GROUPS OF SYMMETRIES OF GEODESIC FLOWS ON CLOSED SURFACES

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1. Introduction. Basic Result.

Let M be a compact two-dimensional Riemannian manifold. The Riemannian metric

$$ds^2 = \frac{1}{2} \sum g_{ij}(q)\, dq_i\, dq_j \tag{1}$$

generates a Hamiltonian system in the cotangent bundle T^*M with Hamiltonian function

$$H = \frac{1}{2} \sum g^{ij} p_i p_j, \tag{2}$$

where $\|g^{ij}\|$ is the inverse matrix to $\|g_{ij}\|$. Under the natural projection $T^*M \to M$ the phase trajectories of the system with Hamiltonian (2) go into geodesic lines of the metric (1). The restriction of the Hamiltonian system to the invariant surface H = 1 is usually called the geodesic flow on the Riemannian surface (M, ds).

From the point of view of classical dynamics the system with Hamiltonian function (2) describes the motion of an inertial mechanical system; M is the configuration space and

$$T = \frac{1}{2} \sum g_{ij} \dot{q}_i \dot{q}_j$$

is the kinetic energy.

Let v be the Hamiltonian vector field on T^*M corresponding to the Hamiltonian function (2). A vector field u (defined on T^*M) is called a field of symmetries of the Hamiltonian system if $[u, v] \equiv 0$, where $[\cdot, \cdot]$ is the commutator of vector fields. Equivalent definition: the phase flow of the dynamical system generated by the field u carries solutions of the Hamiltonian system into solutions of the same system.

It is clear that there is always a trivial field of symmetries $u = \alpha v$, α = const. More generally, if $u = \lambda v$ is a field of symmetries, then the function $\lambda(p, q)$ is an integral of the Hamiltonian system with Hamiltonian (2).

According to the rectivication theorem, in a small neighborhood of each non-singular point in T^*M (when $p \neq 0$) the Hamiltonian system always has a three-parameter family of nontrivial fields of symmetries. From this point of view the problem of existence of fields of symmetries is meaningful when the fields are defined everywhere on T^*M.

Let F be a function on T^*M, v_F be a Hamiltonian vector field with Hamiltonian F. Since

$$[v_H, v_F] = v_{\{H, F\}}$$

($\{\cdot, \cdot\}$ being the Poisson bracket), to each integral F of the Hamiltonian equations with Hamiltonian H corresponds a field of symmetries $u = v_F$. The fields v_H and v_F are independent if the functions H and F are independent. Hence the problem of existence of nontrivial fields of symmetries contains as a special case the problem of the existence of supplementary integrals independent of the function H.

Let M have the structure of a real analytic manifold and the coefficients of the Riemannian metric (1) be local analytic functions on M. It is proved in [1] that if the Euler characteristic of M is negative, then the Hamiltonian system with Hamiltonian (2) has no integral which is analytic on T^*M and independent of the function H. It turns out that the analogous result also holds in the more general problem of groups of symmetries. We shall assume that the field of symmetries u is continuously differentiable in $(p, q) \in T^*M$ and analytic in p. More precisely, the operator of differentiation L_u along the field u

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carries analytic functions defined on T^*M into functions of class C^1, depending analytically on the momenta p_1, p_2.

THEOREM. Let $\chi(M) < 0$. Then $u = \lambda v$, where λ is an analytic function of H.

This result contains as a special case the basic theorem of [1]. There are many examples of integrable systems when $\chi(M) \geq 0$.

Let us assume that the curvature of the two-dimensional Riemannian manifold (M, ds) is everywhere negative. Then the geodesic flow will be a U-system [2]. In this case one can assert more: the geodesic flow does not admit nontrivial fields of symmetries of class C^1 (see [3]). The proof of this fact uses the everywhere denseness of the set of hyperbolic long-periodic closed trajectories. We note that ergodic systems do not admit nonconstant integrals but nontrivial symmetry groups may exist.

Remark. One should keep in mind that analytic dynamical systems can have fields of symmetries of finite smoothness. An example is a system of differential equations on the two-dimensional torus $T^2 = \{x_1, x_2 \bmod 2\pi\}$ of the following form:

$$\dot{x}_1 = X, \quad \dot{x}_2 = \gamma X, \tag{3}$$

where $\gamma \in \mathbb{R}$, X is a positive analytic function on T^2. One can show that for suitable choice of the function X the real axis $\mathbb{R} = \{\gamma\}$ splits into disjoint sets m_ω, m_∞, ... m_k, ..., such that for $\gamma \in m_\omega$ the system (3) has a nontrivial analytic field of symmetries, for $\gamma \in m_\infty$ the system (3) admits an infinitely differentiable field of symmetries but has no analytic symmetries, ..., for $\gamma \in m_k$ there is a field of symmetries of class C^k but there are no nontrivial fields of symmetries of class of smoothness C^{k+1}, ... All the sets m_ω, m_∞, ..., m_k, ... are everywhere dense on the real line and have the cardinality of the continuum; the measure of the set $\mathbb{R} \setminus m_\omega$ is zero.

A. N. Kolmogorov showed [4] that systems on the two-dimensional torus having no singular points and admitting an integral invariant reduce to the form (3).

2. Proof of the Theorem

We shall assume that M is orientable; if not one can pass to a regular two-sheeted covering. We endow M with a complex-analytic structure. For this, we cover M by charts with local isothermal coordiantes q_1, q_2; passage from chart to chart is defined by a local holomorphic function of the complex variable $q_1 + iq_2$. In these coordiantes the Hamiltonian (2) assumes the form

$$H = \Lambda(q_1, q_2)(p_1^2 + p_2^2)/2. \tag{4}$$

A complex-analytic structure on M was already used by Virkhoff for solving the problem of conditions for the existence of integrals which are polynomial in the momenta of degree at most two [5]. Birkhoff's method lies at the base of the new proof found by V. N. Kolokol'tsov of the theorem on the nonexistence of supplementary analytic integral under the condition that $\chi(M) < 0$ [6].

The operator of differentiation along a Hamiltonian vector field corresponding to the Hamiltonian function (4) assumes the form

$$L_v = \sum \Lambda p_j \frac{\partial}{\partial q_j} - \frac{1}{2} \sum \frac{\partial \Lambda}{\partial q_j}(p_1^2 + p_2^2)\frac{\partial}{\partial p_j}. \tag{5}$$

Let

$$L_u = \sum Q_j \frac{\partial}{\partial q_j} + \sum P_j \frac{\partial}{\partial p_j} \tag{6}$$

be the operator of differentiation along the field of symmetries u. By assumption the functions Q_j, P_j are analytic in the momenta p_1 and p_2. We expand them in series in homogeneous forms in the momenta

$$Q_j = \sum_{n\geq 0} Q_j^{(n)}, \quad P_j = \sum_{n \geq 0} P_j^{(n)}.$$

If the operators (5) and (6) commute, then the operator (5) commutes with each of the operators

$$L^{(n)} = \sum Q_j^{(n-1)} \frac{\partial}{\partial q_j} + \sum P_j^{(n)} \frac{\partial}{\partial p_j}, \quad n \geqslant 0. \tag{7}$$

For n = 0 it is necessary to set $Q_j = 0$. This obvious assertion lets us reduce the problem of analytic field of symmetries to the problem of a field of symmetries with homogeneous polynomial components. Thus, in what follows, instead of the operator (6) we shall consider the operator (7).

The following two auxiliary assertions are used several times below. Let

$$F = \sum_{r+s=n} f_{r,s}(q_1, q_2)\, p_1^r p_2^s$$

be a homogeneous polynomial of degree n with differentiable coefficients. Let $p_1 = 1$, $p_2 = i$. Then F = U + iV, where U and V are real functions of q_1, q_2. Let us assume that U and V satisfy the Cauchy—Riemann conditions. Then $F^* = U + iV$ is a local homomorphic function of $z = q_1 + iq_2$.

<u>LEMMA 1</u> (cf. [6]). Let $z \to w(z)$ be a holomorphic function. Then $F^*(z) = F^*(w(z))(w'(z))^{-n}$.

<u>LEMMA 2</u> (cf. [6]). If $\chi(M) < 0$, then $F^* \equiv 0$.

Indeed let $F^*(z) \not\equiv 0$. Then according to Lemma 1, the differential form

$$(dz)^n/F^*(z) \tag{8}$$

is invariant with respect to holomorphic changes of variables. For n = 1 the form (8) is an ordinary Abelian differential. When n > 1, it is natural to call this form an n-differential.

It is well known that for any Ableian differential on a compact Riemann surface M the difference between the number of zeros and the number of poles is equal to $-\chi(M)$. For an n-differential this difference is obviously equal to $-n\chi(M)$. Since F^* is locally holomorphic, the n-differential (8) has no zeros. Since $\chi < 0$, its number of poles is negative. We have obtained a contradiction.

Let P_j^* (Q_j^*) be the value of the polynomial P_j (Q_j) for $p_1 = 1$, $p_2 = i$.

<u>LEMMA 3.</u> $R = \Lambda(P_1^* + iP_2^*)$ is a holomorphic function of $z = q_1 + iq_2$.

<u>Proof.</u> We calculate the commutator $[L_u, L_v]$ and we equate the coefficients of $\partial/\partial p_1$ and $\partial/\partial p_2$ to zero. As a result we get the relations

$$\begin{gathered}\Lambda \frac{\partial P_1^*}{\partial q_1} + \frac{\partial \Lambda}{\partial q_1} P_1^* + i\left(\Lambda \frac{\partial P_1^*}{\partial q_2} + \frac{\partial \Lambda}{\partial q_1} P_2^*\right) = 0, \\ i\left(\Lambda \frac{\partial P_2^*}{\partial q_2} + \frac{\partial \Lambda}{\partial q_2} P_2^*\right) + \Lambda \frac{\partial P_2^*}{\partial q_1} + \frac{\partial \Lambda}{\partial q_2} P_1^* = 0.\end{gathered} \tag{9}$$

From this it follows that

$$\partial R/\partial q_1 + i\, \partial R/\partial q_2 = 0,$$

which is a criterion for the function R to be holomorphic. The lemma is proved.

According to Lemma 2, $R \equiv 0$. Thus, $P_1^* + iP_2^* = 0$. From (9) we get the two equations

$$\partial P_j^*/\partial q_1 + i\, \partial P_j^*/\partial q_2 = 0, \quad j = 1, 2.$$

Consequently, P_j^* are local holomorphic functions of $z = q_1 + iq_2$. By Lemma 2, $P_j^* \equiv 0$.

<u>LEMMA 4</u> (cf. [6]). Let $F^* \equiv 0$. Then $F = HF'$ where F' is a homogeneous polynomial of degree n − 2.

Since $P_j^* \equiv 0$, one has $P_j = HP_j'$. Now we show that the polynomials Q_j are also evenly divisible by the polynomial H.

<u>LEMMA 5.</u> $S = Q_1^* + iQ_2^*$ is a holomorphic function.

Proof. We calculate the commutator $[L_u, L_v]$, equate the coefficients of $\partial/\partial q_1$, $\partial/\partial q_2$ to zero, and use the identities $P_j^* \equiv 0$. As a result we get

$$Q_1^* \frac{\partial\Lambda}{\partial q_1} + Q_2^* \frac{\partial\Lambda}{\partial q_2} - \Lambda\frac{\partial Q_1^*}{\partial q_1} - i\Lambda\frac{\partial Q_1^*}{\partial q_2} = 0,$$
$$i\left(Q_1^* \frac{\partial\Lambda}{\partial q_1} + Q_2^* \frac{\partial\Lambda}{\partial q_2}\right) - \Lambda\frac{\partial Q_2^*}{\partial q_1} - i\Lambda\frac{\partial Q_2^*}{\partial q_2} = 0. \qquad (10)$$

It follows from this that

$$\partial S/\partial q_1 + i\,\partial S/\partial q_2 = 0.$$

Consequently, S is a local holomorphic function. Lemma 5 is proved.

By Lemma 2, $S \equiv 0$. But then from (10) we get the two relations

$$\frac{\partial}{\partial q_1}\frac{Q_j^*}{\Lambda} + i\frac{\partial}{\partial q_2}\frac{Q_j^*}{\Lambda} = 0, \quad j = 1, 2.$$

Consequently, Q_j^*/Λ are holomorphic functions which are equal to zero according to Lemma 2. Applying Lemma 4 we get that $Q_j = HQ_j'$.

Thus, $u = Hu'$. Since the factor H is an integral of the equation of motion, u' is also a field of symmetries. Its degree in the momenta is two less than the degree of the field u. By decreasing induction on n the original problem reduces to the problem of the existence of a field of symmetries with degree 0 or 1.

The case n = 0 is trivial. For n = 1 obviously $Q_j = Q_j^* = 0$. Let $P_j = \xi_j p_1 + \eta_j p_2$. As was shown above, the functions $P_j^* = \xi_j + i\eta_j \equiv 0$. Consequently, $\xi_j = \eta_j \equiv 0$. Hence for n = 1 the field of symmetries is identically equal to zero.

The theorem is completely proved.

3. Concluding Remarks

Apparently Theorem 1 of Sec. 1 is valid in the more general problem of motion of reversible systems in a potential force field with analytic potential energy $V \neq$ const. More precisely, if $\chi(M) < 0$, then a Hamiltonian system has no nontrivial analytic fields of symmetries on a three-dimensional energy surface $T + V = h$, where $h > \max_M V$.

Let M' be a connected and geodesically convex subdomain with boundary of a Riemannian manifold M. Apparently, if $\chi(M') < 0$, then a Hamiltonian system also has no nontrivial fields of symmetries. Under these assumptions in [7] the bsence of a supplementary analytic integral is proved. From this would follow the absence of groups of symmetries in the problem of n fixed gravitating centers for n > 2 (cf. with [7]).

Finally, it would be desirable to establish multidimensional versions of the theorem of Sec. 1. Topological obstructions to the complete integrability of geodesic flows in the case $\dim M > 2$ are found in [8].

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