TENSOR INVARIANTS OF QUASIHOMOGENEOUS SYSTEMS OF DIFFERENTIAL EQUATIONS, AND THE KOVALEVSKAYA-LYAPUNOV ASYMPTOTIC METHOD

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1. Quasihomogeneous Systems, Tensor Conservation Laws. A system of n differential equations

$$\dot{x}^i = v^i(x^1, \dots, x^n), \quad 1 \leqslant i \leqslant n, \tag{1}$$

is called quasihomogeneous with quasihomogeneity exponents $g_1, \dots, g_n$, if

$$v^i(\alpha^{g_1}x^1, \dots, \alpha^{g_n}x^n) = \alpha^{g_i+1}v^i(x^1, \dots, x^n) \tag{2}$$

for all values of x and $\alpha > 0$. In other words, Eqs. (1) are invariant under the substitution

$$x^i \to \alpha^{g_i}x^i, \quad t \to t/\alpha.$$

An example is a system with homogeneous quadratic right-hand members: in it, $g_1 = \dots = g_n = 1$. Among others, the Euler–Poincaré equations describing geodesics on Lie groups with invariant metrics fall into this class. A popular example from dynamics is Kirchoff's problem on the motion of a rigid body in an unbounded volume of an ideal liquid. Quasihomogeneous systems are also exemplified by the equations of the problem of many gravitating bodies and by the Euler–Poisson equations describing the rotation of a heavy rigid body about a fixed point. These remarks show that it is expedient to consider quasihomogeneous systems from the viewpoint of applications.

We recall that a tensor field T(x) is an invariant ("conservation law") for system (1) if $L_vT = 0$, where L_v is the Lie derivative along the vector field v. Here is the explicit expression of the Lie derivative for a tensor field of type (p, q):

$$L_vT^{i_1\dots i_p}_{j_1\dots j_q} = v^s\frac{\partial}{\partial x^s}T^{i_1\dots i_p}_{j_1\dots j_q} + T^{i_1\dots i_p}_{kj_2\dots j_q}\frac{\partial v^k}{\partial x^{j_1}} + \dots + T^{i_1\dots i_p}_{j_1\dots j_{q-1}k}\frac{\partial v^k}{\partial x^{j_q}} - T^{li_2\dots i_p}_{j_1\dots j_q}\frac{\partial v^{i_1}}{\partial x^l} - \dots - T^{i_1\dots i_{p-1}l}_{j_1\dots j_q}\frac{\partial v^{i_p}}{\partial x^l}. \tag{3}$$

The scalar invariants are the integrals of system (1), the invariant vector fields are symmetry fields (they commute with the field v), and the invariant exterior forms generate integral invariants of that system.

We shall call a tensor field T of type (p, q) quasihomogeneous of degree m with quasihomogeneity exponents $g_1, \dots, g_n$, if

$$T^{i_1\dots i_p}_{j_1\dots j_q}(\lambda^{g_1}x^1, \dots, \lambda^{g_n}x^n) = \lambda^{m-g_{j_1}-\dots-g_{j_q}+g_{i_1}+\dots+g_{i_p}}T^{i_1\dots i_p}_{j_1\dots j_q}(x). \tag{4}$$

An equivalent definition:

$$T^{i_1\dots i_p}_{j_1\dots j_q}\frac{x^{j_1}\dots x^{j_q}}{x^{i_1}\dots x^{i_p}}$$

are quasihomogeneous functions of $x^1, \dots, x^n$ of degree m.

The introduction of quasihomogeneous tensor fields is expedient in view of the fact that all the summands in expression (3) of the Lie derivative are quasihomogeneous functions of one and the same degree. In particular, in solving a problem on invariant tensor fields of quasihomogeneous systems we can confine ourselves to considering quasihomogeneous tensors: When the components of an invariant tensor are expanded in series in quasihomogeneous forms, the summands of the same degree obviously represent a quasihomogeneous invariant tensor field.

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2. Kovalevskaya–Lyapunov Method. In Kovalevskaya's classical work [1] the problem was solved of conditions for the meromorphy of the complete solution of the equations of rotation of a heavy rigid body about a fixed point. The method of Kovalevskaya was developed in Lyapunov's work [2], which made it possible to solve the more general problem on the single-valuedness of the general solution as a function of complex time. Following [3], we shall apply the Kovalevskaya–Lyapunov method to quasihomogeneous systems (1). New necessary entities will be introduced in the course of our consideration.

Equations (1) have particular solutions of the form

$$x^1 = c_1 t^{-g_1}, \dots, x^n = c_n t^{-g_n}. \tag{5}$$

The constant coefficients (complex in the general case) c_i satisfy the system of algebraic equations

$$v^i(c_1, \dots, c_n) = -g_i c_i.$$

As a rule, this system has nontrivial solutions.

Let us write down variational equations for particular solution (5):

$$\dot{\xi}^i = \frac{\partial v^i}{\partial x^j}(c_1 t^{-g_1}, \dots, c_n t^{-g_n}) \xi^j. \tag{6}$$

Differentiating identity (2) with respect to x^j, we obtain the equality

$$\frac{\partial v^i}{\partial x^j}(\alpha^{g_1} x^1, \dots \alpha^{g_n} x^n) = \alpha^{g_i - g_j + 1} \frac{\partial v^i}{\partial x^j}(x^1, \dots, x^n). \tag{7}$$

Substituting $\alpha = 1/t$ into (7), rewrite Eqs. (6):

$$\dot{\xi}^i = \frac{\partial v^i}{\partial x^j}(c)\, t^{g_j - g_i - 1} \xi^j, \quad 1 \leqslant i \leqslant n.$$

This linear system has the particular solutions

$$\xi^1 = \varphi^1 t^{\rho - g_1}, \dots, \xi^n = \varphi^n t^{\rho - g_n},$$

where ρ is an eigenvalue and φ is an eigenvector of the matrix $K = \|K_j^i\|$, where

$$K_j^i = \frac{\partial v^i}{\partial x^j}(c) + \delta_j^i g_i.$$

Here δ_j^i is the Kronecker symbol.

The matrix K first appeared in [1]. That is why it is called Kovalevskaya's matrix and its eigenvalues, Kovalevskaya's exponents (see [3]).

Suppose that all g_i are positive integers. It can be shown (see [3]) that if the general solution of system (1) can be represented by single-valued (meromorphic) functions of complex time, then Kovalevskaya's exponents ρ_j are integers (respectively, nonnegative integers).

In Yoshida's work [3], the problem is considered of the existence of quasihomogeneous integrals of system (1) [invariant tensors of type (0, 0)], and the following theorem is proved:

THEOREM. Let f be a quasihomogeneous integral of degree m with quasihomogeneity exponents $g_1, \dots, g_n$, and let $df(c) \neq 0$. Then $\rho = m$ is Kovalevskaya's exponent.

This result has established a remarkable connection between the meromorphy property of the general solution and the existence of nonconstant integrals. Note that if there are k quasihomogeneous integrals $f_1, \dots, f_k$ of one and the same degree m and their differentials $df_1, \dots, df_k$ are linearly independent at the point $x = c$, then $\rho = m$ is Kovalevskaya's exponent of multiplicity $\geq k$.

3. Main Theorem. Our goal is to extend Yoshida's result to tensor invariants of arbitrary type. We shall consider tensor fields of type (p, q), $p + q \geq 1$.

THEOREM. Let system (1) admit a quasihomogeneous tensor invariant of degree m, and let $T(c) \neq 0$. Then there can be found integers $1 \leq i_1, \ldots, i_p, j_1, \ldots, j_q \leq n$ such that

$$\rho_{i_1} + \ldots + \rho_{i_p} - \rho_{j_1} - \ldots - \rho_{j_q} + m = 0. \tag{8}$$

Several consequences can be deduced from this result. First, let us prove Yoshida's theorem from Sec. 2. Let f be a quasihomogeneous integral of degree m, and let a point x − c not be its critical point. Consider the tensor field of type (0, 1):

$$T_i = \partial f / \partial x^i.$$

By the definition in Sec. 1, it is quasihomogeneous of degree m. Since the Lie derivative commutes with the operation of differentiation, the tensor field T_i is an invariant for system (1). It remains to apply the main theorem and make use of relation (8).

Consider now a vector field $u = \{u^i\}$ commuting with the initial field $v = \{v^i\}$. This is a tensor field of type (1, 0) and an invariant for system (1).

COROLLARY 1. Let system (1) admit a quasihomogeneous symmetry field of degree m, and let $u(c) \neq 0$. Then $\rho = -m$ is Kovalevskaya's exponent.

This proposition shows an interesting connection between a condition for the single-valuedness of the general solution of a quasihomogeneous system and the existence of nontrivial symmetries (see Sec. 2). If system (1) admits k quasihomogeneous symmetry fields $u_1, \ldots, u_k$ of the same degree m and linearly independent at the point $x = c$, then $\rho = -m$ is Kovalevskaya's exponent of multiplicity $\geq k$.

COROLLARY 2. If $c \neq 0$, then $\rho = -1$ is Kovalevskaya's exponent.

Indeed, $u \equiv v$ is a quasihomogeneous symmetry field of degree $m = 1$. It remains to take note that $v^i(c) = -g_i c_i$ and $g_i \neq 0$.

This result can be obtained in another way. Since system (1) is autonomous, it has the family of solutions

$$x^i(t, a) = c_i/(t+a)^{g_i}, \quad 1 \leqslant i \leqslant n,$$

where a is a real parameter. The derivatives

$$\left.\frac{\partial x^i}{\partial a}\right|_{a=0} = -\frac{c_i g_i}{t^{g_i+1}}$$

satisfy variational equation (6). Consequently, $\rho = -1$ is an eigenvalue of the matrix K, and $(-c_1 g_1, \ldots, -c_n g_n) \neq 0$ is its eigenvector.

4. Proof of the Main Theorem. Set the right-hand side of (3) equal to zero and substitute solution (5) for x into the obtained identity. Then

$$v^s \left.\frac{\partial T^{i_1\ldots i_p}_{j_1\ldots j_q}}{\partial x^s}\right|_{x=ct^{-g}} = -\frac{1}{t}\sum_s g_s \frac{c_s}{t^{g_s}} \frac{\partial T^{i_1\ldots i_p}_{j_1\ldots j_q}}{\partial x^s}(ct^{-g}) = -\frac{1}{t} g_s x^s \left.\frac{\partial T^{i_1\ldots i_p}_{j_1\ldots j_q}}{\partial x^s}\right|_{x=ct^{-g}} =$$

$$= -\frac{1}{t}(m - g_{j_1} - \ldots - g_{j_q} + g_{i_1} + \ldots + g_{i_n}) T^{i_1\ldots i_p}_{j_1\ldots j_q}\left(\frac{c}{t^g}\right) = -\frac{m - g_{j_1} - \ldots - g_{j_q} + g_{i_1} + \ldots + g_{i_n}}{t^{m-g_{j_1}-\ldots-g_{j_q}+g_{i_1}+\ldots+g_{i_n}+1}} T^{i_1\ldots i_p}_{j_1\ldots j_q}(c).$$

Here Euler's formula for quasihomogeneous functions has been used [it is required to differentiate identity (4) with respect to λ and then set λ = 1]. Using identity (7), transform the other summands in the right-hand side of (3):

$$T^{i_1\ldots i_p}_{kj_2\ldots j_q}\left(\frac{c}{t^g}\right) \left.\frac{\partial v^k}{\partial x^{j_1}}\right|_{ct^{-g}} = \frac{T^{i_1\ldots i_p}_{kj_2\ldots j_q}(c) \dfrac{\partial v^k}{\partial x^{j_1}}(c)}{t^{m-g_k-g_{j_2}-\ldots-g_{j_q}+g_{i_1}+\ldots+g_{i_n}+g_k-g_{j_1}+1}}, \ldots$$

As a result, we arrive at the equality

$$-(m - g_{j_1} - \ldots - g_{j_q} + g_{i_1} + \ldots + g_{i_n}) T^{i_1\ldots i_p}_{j_1\ldots j_q} +$$

$$+T^{i_1\ldots i_p}_{kj_2\ldots j_q}\frac{\partial v^k}{\partial x^{j_1}}+\ldots+T^{i_1\ldots i_p}_{j_1\ldots j_{q-1}k}\frac{\partial v^k}{\partial x^{j_q}}-T^{li_2\ldots i_p}_{j_1\ldots j_q}\frac{\partial v^{i_1}}{\partial x^l}-\ldots-T^{i_1\ldots i_{p-1}l}_{j_1\ldots j_q}\frac{\partial v^{i_p}}{\partial x^l}=0.$$

Recalling the definition of Kovalevskaya's matrix, we finally obtain

$$-mT^{i_1\ldots i_p}_{j_1\ldots j_q}+T^{i_1\ldots i_p}_{kj_2\ldots j_q}K^k_{j_1}+\ldots+T^{i_1\ldots i_p}_{j_1\ldots j_{q-1}k}K^k_{j_q}-T^{li_2\ldots i_p}_{j_1\ldots j_q}K^{i_1}_l-\ldots-T^{i_1\ldots i_{p-1}l}_{j_1\ldots j_q}K^{i_p}_l=0. \quad (9)$$

Without restriction of generality, we can assume that the matrix K is reduced to a triangular form. Arrange the elements of the tensor T in increasing order of the indices i and decreasing order of j. It is natural to consider system (9) as a system of linear algebraic equations for the elements of T. Under our assumptions, the matrix of the left-hand sides of this system has a triangular form with diagonal elements of the form

$$-m+\rho_{j_1}+\ldots+\rho_{j_q}-\rho_{i_1}-\ldots-\rho_{i_p}, \quad (10)$$

where ρ_μ are Kovalevskaya's exponents. Its determinant is obviously equal to the product of numbers (10). Since it is assumed that the tensor T does not vanish at the point x = c, at least one of those numbers vanishes.

Note that in the typical case of the resuscibility of Kovalevskaya's matrix to the diagonal form diag$[\rho_1, \ldots, \rho_n]$ relations (9) can be simplified:

$$(-m+\rho_{j_1}+\ldots+\rho_{j_q}-\rho_{i_1}-\ldots-\rho_{i_p})T^{i_1\ldots i_p}_{j_1\ldots j_q}(c)=0.$$

Since at least one of the elements of the tensor T is different from zero, we again obtain the conclusion of the main theorem.

5. Hamilton's Quasihomogeneous Equations. Let us now apply the above considerations to Hamilton's equations

$$\dot{x}_k=\partial H/\partial y_k,\ \dot{y}_k=-\partial H/\partial x_k,\ 1\leqslant k\leqslant n, \quad (11)$$

with a quasihomogeneous Hamiltonian of degree h

$$H(\alpha^{g_1}x_1, \alpha^{f_1}y_1, \ldots, \alpha^{g_n}x_n, \alpha^{f_n}y_n)=\alpha^h H(x, y). \quad (12)$$

Here g_k, f_k is a collection of quasihomogeneity exponents. It is not hard to see that Eqs. (11) will be quasihomogeneous in the sense of the definition in Sec. 1 if

$$f_k+g_k=1$$

for all k = 1, ..., n. In this case the numbers ρ_1 = h and ρ_2 = −1 will always be among Kovalevskaya's exponents.

Suppose that Eqs. (11) admit the particular solutions

$$x_k=u_k t^{-g_k},\quad y_k=v_k t^{-f_k},\ 1\leqslant k\leqslant n, \quad (13)$$

and that $\sum(|u_k|+|v_k|)\neq 0$.

Proposition. Let Φ(x, y) be a quasihomogeneous integral of degree m of Eqs. (11), and let it and Hamiltonian (12) be independent at a point (x, y) = (u, v): the rank of the Jacobian matrix of the functions H and Φ be equal to two. Then ρ_1 = m and ρ_2 = h − m − 1 are Kovalevskaya's exponents.

Proof. Since Φ is a quasihomogeneous integral of Eqs. (11), ρ = m is Kovalevskaya's exponent (Yoshida's theorem). The Hamiltonian system of equations

$$x_k'=\partial\Phi/\partial y_k,\ y_k'=-\partial\Phi/\partial x_k,\ 1\leqslant k\leqslant n, \quad (14)$$

is a quasihomogeneous system of degree m + 1 − f_k − g_k = m + 1 − h. Since the functions H and Φ are related by an involution, the Hamiltonian field (14) is a symmetry field for system (11). And since the functions H and Φ are independent at the point (x, y) = (u, v), the fields (11) and (14) are also independent at that point. It remains to apply Corollary 1 from Sec. 3.

Remark. If $m \neq h$, then the condition of the independence of the functions H and Φ can obviously be replaced by a weaker one: $d\Phi \neq 0$ for $x = u$, $y = v$.

Note that the sum $\rho_1 + \rho_2 = h - 1$ does not depend on the degree of the integral. It turns out that this is no accident: Kovalevskaya's exponents for quasihomogeneous Hamiltonian systems can be grouped into pairs whose sums are equal to $h - 1$. This fact is an analogue of the well-known Poincaré–Lyapunov theorem on the recursion property of the characteristic equation for the multipliers of a periodic solution of Hamilton's equations. The proof is based on the simple observation that variational Eqs. (6) are in this case Hamiltonian ones:

$$\dot{\xi}_i = \sum \frac{\partial^2 H}{\partial y_i \partial x_k} \xi_k + \sum \frac{\partial^2 H}{\partial y_i \partial y_k} \eta_k,$$
$$\dot{\eta}_i = -\sum \frac{\partial^2 H}{\partial x_i \partial x_k} \xi_k - \sum \frac{\partial^2 H}{\partial x_i \partial y_k} \eta_k, \quad i = 1, \ldots, n. \tag{15}$$

Here formulas (13) should be substituted into the expressions for the second derivatives of the Hamiltonian H. Let (ξ, η) and (ξ^*, η^*) be two solutions of Eqs. (15). It is easily verifiable that the sum

$$\sum (\xi_k \eta_k^* - \xi_k^* \eta_k) \tag{16}$$

is constant. Put

$$\xi_k = \frac{\varphi_k}{t^{\rho_1 - g_k}} \quad \eta_k = \frac{\psi_k}{t^{\rho_1 - f_k}}, \quad \xi_k^* = \frac{\varphi_k^*}{t^{\rho_2 - g_k}}, \quad \eta_k^* = \frac{\psi_k^*}{t^{\rho_2 - f_k}}.$$

Then (16) takes the form

$$t^{f_k + g_k - \rho_1 - \rho_2} \sum (\varphi_k \psi_k^* - \varphi_k^* \varphi_k).$$

This sum is independent of time in these two cases: 1) $\rho_1 + \rho_2 = f_k + g_k = h - 1$; 2) the sum

$$\sum (\varphi_k \psi_k^* - \varphi_k^* \psi_k) = 0. \tag{17}$$

In the second case the vectors

$$\mu = (\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n), \quad \mu^* = (\varphi_1^*, \ldots, \varphi_n^*, \psi_1^*, \ldots, \psi_n^*)$$

are "skeworthogonal." Suppose that the vector μ is skeworthogonal to all the eigenvectors of the matrix K. Since these vectors constitute a basis in C^{2n}, μ is skeworthogonal to all vectors of C^{2n}. But then $\mu = 0$. Thus, there always can be found a vector μ^* such that sum (17) is different from zero, as required.

Suppose that the degrees of the Hamiltonian and an additional integral are integers. Then another pair of integers appears among Kovalevskaya's exponents. One of them is the degree of the new integral, the other is the degree, taken with the opposite sign, of the Hamiltonian symmetry field generated by that integral.

LITERATURE CITED

1. S. V. Kovalevskaya, "A problem on the rotation of a rigid body about a fixed point," in: Scientific Works [in Russian], Nauka, Moscow (1948), pp. 153-220.
2. A. M. Lyapunov, "On a certain property of the differential equations in a problem on the motion of a heavy rigid body having a fixed point," in: Collected Works [in Russian] Vol. 1, Izd. Akad. Nauk SSSR, Moscow (1954), pp. 402-417.
3. H. Yoshida, "Necessary conditions for the existence of algebraic first integrals," Celestial Mechanics, 31, No. 4, 363-379 (1983).