KEPLER'S PROBLEM IN CONSTANT CURVATURE SPACES

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Abstract. In this article the generalization of the motion of a particle in a central field to the case of a constant curvature space is investigated. We found out that orbits on a constant curvature surface are closed in two cases: when the potential satisfies Laplace-Beltrami equation and can be regarded as an analogue of the potential of the gravitational interaction, and in the case when the potential is the generalization of the potential of an elastic spring. Also the full integrability of the generalized two-centre problem on a constant curvature surface is discovered and it is shown that integrability remains even if elastic "forces" are added.

Key words: Central field, closed orbits, spheroconical coordinates.

1. Introduction

The potential of the gravitational interaction has the two following fundamental properties. On the one hand it is a harmonic function in three-dimensional space (i.e. satisfies Laplace's equation), on the other hand only this potential (and the potential of an elastic spring) generates the central field where all bounded orbits are closed (Bertrand's theorem, see for example [5]). It appears that even in the more general situation of the motion in a constant curvature space these properties remain.

As a matter of convenience we will examine the motion on a three-dimensional sphere of unit radius. The case of Lobachevski's space can be investigated in the same way.

Let a material particle m of unit mass move in a field of force with the potential V depending only on the distance between the particle and some fixed point M of the three-dimensional sphere. This problem may be seen as an analogue of the classical problem of motion in a central field. Let ϑ be the length of the arc of the great circle, connecting the points m and M; ϑ is measured in radians. Then V is a function depending only on the angle ϑ. Laplace equation is replaced by Laplace-Beltrami equation:

$$\Delta V = \sin^{-2} \vartheta \frac{\partial}{\partial \vartheta} \left(\sin^2 \vartheta \frac{\partial V}{\partial \vartheta} \right) = 0 .$$

Its solution is

$$V = -\gamma \frac{\cos \vartheta}{\sin \vartheta} + \alpha , \qquad \alpha, \gamma = \text{const} . \tag{1}$$

The constant α is unessential. To be more definite let us assume that $\gamma > 0$. The parameter γ plays the role of the gravitational constant.

In addition to the attracting center at the point M this field has a repulsive center at the antipodal point M'. If we consider this field of force as a stationary field of fluid's velocities, then the value of the flow through the boundary of any closed region, which does not contain attracting or repulsive centers, equals zero. These two singularities M and M' can be treated as a source and a sink.

In the general situation, when V is an arbitrary function of ϑ, the trajectories of m lie on two-dimensional spheres containing points M and M'. Indeed,let us imbed the three-dimensional sphere S^3 into a four-dimensional euclidean space. The force which acts on the point m lies in the two-dimensional plane π, passing through three points m, M, and M'. Let us consider the three-dimensional plane Π which contains π and the plane passing through the point m parallel to its initial velocity. Since the force and the initial velocity belong to Π and since the equations of motion remain invariant under the reflection relative to the plane Π, then the particle m never leaves this plane. The intersection of S^3 and Π is the two-dimensional sphere sought for.

2. The Generalized Bertrand's Problem

Now let us consider the generalized Bertrand's problem: among all analytical potentials it is necessary to find the one in the field of which all the orbits of the particle m on a two-dimensional sphere are closed.

Relative to the spherical coordinates ϑ, φ, Lagrangian L is given by

$$L = \tfrac{1}{2}\left(\dot\vartheta^2 + \sin^2\vartheta\,\dot\varphi^2\right) - V(\vartheta)\,.$$

It is obvious that the coordinate φ is ignorable. Let us introduce the constant of the angular momentum integral

$$\sin^2\vartheta\,\dot\varphi = C\,,$$

and write out Routh's function:

$$R_C = \frac{\dot\vartheta^2}{2} - W_C\,, \qquad \text{where } W_C = \frac{C^2}{2\sin^2\vartheta} + V(\vartheta)\,.$$

Variation of ϑ is described by the Lagrange's equation with Lagrangian R:

$$\ddot\vartheta + W'_\vartheta = 0\,. \tag{2}$$

Let us consider the motion with energy

$$\frac{\dot\vartheta^2}{2} + W_C = h\,.$$

Let σ be the tangent plane for the sphere at the point M, O be the center of the sphere, m' be the point of intersection of the line Om and the plane σ. Let us put $\operatorname{tg}(\vartheta) = r$, then for $0 < \vartheta < \pi/2$ we have $r = |Mm'|$. It appears that the trajectory of m' coincides with the trajectory of a particle in the central field with the potential $U(r)$, where $U(\operatorname{tg}\vartheta) = V(\vartheta)$.

To prove it we will use Clairaut's method. Let $C \neq 0$, then the coordinate φ varies monotonically and it can be used as new time. Let us put

$$\rho = 1/r = \operatorname{ctg}(\vartheta)\,.$$

It is clear that

$$\dot\rho = -\dot\vartheta/\sin^2\vartheta\,, \quad \rho' = -\dot\vartheta/C\,, \quad \rho'' = -\ddot\vartheta\sin^2\vartheta/C^2\,, \tag{3}$$

where prime denotes the derivative with respect to φ.

Equation (2) with provision for identities (3) reduces to the equation

$$\rho'' + \rho = \frac{1}{C^2\rho^2}\,U'\Big(\frac{1}{\rho}\Big)\,,$$

and the energy integral can be rewritten as

$$\frac{C^2}{2}\,(\rho'^2 + \rho^2) + U\Big(\frac{1}{\rho}\Big) = h - \frac{C^2}{2}\,. \tag{4}$$

Thus, the equation for orbits has the same form as in the case of a particle on a plane, moving in the central field of force with the potential $U(r)$. The only difference is that in the case of a plane, the constant h would appear alone in the right part of the energy integral (4).

According to Bertrand's theorem orbits will be closed in two cases:

$$U_1(r) = -\frac{\gamma}{r}\,, \quad U_2(r) = \frac{kr^2}{2}\,; \qquad k,\gamma > 0\,.$$

Relative to the original coordinates we are getting two potentials:

$$V_1(\vartheta) = -\gamma\operatorname{ctg}\vartheta\,, \quad V_2(\vartheta) = \frac{k\operatorname{tg}^2\vartheta}{2}\,; \qquad k,\gamma > 0$$

and if $h_1 - C_1^2/2 < 0$ then Bohlin's transform $\rho \to \rho^2, \varphi \to 2\varphi$ transfers trajectories of the problem with the potential V_1 into the trajectories of the problem with the potential V_2, where $k = -(h_1 - C_1^2/2)$ and $h_2 - C_2^2/2 = \gamma$.

It is natural enough to consider V_1 as the analogue of the potential of gravitational interaction and V_2 as the analogue of the potential of an elastic spring.

Let us note that the function V_1 is defined by the formula which coincides with

(1). In this case

$$r = \operatorname{tg}\vartheta = \frac{p}{1 + e\cos(\varphi - \varphi_0)},$$

where

$$p = C^2/\gamma,$$
$$e = \sqrt{1 + \frac{2C^2}{\gamma^2}\left(h - \frac{C^2}{2}\right)}.$$

When $0 \le e \le 1$ the trajectory of m' is an ellipse; when $e = 1$ the trajectory of m' is a hyperbola and the trajectory of m is tangent to the equator; when $e > 1$ the trajectory of m' is a parabola and the trajectory of m being closed intersects the equator.

Let us consider the cone with the generatrix passing through the trajectory of m'. Then the trajectory of m is the line of intersection of the cone and the sphere. This result can be regarded as an analogue of the first Kepler's law.

Just as in Kepler's problem the orbital period T depends only on the constant of energy. In fact, it can be calculated that

$$T = \frac{1}{C}\int_0^T \sin^2\vartheta(t)\,\frac{d\vartheta}{dt}\,dt = \int_{-\pi}^{\pi} \sin^2\vartheta(\varphi)d\varphi = \frac{p^2}{C}\int_{-\pi}^{\pi} \frac{d\varphi}{p^2 + (1 + e\cos\varphi)^2}$$
$$= \frac{\pi}{\sqrt{\gamma}}\sqrt{\frac{h}{\gamma} + \sqrt{\frac{h^2}{\gamma^2} + 1}}\left(\sqrt{\frac{h^2}{\gamma^2} + 1}\right)^{-1}.$$

Remark. According to Birkhoff [2] ellipses are boundaries of plane billiards with an additional integral being quadratic in regard to velocities. It appears that closed trajectories of Kepler's problem on two-dimensional constant curvature surfaces are boundaries of integrable billiards as well (see for example [1]).

3. The n-Body Problem

The above consideration leads to the natural generalization of the many-body problem: n material particles move in a three-dimensional constant curvature space and their interaction is generated by the potential (1). At the limit when the radius of curvature tends to infinity we have the classical n-body problem.

The two-body problem is especially interesting. As distinct from the plane case it can not be reduced to Kepler's problem. Orbits are much more likely not to be closed in this case. This circumstance allows us to investigate the advance of perihelion relative to the space curvature.

Investigation of particular solutions of the n-body problem and its simplifications is also a matter of considerable theoretical interest.

Since we have a six-parametric movement group in a constant curvature space it is reasonable enough to pose the problem of the rigid body motion (including the motion in ideal liquid). The equations of motion will have the form of Euler-Poincaré equations on Lie algebra of this group.

4. The Two-Centre Problem

It appeears that the generalized two-centre problem is integrable as well. It can be solved by the method of the separation of variables in the spheroconical coordinate system (cf. [4]).

Let us assume that the gravitating centers are placed in points with coordinates $(\alpha,\beta,0)$ and $(-\alpha,\beta,0)$; $\alpha > 0$, $\beta > 0$, $\alpha^2+\beta^2 = 1$. Let x, y, z be Descartes coordinates of the particle moving on the unit sphere. We define ξ and η as roots of the equation

$$f(\lambda^2) = \frac{x^2}{\lambda^2-\alpha^2} + \frac{y^2}{\lambda^2+\beta^2} + \frac{z^2}{\lambda^2} = \frac{(\lambda^2-\xi^2)(\lambda^2+\eta^2)}{(\lambda^2-\alpha^2)(\lambda^2+\beta^2)\lambda^2} . \tag{5}$$

It is clear that

$$0 < \xi^2 < \alpha^2 , \qquad 0 < \eta^2 < \beta^2 .$$

Variables ξ and η can be regarded as curvilinear coordinates on the sphere. Since

$$\frac{x^2}{\xi^2-\alpha^2} + \frac{y^2}{\xi^2+\beta^2} + \frac{z^2}{\xi^2} = 0 , \qquad \frac{x^2}{\eta^2-\alpha^2} + \frac{y^2}{\eta^2+\beta^2} + \frac{z^2}{\eta^2} = 0 ,$$

$$x^2+y^2+z^2 = 1 ,$$

it follows that the coordinate lines are the lines of intersection of the sphere and confocal cones. The formulas of coordinate transforms can be easily found from (5):

$$\begin{aligned} x^2 &= \operatorname{res}_{\alpha^2} f(\lambda^2) = (\alpha^2-\xi^2)(\alpha^2+\eta^2)/\alpha^2 , \\ y^2 &= \operatorname{res}_{-\beta^2} f(\lambda^2) = (\beta^2+\xi^2)(\beta^2-\eta^2)/\beta^2 , \\ z^2 &= \operatorname{res}_0 f(\lambda^2) = \xi^2\eta^2/\alpha^2\beta^2 . \end{aligned} \tag{6}$$

Coordinates ξ and η are orthogonal. Let p_ξ and p_η be canonically conjugate momenta. The kinetic energy T relative to the new coordinates is given by

$$T = \frac{1}{2}\left[\frac{(\alpha^2-\xi^2)(\beta^2+\xi^2)}{\xi^2+\eta^2}\,p_\xi^2 + \frac{(\alpha^2+\eta^2)(\beta^2-\eta^2)}{\xi^2+\eta^2}\,p_\eta^2\right].$$

The potential energy V is defined as

$$V = \gamma_+ \frac{\cos\vartheta_+}{\sin\vartheta_+} + \gamma_- \frac{\cos\vartheta_-}{\sin\vartheta_-} = \frac{\gamma_+ \cos\vartheta_+ \sin\vartheta_- + \gamma_- \cos\vartheta_- \sin\vartheta_+}{\sin\vartheta_+ \cos\vartheta_-},$$

where $\vartheta_\pm$ are the angles between the radii-vectors of gravitating centers and the moving particle; $\gamma_\pm = \mathrm{const}$.

It is clear that

$$\cos\vartheta_\pm = \pm\,\alpha x + \beta y = \sqrt{(\beta^2+\xi^2)(\beta^2-\eta^2)} \pm \sqrt{(\alpha^2-\xi^2)(\alpha^2+\eta^2)}\,.$$

Then

$$\sin\vartheta_\pm = \sqrt{(\alpha^2+\eta^2)(\beta^2+\xi^2)} \mp \sqrt{(\alpha^2-\xi^2)(\beta^2-\eta^2)}\,.$$

Further,

$$\sin\vartheta_+ \sin\vartheta_- = \xi^2+\eta^2\,,$$

$$\cos\vartheta_+ \sin\vartheta_- = \sqrt{(\alpha^2-\xi^2)(\beta^2+\xi^2)} + \sqrt{(\alpha^2+\eta^2)(\beta^2-\eta^2)}\,,$$

$$\sin\vartheta_+ \cos\vartheta_- = \sqrt{(\alpha^2-\xi^2)(\beta^2+\xi^2)} - \sqrt{(\alpha^2+\eta^2)(\beta^2-\eta^2)}\,.$$

Therefore,

$$V = \frac{u(\xi)+v(\eta)}{\xi^2+\eta^2}$$

and the variables have been separated.

Thus, the remarkable result of Euler that the two-centre problem is integrable holds true in constant non-zero curvature space as well. On the other hand there is a well known Lagrange's generalization of Euler's result: if the source of elastic attraction or repulsion is placed at the mean point between the gravitating centers, the equations of motion remain integrable (see [3]). Let us show that this Lagrange's theorem holds true in the non-zero curvature space.

For this end place the sources of elastic attraction or repulsion at the points with coordinates:

$$(\pm 1,0,0)\,,\quad (0,\pm 1,0)\,,\quad (0,0,\pm 1)\,. \tag{7}$$

Then the following terms should be added to the potential energy:

$$\frac{1}{2}\sum_{i=1}^{3} k_i \operatorname{tg}^2 \vartheta_i\,, \tag{8}$$

where k_i = const, ϑ_i are the angles between the radii-vectors of the points (7) and the particle. Let us show that this problem can be solved by the separation of spheroconical variables ξ and η.

In fact, the function (8) up to an unessential constant can be rewritten as

$$a/x^2 + b/y^2 + c/z^2\,, \qquad a, b, c = \text{const}\,.$$

Using formulas of coordinate transforms (6), we can find the expression for the potential of the elastic interaction relative to the spheroconical coordinates:

$$\frac{a\,\alpha^2}{(\alpha^2-\xi^2)(\alpha^2+\eta^2)} + \frac{b\,\beta^2}{(\beta^2+\xi^2)(\beta^2-\eta^2)} + \frac{c\,\alpha^2\beta^2}{\xi^2\eta^2} =$$
$$= \frac{1}{\xi^2+\eta^2}\Big[a\,\alpha^2\Big(\frac{1}{\alpha^2-\xi^2} - \frac{1}{\alpha^2+\eta^2}\Big) + b\,\beta^2\Big(\frac{1}{\beta^2-\eta^2} - \frac{1}{\beta^2+\xi^2}\Big) +$$
$$+\, c\,\alpha^2\beta^2\Big(\frac{1}{\xi^2} + \frac{1}{\eta^2}\Big)\Big]\,.$$

This identity shows that the variables ξ and η have been separated.

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