$$

It is clear that the homogeneous function

$$F_4 + F_2 \frac{T}{h - V} + F_0 \left(\frac{T}{h - V}\right)^2$$

of degree 4 in the velocities is an additional integral of the metric (2.2).

In the Goryachev-Chaplygin case the integral has the form F_3+F_1. In an analogous way one constructs a Riemannian metric on the sphere for which the equations of the geodesics have an integral of degree three.

We do not know any examples of Riemannian metrics on the two-dimensional sphere that admit nontrivial integrals of degree ≥ 5. It is possible that such metrics simply do not exist.

Remark. On the two-dimensional torus the Liouville metrics are well known, for which the geodesic flows are integrated via linear or quadratic integrals. A Riemannian metric is said to be *nondegenerately integrable* (see [9]) if its geodesic flow is integrable via a smooth integral such that all the critical points of its manifold on a constant energy surface are nondegenerate. A. T. Fomenko conjectured that on a two-dimensional torus any nondegenerately integrable metric is Liouville. In particular, this means that any polynomial integral reduces to an integral of degree ≤ 2. This conjecture has not yet been proved. However, T. Z. Nguyen, L. S. Polyakova, and V. V. Kalashnikov, Jr., have proved that the conjecture is true at least as regards complexity of metrics (the concept of complexity was introduced in [9], Definition 8). Namely: the complexity of the geodesic flow of an arbitrary nondegenerately integrable smooth Riemannian metric on the 2-torus is equal to the complexity of the geodesic flow of some Liouville metric. A similar assertion holds for metrics on the 2-sphere. Therefore, as to complexity, the integrable metrics with linear and quadratic integrals "approximate" an arbitrary integrable Riemannian metric with an arbitrary smooth integral.

We now proceed to study fields of symmetries of a general form. In [7] the case when $\chi(M) < 0$ was considered. It was proved that any field of symmetries u has the form $f(H)v$. Thus, there are no nontrivial symmetries, and, in particular, any field of symmetries is obviously Hamiltonian. The function

$$\int f(H)\,dH$$

is the corresponding Hamiltonian.

For $\chi(M) = 0$ there are more possibilities. We have

Theorem 2. *Any field of symmetries of degree one is Hamiltonian.*

It is generated by a linear integral and therefore is Noetherian.

Theorem 3. *If the Gaussian curvature of the metric on the torus is identically equal to zero, then any field of symmetries of degree two is Hamiltonian.*

An analytic criterion for the metric (1.4) on the torus to be Euclidean is that $\Lambda = \text{const}$. As the example of field (1.5) shows, the assumption $\Lambda \neq \text{const}$ is essential in Theorem 3.

Theorem 2 does not hold for fields of symmetries of degree ≥ 3. Indeed, let Λ be a nonconstant periodic function only of the angular variable q_2. Then the vector field u of degree one defined by the equations

$$q_1' = 1, \qquad q_2' = 0, \qquad p_1' = 0, \qquad p_2' = 0,$$

is a field of symmetries. Of course, it is Hamiltonian. However the field (Hu) of degree three, which is also a field of symmetries, is not Hamiltonian.

We do not exclude the possibility that if a Hamiltonian system with Hamiltonian (1.4) admits a polynomial field of symmetries of degree n, not collinear to the field v, then there must exist an additional integral in the momenta of degree $\leq n$.

The problem of the structure of symmetries of reversible systems on the sphere is more complicated and is not considered here.

3. Irreversible systems

We complicate the problem by replacing the standard symplectic structure

$$\omega = \sum dp_k \wedge dq_k$$

on T^*M by the closed nondegenerate 2-form $\omega + \varphi$, where φ is a 2-form on M. In the local coordinates q_1, q_2 it has the form

$$\lambda(q_1, q_2)\, dq_1 \wedge dq_2 .$$

If we replace ω by $\omega+\varphi$ in (1.1), then we obtain the following for Hamiltonian's equations:

$$\begin{aligned} \dot q_1 &= \frac{\partial H}{\partial p_1}, & \dot q_2 &= \frac{\partial H}{\partial p_2}, \\ \dot p_1 &= -\frac{\partial H}{\partial q_1} + \lambda \frac{\partial H}{\partial p_2}, & \dot p_2 &= -\frac{\partial H}{\partial q_2} - \lambda \frac{\partial H}{\partial p_1}. \end{aligned} \tag{3.1}$$

Following Birkhoff [3], such systems are termed *irreversible.*

The terms in equations (3.1) that contain λ are usually called gyroscopic forces. Their nature can be different. Gyroscopic forces appear, for example, in passing to a rotating system of coordinates, and also in the description of the motion of charged bodies in magnetic fields.

Obviously the equations (3.1) admit the energy integral H. The problem of the existence of other integrals, polynomial in the momenta p_1, p_2, for the equations (3.1) on closed surfaces was considered by Bolotin in [8]. In particular, for the case of the two-dimensional torus he established that if equations (3.1) with Hamiltonian (1.4) have a polynomial integral independent of H, then

$$\int_{\mathbf{T}^2} \lambda\, dq_1\, dq_2 = 0 . \tag{3.2}$$

We shall study the more generalized problem of conditions for the existence of vector fields of symmetries with polynomial components for equations (3.1). In contrast to the reversible case, here the fields of symmetries will not be homogeneous. They can be represented as a finite sum of homogeneous fields

$$u = u_m + u_{m-1} + \cdots , \qquad \deg u_k = k ,$$

arranged in order of decreasing degree. The degree of the field u is

$$\deg u_m = m .$$

It is clear that u_m is a field of symmetries of a reversible system (when $\lambda = 0$). This simple remark allows us to use the results of §2.

Theorem 4. *A nonzero field of symmetries of degree one is always locally Hamiltonian. It is Hamiltonian only if condition* (3.2) *holds.*

For $m \geq 2$ the situation is different. We have

Theorem 5. *Assume that the Hamiltonian field* (3.1) *admits a field of symmetries of degree* $m \geq 2$, *where the highest homogeneous pieces of the vector fields* v *and* u *are linearly independent for*

$$p_1 = 1, \qquad p_2 = i \qquad (i^2 = -1).$$

Then (3.2) *holds.*

It appears that in Theorem 5 it is sufficient to require that the fields u and v be independent. Suppose u is a Hamiltonian field with Hamiltonian

$$F = F_m + F_{m-1} + \cdots .$$

Then the condition of independence of the vectors v and u_m at the point

$$p_1 = 1, \qquad p_2 = i$$

is equivalent to the following: the polynomial F_m is not divisible by H.

The proofs of Theorems 1–5 are given below.

4. Additional assertions

In this section we consider the reversible case and do not impose any restrictions on the degree $n = \deg u$ of the field of symmetries u.

Let F be a homogeneous polynomial in p_1 and p_2. We denote by F^* its value for $p_1 = 1$, $p_2 = i$ $(i^2 = -1)$.

Lemma 1 ([7]). *$\Lambda(P_1^* + iP_2^*)$ is a holomorphic function of $z = q_1 + iq_2$.*

Since Λ, P_1^*, P_2^* are smooth complex-valued functions on the torus

$$\mathbb{T}^2 = \{q_1, q_2 \bmod 2\pi\},$$

they are bounded. Hence, according to Lemma 1 and Liouville's theorem,

$$\Lambda(P_1^* + iP_2^*) = \gamma_1 + i\gamma_2, \tag{4.1}$$

where γ_1, γ_2 are some real constants.

Lemma 2. $\gamma_1 = \gamma_2 = 0$.

For the proof we compute the commutator $[L_u, L_v]$ and set the coefficients of $\partial/\partial q_1$ and $\partial/\partial q_2$ equal to zero. As a result we obtain the relations

$$p_1 \sum Q_k \frac{\partial \Lambda}{\partial q_k} - \Lambda \sum p_k \frac{\partial Q_1}{\partial q_k} + \tfrac{1}{2}(p_1^2 + p_2^2) \sum \frac{\partial \Lambda}{\partial q_k}\frac{\partial Q_1}{\partial p_k} + P_1\Lambda = 0, \tag{4.2}$$

$$p_2 \sum Q_k \frac{\partial \Lambda}{\partial q_k} - \Lambda \sum p_k \frac{\partial Q_2}{\partial q_k} + \tfrac{1}{2}(p_1^2 + p_2^2) \sum \frac{\partial \Lambda}{\partial q_k}\frac{\partial Q_2}{\partial p_k} + P_2\Lambda = 0. \tag{4.3}$$

The index of summation k takes the values 1 and 2.

Setting $p_1 = 1$, $p_2 = i$ in (4.2) and (4.3), we find the equalities

$$\sum Q_k^* \frac{\partial \Lambda}{\partial q_k} + P_1^*\Lambda - \Lambda \frac{\partial Q_1^*}{\partial q_1} - \Lambda i \frac{\partial Q_1^*}{\partial q_2} = 0, \tag{4.4}$$

$$i \sum Q_k^* \frac{\partial \Lambda}{\partial q_k} + P_2^*\Lambda - \Lambda \frac{\partial Q_2^*}{\partial q_1} - \Lambda i \frac{\partial Q_2^*}{\partial q_2} = 0. \tag{4.5}$$

We multiply (4.5) and i and add it to (4.4). As a result we obtain

$$P_1^* + iP_2^* = \frac{\partial}{\partial q_1}(Q_1^* + iQ_2^*) + i\frac{\partial}{\partial q_2}(Q_1^* + iQ_2^*),$$

or, using (4.1),

$$\frac{\gamma_1 + i\gamma_2}{\Lambda} = \frac{\partial}{\partial q_1}(Q_1^* + iQ_2^*) + i\frac{\partial}{\partial q_2}(Q_1^* + iQ_2^*). \tag{4.6}$$

Averaging over the torus, we arrive at the equality

$$(\gamma_1 + i\gamma_2)\int_0^{2\pi}\int_0^{2\pi}\frac{dq_1\,dq_2}{\Lambda} = 0.$$

Hence,

$$\gamma_1 + i\gamma_2 = 0,$$

as required.

Lemma 3.

$$Q_1^* + iQ_2^* = c_1 + ic_2,$$

where c_1, c_2 *are real constants.*

Indeed, equality (4.6) together with the conclusion of Lemma 2 is a criterion for the function $Q_1^* + iQ_2^*$ to be holomorphic. It remains to use Liouville's theorem.

Lemma 4. $P_1^* + P_2^* = 0$.

For the proof we compute the commutator $[L_u, L_v]$ and set the coefficients of $\partial/\partial p_1$ and $\partial/\partial p_2$ equal to zero. As a result we obtain the relations

$$\begin{aligned}(4.7)\qquad & -\tfrac{1}{2}(p_1^2+p_2^2)\left(Q_1\frac{\partial^2\Lambda}{\partial q_1^2} + Q_2\frac{\partial^2\Lambda}{\partial q_1\,\partial q_2}\right) - \frac{\partial\Lambda}{\partial q_1}\sum P_k p_k \\ & - \Lambda\sum p_k\frac{\partial P_1}{\partial q_k} + \tfrac{1}{2}(p_1^2+p_2^2)\sum\frac{\partial\Lambda}{\partial q_k}\frac{\partial P_1}{\partial p_k} = 0,\end{aligned}$$

$$\begin{aligned}(4.8)\qquad & -\tfrac{1}{2}(p_1^2+p_2^2)\left(Q_1\frac{\partial^2\Lambda}{\partial q_1\,\partial q_2} + Q_2\frac{\partial^2\Lambda}{\partial q_2^2}\right) - \frac{\partial\Lambda}{\partial q_2}\sum P_k p_k \\ & - \Lambda\sum p_k\frac{\partial P_2}{\partial q_2} + \tfrac{1}{2}(p_1^2+p_2^2)\sum\frac{\partial\Lambda}{\partial q_k}\frac{\partial P_2}{\partial p_k} = 0.\end{aligned}$$

We set $p_1 = 1$ and $p_2 = i$ and use Lemma 2. Then from (4.7) and (4.8) we obtain the equalities

$$\frac{\partial P_k^*}{\partial q_1} + i\frac{\partial P_k^*}{\partial q_2} = 0, \qquad k = 1, 2.$$

Hence, P_k^* is a holomorphic function of

$$z = q_1 + iq_2.$$

By Liouville's theorem,

$$P_k^* = \mu_k = \text{const} \qquad (\mu_k \in \mathbb{C}).$$

Using Lemma 3, from the relation (4.4) it is easy to derive the equality

$$\mu_1 M = \frac{\partial Q_1^* M}{\partial q_1} + \frac{\partial Q_2^* M}{\partial q_2},$$

where $M = 1/\Lambda$. Hence,

$$\mu_1\int_0^{2\pi}\int_0^{2\pi} M\,dq_1\,dq_2 = 0.$$

This implies $\mu_1 = 0$. Analogously we can derive that $\mu_2 = 0$. This completes the proof of Lemma 4.

At the same time we have obtained the relation

$$(4.9)\qquad \frac{\partial Q_1^* M}{\partial q_1} + \frac{\partial Q_2^* M}{\partial q_2} = 0,$$

which will be used later on.

We set

$$Q_1^* = \Phi_1 + i\Psi_1, \qquad Q_2^* = \Phi_2 + i\Psi_2,$$

where Φ_k, Ψ_k are single-valued functions on $\mathbb{T}^2$. By Lemma 3,

$$\Phi_1 - \Psi_2 = c_1, \qquad \Psi_1 + \Phi_2 = c_2. \tag{4.10}$$

Equality (4.9) splits into two:

$$\frac{\partial \Phi_1 M}{\partial q_1} + \frac{\partial \Phi_2 M}{\partial q_2} = 0, \qquad \frac{\partial \Psi_1 M}{\partial q_1} + \frac{\partial \Psi_2 M}{\partial q_2} = 0. \tag{4.11}$$

Lemma 5. *The function* $\sigma = \Psi_1 - \Phi_2$ *satisfies the equation*

$$2c_1 \frac{\partial^2 M}{\partial q_1 \partial q_2} = c_2 \left(\frac{\partial^2 M}{\partial q_1^2} - \frac{\partial^2 M}{\partial q_2^2} \right) + \Delta(\sigma M), \tag{4.12}$$

where Δ *is the Laplace operator.*

Equation (4.12) is simple to derive from (4.10) and (4.11). It plays an essential role in what follows.

Theorem 6 ([2]). *Let* F *be a homogeneous polynomial of degree* $n \geq 2$ *in the momenta and suppose that* $F^* = 0$. *Then*

$$F = (p_1^2 + p_2^2) G,$$

where G *is a polynomial in* p_1 *and* p_2 *of degree* $n - 2$.

It follows from Lemmas 4 and 6 that

$$P_i = (p_1^2 + p_2^2) K_i, \qquad i = 1, 2, \tag{4.13}$$

where the K_i are polynomials in the momenta of degree $n - 2$. If we had proved that

$$Q_1^* = Q_2^* = 0$$

(cf. Lemma 3), then the field of symmetries u could be represented as a product Hw, where w is a field of symmetries of degree $n - 2$. By induction the problem about homogeneous fields of symmetries would reduce to a problem about fields of symmetries of degree $n \leq 2$. Unfortunately, $Q_k^* \neq 0$ in the general case.

Lemma 7. *The following equality holds*:

$$\frac{1}{2} \frac{\partial Q_1^*}{\partial q_1} + \frac{1}{2} \frac{\partial Q_2^*}{\partial q_2} + K_i^* + iK_i^* = \nu_1 + i\nu_2 = \text{const}. \tag{4.14}$$

Proof. It is clear that

$$\begin{aligned} \frac{\partial P_i}{\partial q_j} &= (p_1^2 + p_2^2) \frac{\partial K_i}{\partial q_j}, \\ \frac{\partial P_i}{\partial p_j} &= (p_1^2 + p_2^2) \frac{\partial K_i}{\partial p_j} + 2p_j K_i. \end{aligned} \tag{4.15}$$

Substituting (4.13) and (4.15) in (4.7) and (4.8), cancelling by $p_1^2 + p_2^2$ and then substituting $p_1 = 1$, $p_2 = i$, we obtain two equalities:

$$\begin{aligned} \tfrac{1}{2} Q_1^* \frac{\partial^2 \Lambda}{\partial q_1^2} + \tfrac{1}{2} Q_2^* \frac{\partial^2 \Lambda}{\partial q_1 \partial q_2} + iK_2^* \frac{\partial \Lambda}{\partial q_1} + \Lambda \frac{\partial K_1^*}{\partial q_1} + i\Lambda \frac{\partial K_1^*}{\partial q_2} - iK_1^* \frac{\partial \Lambda}{\partial q_2} &= 0, \\ \tfrac{1}{2} Q_1^* \frac{\partial^2 \Lambda}{\partial q_1 \partial q_2} + \tfrac{1}{2} Q_2^* \frac{\partial^2 \Lambda}{\partial q_2^2} + K_1^* \frac{\partial \Lambda}{\partial q_2} + \Lambda \frac{\partial K_2^*}{\partial q_1} + i\Lambda \frac{\partial K_2^*}{\partial q_2} - K_2^* \frac{\partial \Lambda}{\partial q_1} &= 0. \end{aligned} \tag{4.16}$$

Again setting $M = 1/\Lambda$, we transform the sum

$$Q_1^* M \frac{\partial^2 \Lambda}{\partial q_1^2} + Q_2^* M \frac{\partial^2 \Lambda}{\partial q_1 \partial q_2} = \frac{\partial}{\partial q_1}\left(Q_1^* M \frac{\partial \Lambda}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(Q_2^* M \frac{\partial \Lambda}{\partial q_1}\right) - \left(\frac{\partial Q_1^* M}{\partial q_1} + \frac{\partial Q_2^* M}{\partial q_2}\right)\frac{\partial \Lambda}{\partial q_1}.$$

The last term is equal to zero in view of (4.9). Analogously,

$$Q_1^* M \frac{\partial^2 \Lambda}{\partial q_1 \partial q_2} + Q_2^* M \frac{\partial^2 \Lambda}{\partial q_2^2} = \frac{\partial}{\partial q_1}\left(Q_1^* M \frac{\partial \Lambda}{\partial q_2}\right) + \frac{\partial}{\partial q_2}\left(Q_2^* M \frac{\partial \Lambda}{\partial q_2}\right).$$

We multiply the first sequence of (4.16) by M, the second by iM, add them, and use the relations just obtained. As a result we get the equality

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial q_1}\left(Q_1^* M \frac{\partial \Lambda}{\partial q_1} + iQ_1^* M \frac{\partial \Lambda}{\partial q_2}\right) + \frac{1}{2}\frac{\partial}{\partial q_2}\left(Q_2^* M \frac{\partial \Lambda}{\partial q_1} + iQ_2^* M \frac{\partial \Lambda}{\partial q_2}\right)\\ &\quad + \frac{\partial}{\partial q_1}(K_1^* + iK_2^*) + i\frac{\partial}{\partial q_2}(K_1^* + iK_2^*) = 0.\end{aligned} \tag{4.17}$$

We transform the sum

$$\begin{aligned}&\frac{\partial}{\partial q_2}\left(Q_2^* M \frac{\partial \Lambda}{\partial q_1}\right) + i\frac{\partial}{\partial q_1}\left(Q_1^* M \frac{\partial \Lambda}{\partial q_2}\right)\\ &= i\frac{\partial}{\partial q_1}\left[(Q_1^* + iQ_2^*)M\frac{\partial \Lambda}{\partial q_2} - iQ_2^* M \frac{\partial \Lambda}{\partial q_2}\right]\\ &\quad - i\frac{\partial}{\partial q_2}\left[(Q_1^* + iQ_2^*)M\frac{\partial \Lambda}{\partial q_1} - Q_1^* M \frac{\partial \Lambda}{\partial q_1}\right]\\ &= \frac{\partial}{\partial q_1}\left(Q_2^* M \frac{\partial \Lambda}{\partial q_2}\right) + i\frac{\partial}{\partial q_2}\left(Q_1^* M \frac{\partial \Lambda}{\partial q_1}\right)\\ &\quad - i(c_1 + ic_2)\left[\frac{\partial}{\partial q_2}\left(M\frac{\partial \Lambda}{\partial q_1}\right) - \frac{\partial}{\partial q_1}\left(M\frac{\partial \Lambda}{\partial q_2}\right)\right].\end{aligned} \tag{4.18}$$

Here we used the identity

$$Q_1^* = iQ_2^* = c_1 + ic_2$$

(Lemma 3). Since $M\Lambda \equiv 1$, the last term in (4.18) is equal to zero. Using this remark the sum of the first two terms in (4.17) takes the following form:

$$\frac{\partial}{\partial q_1}\left(Q_1^* M \frac{\partial \Lambda}{\partial q_1} + Q_2^* M \frac{\partial \Lambda}{\partial q_2}\right) + i\frac{\partial}{\partial q_2}\left(Q_1^* M \frac{\partial \Lambda}{\partial q_1} + Q_2^* M \frac{\partial \Lambda}{\partial q_2}\right).$$

However, using (4.9),

$$\begin{aligned}Q_1^* M \frac{\partial \Lambda}{\partial q_1} + Q_2^* M \frac{\partial \Lambda}{\partial q_2} &= \frac{\partial Q_1^* M\Lambda}{\partial q_1} + \frac{\partial Q_2^* M\Lambda}{\partial q_2} - \Lambda\left(\frac{\partial Q_1^* M}{\partial q_1} + \frac{\partial Q_2^* M}{\partial q_2}\right)\\ &= \frac{\partial Q_1^*}{\partial q_1} + \frac{\partial Q_2^*}{\partial q_2}.\end{aligned} \tag{4.19}$$

Thus, we finally obtain

$$\frac{\partial}{\partial q_1}\left[\frac{1}{2}\frac{\partial Q_1^*}{\partial q_1} + \frac{1}{2}\frac{\partial Q_2^*}{\partial q_2} + K_1^* + iK_2^*\right] + i\frac{\partial}{\partial q_2}\left[\frac{1}{2}\frac{\partial Q_1^*}{\partial q_1} + \frac{1}{2}\frac{\partial Q_2^*}{\partial q_2} + K_1^* + iK_2^*\right] = 0.$$

Hence, by the Cauchy-Riemann theorem,

$$\frac{1}{2}\frac{\partial Q_1^*}{\partial q_1} + \frac{1}{2}\frac{\partial Q_2^*}{\partial q_2} + K_1^* + iK_2^*$$

is a holomorphic function of q_1+iq_2. Since it is bounded, it is constant. This proves the lemma.

Let F be a polynomial in p_1, p_2. For brevity we write $\partial F^*/\partial p_k$ instead of $(\partial F/\partial p_k)^*$. Obviously,

$$\frac{\partial^2 F^*}{\partial p_k\,\partial q_j}=\frac{\partial^2 F^*}{\partial q_j\,\partial p_k}.$$

Lemma 8. $\nu_1=\nu_2=0$.

Proof. Using (4.13), we differentiate (4.2) with respect to p_1 and (4.3) with respect to p_2, and substitute $p_1=1$, $p_2=i$. As a result we obtain the relations

$$\begin{aligned}\sum Q_k^*\frac{\partial\Lambda}{\partial q_k}&+\sum\frac{\partial Q_k^*}{\partial p_1}\frac{\partial\Lambda}{\partial q_k}+\sum\frac{\partial Q_1^*}{\partial p_k}\frac{\partial\Lambda}{\partial q_k}\\&+2K_1^*\Lambda-\Lambda\frac{\partial Q_1^*}{\partial q_1}-\Lambda\frac{\partial^2 Q_1^*}{\partial q_1\,\partial p_1}-i\Lambda\frac{\partial^2 Q_1^*}{\partial q_2\partial p_1}=0,\end{aligned}\tag{4.20}$$

$$\begin{aligned}\sum Q_k^*\frac{\partial\Lambda}{\partial q_k}&+i\sum\frac{\partial Q_k^*}{\partial p_2}\frac{\partial\Lambda}{\partial q_k}+i\sum\frac{\partial Q_2^*}{\partial p_k}\frac{\partial\Lambda}{\partial q_k}\\&+2iK_2^*\Lambda-\Lambda\frac{\partial Q_2^*}{\partial q_2}-i\Lambda\frac{\partial^2 Q_2^*}{\partial q_2\partial p_2}-\Lambda\frac{\partial^2 Q_2^*}{\partial q_1\partial p_2}=0.\end{aligned}\tag{4.21}$$

Since Q_1 and Q_2 are homogeneous polynomials in the momenta of degree $n-1$, by Euler's theorem

$$\sum\frac{\partial Q_i}{\partial p_j}p_j=(n-1)Q_i,\qquad i=1,2.$$

Substituting $p_1=1$, $p_2=i$, we obtain the relations

$$\frac{\partial Q_k^*}{\partial p_1}+i\frac{\partial Q_k^*}{\partial p_2}=(n-1)Q_k^*,\qquad k=1,2.\tag{4.22}$$

We multiply equations (4.20), (4.21) by M, add them and use formulas (4.19) and (4.22). After simple transformations we obtain the relation

$$\begin{aligned}K_1^*+iK_2^*=&-\frac{1}{2}\frac{\partial Q_1^*}{\partial q_1}-\frac{1}{2}\frac{\partial Q_2^*}{\partial q_2}-\frac{n-1}{2}\frac{\partial Q_1^*}{\partial q_1}-\frac{n-1}{2}\frac{\partial Q_2^*}{\partial q_2}\\&+\frac{1}{2}\frac{\partial}{\partial q_1}\left(\frac{\partial Q_1^*}{\partial p_1}+\frac{\partial Q_2^*}{\partial p_2}\right)+\frac{i}{2}\frac{\partial}{\partial q_2}\left(\frac{\partial Q_1^*}{\partial p_1}+\frac{\partial Q_2^*}{\partial p_2}\right)\\&-\frac{1}{2}M\frac{\partial\Lambda}{\partial q_1}\left(\frac{\partial Q_1^*}{\partial p_1}+i\frac{\partial Q_2^*}{\partial p_1}\right)-\frac{1}{2}M\frac{\partial\Lambda}{\partial q_2}\left(\frac{\partial Q_1^*}{\partial p_2}+i\frac{\partial Q_2^*}{\partial p_2}\right).\end{aligned}\tag{4.23}$$

We set

$$r=\frac{\partial Q_1^*}{\partial p_2}-\frac{\partial Q_2^*}{\partial p_1}.$$

Using (4.22) we obtain the relations

$$\begin{aligned}\frac{\partial Q_1^*}{\partial p_1}+\frac{\partial Q_2^*}{\partial p_2}&=\frac{\partial Q_1^*}{\partial q_1}+i\frac{\partial Q_1^*}{\partial p_2}-i\left(\frac{\partial Q_2^*}{\partial p_1}+i\frac{\partial Q_2^*}{\partial p_2}\right)-ir\\&=(n-1)Q_1^*-i(n-1)Q_2^*-ir,\\\frac{\partial Q_1^*}{\partial p_1}+i\frac{\partial Q_2^*}{\partial p_1}&=(n-1)Q_1^*-ir,\\\frac{\partial Q_1^*}{\partial p_2}+i\frac{\partial Q_2^*}{\partial p_2}&=(n-1)Q_2^*+r.\end{aligned}$$

Substituting these relations in (4.23) and using (4.19), we obtain
(4.24)

$$K_1^* + iK_2^* = -\frac{1}{2}\frac{\partial Q_1^*}{\partial q_1} - \frac{1}{2}\frac{\partial Q_2^*}{\partial q_2} - \frac{n-1}{2}\frac{\partial}{\partial q_1}(Q_1^* + iQ_2^*) + \frac{n-1}{2}i\frac{\partial}{\partial q_2}(Q_1^* + iQ_2^*) - \frac{i}{2}\frac{\partial r}{\partial q_1} + \frac{1}{2}\frac{\partial r}{\partial q_2} + \frac{i}{2}M\frac{\partial \Lambda}{\partial q_1}r - \frac{1}{2}M\frac{\partial \Lambda}{\partial q_2}r.$$

By Lemma 3,

$$Q_1^* + iQ_2^* \equiv \text{const}.$$

It is easy to verify that the sum of the last four terms in (4.24) is equal to

$$-\frac{i}{2}\Lambda\left(\frac{\partial rM}{\partial q_1} + i\frac{\partial rM}{\partial q_2}\right).$$

As a result, (4.24) takes the form (after using (4.14))

$$-\frac{i}{2}\Lambda\left(\frac{\partial rM}{\partial q_1} + i\frac{\partial rM}{\partial q_2}\right) = \nu_1 + i\nu_2$$

or

$$-\frac{i}{2}\Lambda\left(\frac{\partial rM}{\partial q_1} + i\frac{\partial rM}{\partial q_2}\right) = (\nu_1 + i\nu_2)M.$$

Averaging over the two-dimensional torus, we obtain the relation

$$\frac{\nu_1 + i\nu_2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} M\,dq_1\,dq_2 = 0.$$

Since $M > 0$, it follows that $\nu_1 = \nu_2 = 0$, and the lemma is proved.

At the same time we have obtained the equality

$$\frac{\partial rM}{\partial q_1} + i\frac{\partial rM}{\partial q_2} = 0,$$

which is the condition for the function rM to be holomorphic. By Liouville's theorem,

$$rM = c_3 + ic_4.$$

Thus, the following result holds.

Lemma 9.

$$\frac{\partial Q_1^*}{\partial p_2} - \frac{\partial Q_2^*}{\partial p_1} = (c_3 + ic_4)\Lambda. \tag{4.25}$$

We note in conclusion that equality (4.14) (taking Lemma 8 into account) can be rewritten in the following equivalent form:

$$\frac{\partial Q_1^*}{\partial q_1} + \frac{\partial Q_2^*}{\partial q_2} + \frac{\partial P_1^*}{\partial p_1} + \frac{\partial P_2^*}{\partial p_2} = 0. \tag{4.26}$$

We summarize the results of this section:

$$P_1^* = P_2^* = 0, \qquad Q_1^* + iQ_2^* = c_1 + ic_2;$$

moreover, the functions Q_1^* and Q_2^* satisfy equations (4.9), (4.12), (4.25), and (4.26).

5. Proof of Theorem 2

For $n = 1$ the functions Q_k do not depend on the momenta (so that $Q_k^* = Q_k$), and

$$P_k = a_k p_1 + b_k p_2, \qquad k = 1, 2.$$

By Lemma 4,

$$P_k^* = a_k + ib_k \equiv 0.$$

Hence,

$$a_k = b_k \equiv 0 \quad \text{and} \quad P_k \equiv 0 \qquad (k = 1, 2).$$

Since the Q_k are real functions, then, by Lemma 3,

$$Q_k \equiv c_k = \text{const}; \qquad k = 1, 2.$$

The field u with differentiation operator (Lie derivative)

$$L_u = c_1 \frac{\partial}{\partial q_1} + c_2 \frac{\partial}{\partial q_2}$$

is Hamiltonian; its Hamiltonian is the linear function

$$F = c_1 p_1 + c_2 p_2,$$

which is an integral of the equations of motion.

By (4.9), the function $M = 1/\Lambda$ satisfies the equation

$$c_1 \frac{\partial M}{\partial q_1} + c_2 \frac{\partial M}{\partial q_2} = 0.$$

Since $u \neq 0$, it follows that $c_1^2 + c_2^2 \neq 0$.

If the quotient c_1/c_2 is irrational, then $M \equiv \text{const}$. Let $c_1/c_2 = k/l$, where k and l are relatively prime integers. In this case M is a function only of the variable

$$\varphi_1 = lq_1 - kq_2. \tag{5.1}$$

By Bézout's theorem, there are two integers r and s such that

$$kr + ls = 1.$$

We set

$$\varphi_2 = rq_1 + sq_2. \tag{5.2}$$

The relations (5.1), (5.2) define an automorphism of the two-dimensional torus. Extending this transformation to a canonical transformation, we arrive at the case when the Hamiltonian H does not depend on the angular coordinate φ_2. Thus, if there is a nontrivial field of symmetries of degree one, then there is an ignorable cyclic coordinate.

6. Proof of Theorem 3

For $n = 2$ the functions Q_1 and Q_2 are linear in the momenta:

$$Q_k = a_k p_1 + b_k p_2, \qquad k = 1, 2.$$

It is clear that

$$\frac{\partial Q_1^*}{\partial p_2} = b_1, \qquad \frac{\partial Q_2^*}{\partial p_1} = a_2.$$

Hence, $\sigma = b_1 - a_2$, where σ is the function from Lemma 5. By (4.25),

$$\sigma M = c_3 + ic_4 \equiv \text{const}, \qquad \Delta(\sigma M) = 0.$$

Thus, equation (4.12) is simplified:

$$2c_1 \frac{\partial^2 M}{\partial q_1 \partial q_2} = c_2 \left(\frac{\partial^2 M}{\partial q_1^2} - \frac{\partial^2 M}{\partial q_2^2} \right). \tag{6.1}$$

We assume that

$$c_1^2 + c_2^2 \neq 0 .$$

The case $c_1 = c_2 = 0$ will be considered below. We use Fourier's method. Let

$$M = \sum M_{m_1 m_2} e^{i(m_1 q_1 + m_2 q_2)}, \qquad (m_1, m_2) \in \mathbb{Z}^2 .$$

Then from (6.1) we obtain a chain of relations

$$[2c_1 m_1 m_2 - c_2(m_1^2 - m_2^2)] M_{m_1 m_2} = 0 . \tag{6.2}$$

We consider the set

$$B = \{(m_1, m_2) \in \mathbb{Z}^2 : M_{m_1 m_2} \neq 0\} .$$

In view of (6.2) for points of B we have the equality

$$\frac{m_1 m_2}{m_1^2 - m_2^2} \equiv \text{const} .$$

Let (n_1, n_2) be another point of B. Since

$$\frac{m_1 m_2}{m_1^2 - m_2^2} = \frac{n_1 n_2}{n_1^2 - n_2^2},$$

it follows that either

$$1) \quad m_1 n_2 = m_2 n_1 = 0$$

or

$$2) \quad m_1 n_1 + m_2 n_2 = 0 .$$

In the first case the points lie on one line passing through the origin, and in the second case these points lie on two lines l_1, l_2 which intersect orthogonally at the origin.

Let

$$(m_1^0, m_2^0) \neq (0, 0)$$

be a point of the integer lattice $\mathbb{Z}^2$, lying on l_1 and closest to the origin. It is clear that all points of $l_1 \cap \mathbb{Z}^2$ have the form

$$(m_1, m_2) = (\lambda m_1^0, \lambda m_2^0), \qquad \lambda \in \mathbb{Z} .$$

Since $(m_2^0, -m_1^0)$ is the point closest to the origin of

$$(l_2 \cap \mathbb{Z}^2) \setminus \{0, 0\},$$

then all the points of $l_2 \cap \mathbb{Z}^2$ have the form

$$(m_1, m_2) = (\lambda m_2^0, -\lambda m_1^0), \qquad \lambda \in \mathbb{Z} .$$

Hence,

$$\begin{aligned} M &= \sum_{(m_1, m_2) \in l_1} M_{m_1 m_2} e^{i(m_1 q_1 + m_2 q_2)} + \sum_{(m_1, m_2) \in l_2} M_{m_1 m_2} e^{i(m_1 q_1 + m_2 q_2)} \\ &= \sum_\lambda M_{\lambda m_1^0, \lambda m_2^0} e^{i\lambda(m_1^0 q_1 + m_2^0 q_2)} + \sum_\lambda M_{\lambda m_2^0, -\lambda m_1^0} e^{i\lambda(m_2^0 q_1 - m_1^0 q_2)} \\ &= f(m_1^0 q_1 + m_2^0 q_2) + g(m_2^0 q_1 - m_1^0 q_2), \end{aligned}$$

where f and g are some 2π-periodic functions.

We pass to new angular coordinates $x_1, x_2 \bmod 2\pi$ according to the formulas

$$x_1 = m_1^0 q_1 + m_2^0 q_2, \qquad x_2 = m_2^0 q_1 - m_1^0 q_2 .$$

We extend this transformation to a canonical transformation $q, p \to x, y$, by setting

$$[(m_1^0)^2 + (m_2^0)^2]y_1 = m_1^0 p_1 + m_2^0 p_2,$$
$$[(m_1^0)^2 + (m_2^0)^2]y_2 = m_2^0 p_1 - m_1^0 p_2.$$

In the new variables x, y the Hamiltonian (1.4) takes the following form:

$$H = \frac{(m_1^0)^2 + (m_2^0)^2}{2[f(x_1) + g(x_2)]}(y_1^2 + y_2^2). \tag{6.3}$$

Thus, the variables x and y are separated. The function (6.3) is the Hamiltonian of a Liouville system with two degrees of freedom. Equation (6.1) appeared in [4] in connection with the problem of the existence of a quadratic integral.

Thus, we may assume that

$$M = F(q_1) + G(q_2),$$

where F and G are 2π-periodic coordinates. Moreover, in correspondence with the assumption of the main theorem (§1), $M \neq \text{const}$.

Equation (6.1) gives us that

$$c_2(F'' - G'') = 0.$$

Here the primes denote the derivative of a function of one variable. Since the functions F and G are periodic and at least one of them is not constant, obviously $c_2 = 0$. Corresponding to the assertion of Lemma 3,

$$a_1 - b_2 = c_1, \qquad a_2 + b_1 = 0. \tag{6.4}$$

Moreover,

$$\sigma M = c_3 + ic_4 = \text{const}.$$

Since the function $\sigma = b_1 - a_2$ is real, $c_4 = 0$. Using equality (6.4) we conclude that

$$c_3 = -2a_2 M = 2b_1 M. \tag{6.5}$$

We now use equality (4.9). From it we get the two relations

$$\frac{\partial M a_1}{\partial q_1} + \frac{\partial M a_2}{\partial q_2} = 0, \qquad \frac{\partial M b_1}{\partial q_1} + \frac{\partial M b_2}{\partial q_2} = 0.$$

Using (6.5) we arrive at the equalities

$$M b_2 = \varphi_1(q_1), \qquad M a_1 = \varphi_2(q_2), \tag{6.6}$$

where φ_1 and φ_2 are periodic functions of one variable.

Using the first equality of (6.4), we obtain a chain of equalities

$$c_1 F + c_1 G = c_1 M = (a_1 - b_2)M = \varphi_2(q_2) - \varphi_1(q_1),$$

from which we get

$$\varphi_1 = -c_1 F + c_5, \qquad \varphi_2 = c_1 G + c_5,$$

where c_5 is some constant that can be set equal to zero. In fact, we set

$$F = \widetilde{F} + \frac{c_5}{c_1}, \qquad G = \widetilde{G} - \frac{c_5}{c_1}.$$

Then

$$\varphi_1 = -c_1 \widetilde{F}, \qquad \varphi_2 = c_1 \widetilde{G}, \qquad \widetilde{F} + \widetilde{G} = F + G.$$

Recalling that $M = 1/\Lambda$, from (6.5) and (6.6) we obtain the desired equalities

$$\begin{gathered} a_1 = c_1 G\Lambda, \qquad a_2 = -c_0\Lambda, \\ b_1 = c_0\Lambda, \qquad b_2 = -c_1 F\Lambda, \qquad c_0 = c_3/2. \end{gathered} \tag{6.7}$$

Finally, using equality (4.14) and Lemma 8 we obtain

$$\frac{1}{2}\frac{\partial}{\partial q_1}(a_1 + ib_1) + \frac{1}{2}\frac{\partial}{\partial q_2}(a_2 + ib_2) + K_1^* + iK_2^* = 0.$$

Since $n = 2$, K_1^* and K_2^* are real functions. Hence,

$$\begin{aligned} K_1^* &= -\frac{1}{2}\left(\frac{\partial a_1}{\partial q_1} + \frac{\partial a_2}{\partial q_2}\right) = \frac{c_0}{2}\frac{\partial \Lambda}{\partial q_2} - \frac{1}{2}c_1 G\frac{\partial \Lambda}{\partial q_1}, \\ K_2^* &= -\frac{1}{2}\left(\frac{\partial b_1}{\partial q_1} + \frac{\partial b_2}{\partial q_2}\right) = -\frac{c_0}{2}\frac{\partial \Lambda}{\partial q_1} + \frac{1}{2}c_1 F\frac{\partial \Lambda}{\partial q_2}. \end{aligned} \tag{6.8}$$

Equalities (6.7) and (6.8) lead to the final formulas for the field of symmetries u:

$$\begin{gathered} q_1' = c_1 G\Lambda p_1 + c_0\Lambda p_2, \qquad q_2' = -c_0\Lambda p_1 - c_1 F\Lambda p_2, \\ p_1' = -\frac{1}{2}\left(c_1 G\frac{\partial \Lambda}{\partial q_1} - c_0\frac{\partial \Lambda}{\partial q_2}\right)(p_1^2 + p_2^2), \\ p_2' = \frac{1}{2}\left(c_1 F\frac{\partial \Lambda}{\partial q_2} - c_0\frac{\partial \Lambda}{\partial q_1}\right)(p_1^2 + p_2^2). \end{gathered} \tag{6.9}$$

Hence, the field u can be represented as the sum of two fields:

$$c_1 u_1 = c_0 u_2.$$

It is easy to check that u_1 is a Hamiltonian field of symmetries with Hamiltonian

$$\Phi = \frac{Gp_1^2 - Fp_2^2}{2(F+G)}.$$

The vector field u_2 must also be a field of symmetries. It is easy to check that the condition that the fields v and u_2 commute is the equality

$$\Lambda\Delta\Lambda - \left(\frac{\partial \Lambda}{\partial q_1}\right)^2 - \left(\frac{\partial \Lambda}{\partial q_2}\right)^2 = 0. \tag{6.10}$$

Since $\Lambda > 0$, we can set $N = \ln \Lambda$. Equation (6.10) is equivalent to the equation $\Delta N = 0$. Since a bounded harmonic function is constant, $\Lambda = \text{const}$. Hence, if Λ is not a constant function, then $c_0 = 0$.

It remains to consider the degenerate case, when

$$c_1 = c_2 = 0.$$

We shall show that in this case the field of symmetries is collinear to the Hamiltonian vector field v.

Indeed, the relation

$$Q_1^* + iQ_2^* = 0$$

leads to the two equalities

$$a_1 - b_2 = a_2 + b_1 = 0. \tag{6.11}$$

Using the relations

$$P_1^* = P_2^* = 0,$$

we find that the functions Q_k and P_k have the following form:

$$\begin{gathered} Q_1 = a_1 p_1 - a_2 p_2, \qquad Q_2 = a_2 p_1 + a_1 p_2, \\ P_k = \xi_k(p_1^2 + p_2^2), \qquad k = 1, 2. \end{gathered}$$

Furthermore, equalities (4.26) and (6.11) give us that

$$\frac{\partial a_1}{\partial q_1}+\frac{\partial a_2}{\partial q_2}+2\xi_1=0, \qquad \frac{\partial a_1}{\partial q_2}-\frac{\partial a_2}{\partial q_1}+2\xi_2=0.$$

Moreover, from (4.25) and (6.11) we obtain the relation

$$a_2=\alpha_2\Lambda, \qquad \alpha_2=\text{const}. \tag{6.12}$$

The condition that the fields u, v commute leads to two equations (see (4.9))

$$\frac{\partial}{\partial q_1}\frac{a_1}{\Lambda}+\frac{\partial}{\partial q_2}\frac{a_2}{\Lambda}=-\frac{\partial}{\partial q_1}\frac{a_2}{\Lambda}+\frac{\partial}{\partial q_2}\frac{a_1}{\Lambda}=0.$$

Using (6.12), they reduce to the following form:

$$\frac{\partial}{\partial q_1}\frac{a_1}{\Lambda}=\frac{\partial}{\partial q_2}\frac{a_1}{\Lambda}=0.$$

Hence, $a_1=\alpha_1\Lambda$, where $\alpha_1=\text{const}$.

Thus, the field of symmetries u has the form

(6.13)

$$q_1'=\Lambda(\alpha_1 p_1-\alpha_2 p_2), \qquad q_2'=\Lambda(\alpha_2 p_1+\alpha_1 p_2),$$

$$p_1'=-\frac{1}{2}\left(\alpha_1\frac{\partial\Lambda}{\partial q_1}+\alpha_2\frac{\partial\Lambda}{\partial q_2}\right)(p_1^2+p_2^2), \qquad p_2'=-\frac{1}{2}\left(\alpha_1\frac{\partial\Lambda}{\partial q_2}-\alpha_2\frac{\partial\Lambda}{\partial q_1}\right)(p_1^2+p_2^2).$$

The terms containing α_1 give a field proportional to the original field v. Therefore, one can set $\alpha_1=0$. But then (6.13) will coincide with the field (6.9), in which it is necessary to set $c_1=0$. However, as shown above, it commutes with the field v only for $\alpha_2=0$.

This completes the proof of Theorem 3.

7. Proof of Theorem 1

We now consider a Hamiltonian field of symmetries generated by a homogeneous Hamiltonian of degree m:

$$F=f_{m,0}p_1^m+f_{m-1,1}p_1^{m-1}p_2+\cdots+f_{0,m}p_2^m.$$

Since $Q_k=\partial F/\partial p_k$ $(k=1,2)$, it follows that

$$Q_1^*=mf_{m,0}+(m-1)f_{m-1,1}i+(m-2)f_{m-2,2}i^2+\cdots,$$
$$Q_2^*=f_{m-1,1}+2f_{m-2,2}i+3f_{m-3,3}i^2+\cdots.$$

We recall (see §4) that $\sigma=\Psi_1-\Phi_2$, where

$$Q_1^*=\Phi_1+i\Psi_1, \qquad Q_2^*=\Phi_2+i\Psi_2.$$

Hence,

$$\sigma=(m-2)f_{m-1,1}-(m-6)f_{m-3,3}+\cdots.$$

Suppose that one of the following conditions holds:

a) the function F is even with respect to p_1 and p_2;

b) F is even with respect to p_2 and odd with respect to p_1.

Then obviously

$$f_{m-1,1}=f_{m-3,3}=\cdots=0$$

and, in particular, $\sigma=0$. But then, according to (4.12), the function $M=\Lambda^{-1}$ satisfies equation (6.1). If $c_1^2+c_2^2\neq 0$, then, as proved in §6, Hamilton's equations admit an integral of degree not greater than two.

We now consider the case $c_1 = c_2 = 0$. Then by Lemma 3

$$Q_1^* + iQ_2^* = \left(\frac{\partial F}{\partial p_1} p_1 + \frac{\partial F}{\partial p_2} p_2\right)^* = mF^* = 0.$$

Hence (Lemma 6), $F = H\Phi$, where Φ is a homogeneous integral of degree $m-2$, having the form a) or b). Applying on decreasing induction m, we arrive at an integral of degree ≤ 2.

In the case when F is even with respect to p_1 and odd with respect to p_2, the function σ, as a rule, is different from zero. However, this case reduces to case b) by a simple renaming of the variables.

8. Proof of Theorem 4

Using the results of §5, a field of symmetries of degree one of an irreversible system has the following form:

$$q_1' = c_1, \quad q_2' = c_2, \quad p_1' = \zeta_1, \quad p_2' = \zeta_2, \tag{8.1}$$

where c_1, c_2 are some constants, and ζ_1 and ζ_2 are functions on the two-dimensional torus. Here, according to §5, the function Λ satisfies the equality

$$c_1 \frac{\partial \Lambda}{\partial q_1} + c_2 \frac{\partial \Lambda}{\partial q_2} = 0.$$

It is easy to check that the conditions for the commutation of the original Hamiltonian vector field v with the field (8.1) lead to the equalities

$$\zeta_1 = \zeta_2 = 0, \qquad c_1 \frac{\partial \lambda}{\partial q_1} + c_2 \frac{\partial \lambda}{\partial q_2} = 0.$$

Thus, the form of a field of symmetries of degree one is the same in both the reversible and irreversible cases. Here the functions λ and Λ satisfy the same equation. If c_1/c_2 is irrational, then $\Lambda = \text{const}$, $\lambda = \text{const}$. Otherwise (when $c_1/c_2 = k/l$, $k, l \in \mathbb{Z}$) we can pass to the new angular coordinates φ_1 and φ_2 according to formulas (5.1) and (5.2). It is clear that the right-hand sides of the equations (3.1) will not depend on the coordinate φ_2, and the field of symmetries has the simplest form

$$\varphi_1' = 0, \quad \varphi_2' = 1, \quad \psi_1' = 0, \quad \psi_2' = 0. \tag{8.2}$$

Here ψ_1 and ψ_2 are canonical variables conjugate to φ_1, φ_2. Is the vector field (8.2) Hamiltonian? This problem reduces to the solvability of the equations (see (3.1))

$$\begin{aligned} 0 &= \frac{\partial \Phi}{\partial \psi_1}, & 1 &= \frac{\partial \Phi}{\partial \psi_2}, \\ 0 &= -\frac{\partial \Phi}{\partial \varphi_1} + \lambda(\varphi_1)\frac{\partial \Phi}{\partial \psi_2}, & 0 &= -\frac{\partial \Phi}{\partial \varphi_2} - \lambda(\varphi_1)\frac{\partial \Phi}{\partial \psi_1}. \end{aligned} \tag{8.3}$$

Obviously, $\Phi = \psi_2 + a$, where a is an as yet unknown function on the torus $\mathbb{T}^2 = \{\varphi_1, \varphi_2 \bmod 2\pi\}$. From the last two equations of the system (8.3) we obtain the desired relations

$$\frac{\partial a}{\partial \varphi_1} = \lambda(\varphi_1), \qquad \frac{\partial a}{\partial \varphi_2} = 0.$$

Hence, a is a function only of φ_1, and $a' = \lambda$. Hence

$$a = \langle \lambda \rangle \varphi_1 + A,$$

where A is a uniquely determined function on the torus, and

$$\langle\lambda\rangle = \frac{1}{4\pi^2}\int_{\mathbf{T}^2} \lambda\, d\varphi_1\, d\varphi_2 .$$

Thus, in the irreversible case a field of symmetries is locally Hamiltonian. The Hamiltonian Φ will be a uniquely determined function in phase space if $\langle\lambda\rangle = 0$.

9. Proof of Theorem 5

We consider a field of third degree $n \geq 2$, defined by the equations

$$q_1' = Q_1 + S_1 + \cdots, \qquad q_2' = Q_2 + S_2 + \cdots,$$
$$p_1' = P_1 + R_1 + T_1 + \cdots, \qquad p_2' = P_2 + R_2 + T_2 + \cdots .$$

Here $P_k, R_k, Q_k, S_k, T_k, \ldots$ $(k = 1, 2)$ are homogeneous polynomials in the momenta p_1, p_2 of degrees $n, n-1, n-1, n-2, n-2, \ldots$, respectively. The dots denote terms of degree $\leq n-2$.

Setting the coefficient of $\partial/\partial p_1$ in the commutator of the operators L_v, L_u equal to zero, we extract the homogeneous terms of degree n and then set $p_1 = 1$, $p_2 = i$. As a result we obtain the relation

$$\begin{aligned} &Q_1^* \frac{\partial \lambda\Lambda}{\partial q_1} i + Q_2^* \frac{\partial \lambda\Lambda}{\partial q_2} i - R_1^* \frac{\partial \Lambda}{\partial q_1} - R_2^* \frac{\partial \Lambda}{\partial q_1} i + P_2^* \lambda\Lambda \\ &\quad - \Lambda \frac{\partial R_1^*}{\partial q_1} - \Lambda i \frac{\partial R_1^*}{\partial q_2} - i\lambda\Lambda \left(\frac{\partial P_1^*}{\partial p_1} + i \frac{\partial P_1^*}{\partial p_2} \right) = 0. \end{aligned} \tag{9.1}$$

By Euler's formula for homogeneous functions, the expression in parentheses is equal to nP_1^*. By Lemma 4, $P_1^* = P_2^* = 0$. Therefore, equation (9.1) takes the form

$$Q_1^* \frac{\partial \lambda\Lambda}{\partial q_1} i + Q_2^* \frac{\partial \lambda\Lambda}{\partial q_2} i - \frac{\partial R_1^* \Lambda}{\partial q_1} - R_2^* \frac{\partial \Lambda}{\partial q_1} i - \Lambda \frac{\partial R_1^*}{\partial q_2} i = 0. \tag{9.2}$$

Setting the coefficient of $\partial/\partial p_2$ equal to zero, we obtain the analogous equation

$$-Q_1^* \frac{\partial \lambda\Lambda}{\partial q_1} - Q_2^* \frac{\partial \lambda\Lambda}{\partial q_2} - i \frac{\partial R_2^* \Lambda}{\partial q_2} - R_1^* \frac{\partial \Lambda}{\partial q_2} - \Lambda \frac{\partial R_2^*}{\partial q_1} = 0. \tag{9.3}$$

We multiply (9.3) by i and add it to (9.2):

$$\frac{\partial (R_1^* + iR_2^*)\Lambda}{\partial q_1} + i \frac{\partial (R_1^* + iR_2^*)\Lambda}{\partial q_2} = 0.$$

This is the Cauchy-Riemann condition for the function $(R_1^* + iR_2^*)\Lambda$ to be holomorphic. Since it is bounded, it is constant:

$$\Lambda(R_1^* + iR_2^*) = \delta_1 + i\delta_2 = \text{const}. \tag{9.4}$$

Now setting the homogeneous terms of degree $n-1$ equal to zero and making analogous transformations, we obtain the relation

$$\frac{\partial (T_1^* + iT_2^*)\Lambda}{\partial q_1} + i \frac{\partial (T_1^* + iT_2^*)\Lambda}{\partial q_2} + \lambda n i [\Lambda(R_1^* + iR_2^*)] = 0. \tag{9.5}$$

If $\delta_1 + i\delta_2 \neq 0$, then by (9.4) the mean value of the function λ over the two-dimensional torus is equal to zero.

We now consider the case when $\delta_1 = \delta_2 = 0$, i.e.,

$$R_1^* + iR_2^* = 0. \tag{9.6}$$

Now setting the terms of degree $n-1$ in the coefficients of $\partial/\partial q_1$ and $\partial/\partial q_2$ equal to zero and using relation (9.6), we obtain the equation

$$\frac{\partial(S_1^* + iS_2^*)}{\partial q_1} + i\frac{\partial(S_1^* + iS_2^*)}{\partial q_2} + \lambda i(n-1)(c_1 + ic_2) = 0. \tag{9.7}$$

If $c_1 + ic_2 \neq 0$, then the mean value of λ is equal to zero. Since for $p_1 - 1$, $p_2 = i$ the functions P_1, P_2 vanish, the highest homogeneous pieces of the vector fields v and u at the point $p_1 = 1$, $p_2 = i$ are equal, respectively, to

$$(\Lambda, \Lambda i, 0, 0) \quad \text{and} \quad (Q_1^*, Q_2^*, 0, 0).$$

These vectors are linearly independent if

$$Q_1^* i - Q_2^* = i(Q_1^* + iQ_2^*) = i(c_1 + ic_2) \neq 0.$$

This completes the proof of the theorem.

Remark. By induction one can show that if the mean value of the function λ over the torus is different from zero, then

$$Q_1^* = iQ_2^* = S_1^* + iS_2^* = \cdots = 0.$$

If the field of symmetries is Hamiltonian, with the Hamiltonian $F_m + F_{m-1} + \cdots$, then it follows from this that all the polynomials F_k $(k \leq m)$ are divisible by H. Hence, in turn, we get the assertion of Bolotin's theorem [8].

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