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POLYNOMIAL INTEGRALS OF DYNAMICAL SYSTEMS WITH ONE-AND-A-HALF DEGREES OF FREEDOM

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It has long been remarked that all the known first integrals of the equations of classical mechanics are polynomials in velocities (or functions of the polynomials). This observation has still not received a full substantiation. Therefore, interest is taken in the analytic and geometric nature of polynomial integrals.

Whittaker [1] and Birkhoff [2] analyzed the conditions of existence of linear and quadratic integrals of systems with two degrees of freedom. All the linear integrals are Noetherian (they are conditioned by the existence of hidden cyclic coordinates), and the presence of quadratic integrals is tied to the possibility of separation of variables (cf. [3]).

From the point of view of a global approach, it is advisable to consider polynomial integrals, the coeffcients of which are single-valued (smooth or analytic) functions on the configuration space of the dynamical system. Different aspects of the problem of the existence of polynomial integrals in the large were considered in [4-6].

The simplest nontrivial problem of such a type is the question of the presence in the system

$$\ddot{x} = -V_x \tag{1}$$

with a potential V(x, t) 2π-periodic in x, of a first integral in the form of a polynomial

$$a_0(x, t) + a_1(x, t)\dot{x} + \ldots + a_n(x, t)\dot{x}^n \tag{2}$$

with (2π-periodic in x) coefficients a_k $(0 \leqslant k \leqslant n)$. This article is devoted to the consideration of this problem.

2. Differentiating function (2) due to system (1), we obtain the following system of equations in first-order partial derivatives for finding the potential V of the integrable system and coefficients a_k:

$$(a_n)_x = 0; \tag{3.n + 1}$$
$$(a_n)_t + (a_{n-1})_x = 0; \tag{3.n}$$
$$(a_{n-1})_t + (a_{n-2})_x = na_nV_x; \tag{3.n−1}$$
$$(a_{n-2})_t + (a_{n-3})_x = (n-1)\,a_{n-1}V_x; \tag{3.n−2}$$
$$\cdots\cdots\cdots\cdots$$
$$(a_1)_t + (a_0)_x = 2a_2V_x; \tag{3.1}$$
$$(a_0)_t = a_1V_x. \tag{3.0}$$

This system consists of n + 2 equations and contains as many unknown functions V, a_0, a_1, ..., a_n.

Let us note that Eq. (1) maintains its form for the substitution $t \mapsto t,\ x \mapsto x + \alpha t,\ V(x, t) \mapsto V(x + \alpha t, t) + f(t)$, where α = const, f is an arbitrary function of time. We will constantly use this trivial calibration.

LEMMA 1. $a_n = a_n^0 = \text{const} \neq 0,\ a_{n-1} = a_{n-1}^0 = \text{const},\ a_{n-2} = na_n^0V$ (after appropriate calibration of V).

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Proof. It follows from (3.n + 1) that the function a_n does not depend on x. Consequently, $a_n = \langle a_n \rangle$, where

$$\langle g \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(x, t)\, dx.$$

Averaging both parts of Eq. (3.n) over x, we obtain that $(\langle a_n \rangle)_t = 0$. Consequently, $a_n = \text{const}$. But then it follows from (3.n) that a_{n-1} does not depend on x, and, therefore, $a_{n-1} = \langle a_{n-1} \rangle$. Averaging Eq. (3.n−1) over x, we arrive at the equation $a_{n-1} = \langle a_{n-1} \rangle = \text{const}$. Consequently, Eq. (3.n−1) acquires the form $(a_{n-2})_x = n a_n^0 V_x$, from which we obtain that the difference $a_{n-2} - n a_n^0 V$ depends only on time. That was what was required.

By dividing the integral (2) by a nonzero constant a_n^0 we arrive at the equation $a_n \equiv 1$. By again implementing the sustitution $x \mapsto x - (a_{n-1}^0/n)\, t$, we obtain an equation of the form (1) with the integral (2), in which $a_{n-1} \equiv 0$. Thus, one can consider that $a_n = 1$ and $a_{n-1} = 0$.

With the help of Lemma 1 it is easy to obtain the conditions of existence of an integral of degree $n \leqslant 3$.

For n = 1, from the relation $a_{n-2} = nV$ we obtain that $V \equiv 0$. Consequently, the linear integral reduces to an integral of the moment of $\dot{x}$.

For n = 2, we obtain from Eq. (3.0) an equation in the potential $V_t = 0$. Whence $V = f(x)$, where $f(\cdot)$ is an arbitrary smooth function with period 2π. The integral (2) turns, in this case, into the usual integral of the energy of an autonomous system with one degree of freedom.

For n = 3, Eqs. (3.0) and (3.1) yield the system

$$(a_0)_x = -3V_t, \quad (a_0)_t = 3VV_x,$$

from which we obtain a second-order equation in the potential

$$V_{tt} + (VV_x)_x = 0. \tag{4}$$

It coincides with the integrable stationary Khokhlov–Zabolotskii equation. We are interested only in the solutions of (4) that are 2π-periodic in x. Applying the Cauchy–Kovalevskaya theorem, we obtain that for arbitrary analytic 2π-periodic functions f and g, there exists an analytic solution V(x, t) of Eq. (4), periodic in x with period 2π and $V(x, 0) = f(x)$, $V_1(x, 0) = g(x)$. In this manner, there is a family of potentials, depending on two arbitrary periodic functions, for which Eq. (1) has a nontrivial integral of the third degree in velocity.

In applications, one often encounters cases when the potential is a trigonometric polynomial in x:

$$V = \sum_{-m}^{m} v_k(t) \exp(ikx), \quad v_{\pm m}(t) \not\equiv 0, \quad m \neq 0. \tag{5}$$

As an example, let us point out the Chaplygin equation

$$\ddot{x} = t^2 \sin x, \tag{6}$$

describing the rotation of a heavy plate in an unbounded volume of an ideal fluid [7]. It is easy to see that Eq. (4) does not have solutions of the form (5). Actually, the left part of (4) will be a trigonometric polynomial with a nontrivial harmonic $-2m^2 v_m^2 \exp(2imx)$. Let us remark that there are potentials of the form (5), for which equation (1) admits a quadratic integral.

Observations of this point can be generalized to the case of polynomial integrals of arbitrary degree.

3. The structure of systems with polynomial integrals are described by

THEOREM 1. For any $n \geqslant 1$ there exists a family of analytic potentials V(x, t), 2π-periodic in x, depending on n − 1 arbitrary analytic 2π-periodic functions, for which Eq. (1) has a polynomial integral of degree n with univalent coefficients.

The proof is based on an application of the Cauchy–Kovalevskaya Theorem. However, this theorem is immediately inapplicable to system (3). Let us transform (3) with the help of Lemma 1 and the convention that $a_n = 1$:

$$nV_t + (a_{n-3})_x = (n-1)\, a^0_{n-1} V_x; \quad (7.n-2)$$
$$(a_{n-3})_t + (a_{n-4})_x = (n-2)\, nVV_x; \quad (7.n-3)$$
$$(a_{n-4})_t + (a_{n-5})_x = (n-3)\, a_{n-3} V_x; \quad (7.n-4)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(a_1)_t + (a_0)_x = 2a_2 V_x; \quad (7.1)$$
$$(a_0)_t = a_1 V_x. \quad (7.0)$$

This system consists of n − 1 equations and contains n − 1 unknown functions $a_0, \ldots, a_{n-3}$ from V. The Cauchy–Kovalevskaya theorem is now applicable to the system (7): for t = 0 it is necessary to give the values of the n − 1 functions $a_0, \ldots, a_{n-3}, V$ in the form of arbitrary 2π-periodic functions of x. Then (7) for small t will have solutions, 2π-periodic in x.

It would be of interest to clarify the question of the solutions of system (7) on the entire time-axis t.

4. Let us analyze the question of the solvability of system (7) in the class of potentials which are trigonometric polynomials in x. We will not assume here that $a^0_{n-1} = 0$. Therefore, in the right part of Eq. (7.n−2) it is necessary to add $(n-1)\, a^0_{n-1} V_x$.

THEOREM 2. Let us assume that Eq. (1) with the potential

$$V = \textstyle\sum_{-m}^{m} v_k(t) \exp(ikx) \quad (8)$$

has a polynomial integral of degree $n \geqslant 1$. Then

1) if n is odd, V does not depend on x,

2) if n is even, $v_m = c\exp(i\beta t)$, где $c \in \mathbf{C}$, $\beta = a^0_{n-1}/n$.

COROLLARY 1. Let us assume that $|v_m| \neq$ const. Then Eq. (1) does not have nontrivial polynomial integrals with unique (2π-periodic in x) coefficients.

COROLLARY 2. The Chaplygin equation (6) does not have univalent polynomial integrals.

The last assertion indicates the nonintegrability of Eq. (6) (cf. [7]).

Corollary 1 shows that dynamic systems of the form (1) with nontrivial polynomial integrals are infinitely sparse. In order to formalize this statement, let us consider systems with potentials (8), in which the coefficients v_k are periodic in x with one and the same period p > 0. One can identify the set of such potentials with the set of Fourier coefficients of the functions $v_k(t)$, $|k| \leqslant m$. It follows from Theorem 2 that β is a multiple of p and all the Fourier coefficients of the function v_m besides the first, vanish. Consequently, the systems (1) with a nontrivial integral, lie in a subspace of infinite codimension.

As the case n = 2 shows (see Sec. 2), the parameter β can assume any values.

COROLLARY 3. Equation (1) with the potential $V = \lambda(t)\sin mx + \mu(t)\cos mx$ $(m \in \mathbf{N})$ has a nontrivial univalent polynomial integral if and only if it has a polynomial integral of degree $\leqslant 2$.

Actually, if Eq. (1) admits an integral of odd degree, then there is a linear integral (conclusion 1) of Theorem 2). But if the degree of the polynomial integral is even, then from conclusion 2), it is easy to deduce that the potential V is a 2π-periodic function of x + αt, α = const. In this case, equation (1) admits a generalized energy integral, the degree of which equals two (see Sec. 2).

For the present it remains unclear whether Corollary 3 for polynomial potentials (8) of a general form is true (cf. [6]).

5. Proof of Theorem 2. If V is a trigonometric polynomial in x, then from relations (3) it is easy to show by induction that the functions $a_{n-2}, \ldots, a_1, a_0$ also are trigonometric polynomials, where

$$a_{n-2} = \bar{A}_{n-2}\exp(-imx) + \ldots + A_{n-2}\exp(imx);$$
$$a_{n-3} = \bar{A}_{n-3}\exp(-imx) + \ldots + A_{n-3}\exp(imx);$$
$$a_{n-4} = \bar{A}_{n-4}\exp(-i2mx) + \ldots + A_{n-4}\exp(i2mx);$$

$$a_{n-5} = \bar{A}_{n-5} \exp(-i2mx) + \ldots + A_{n-5} \exp(i2mx)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Expanding the left and right parts of relations (3) in Fourier series and equating the coefficients in the major harmonics, we obtain a chain of equations for finding A_{n-2}, A_{n-3}, ...:

$$A_{n-2} = nv_m; \quad (9.n-2)$$
$$\dot{A}_{n-2} + imA_{n-3} = (n-1)\, ima^0_{n-1}v_m; \quad (9.n-3)$$
$$2A_{n-4} = (n-2)\, A_{n-2}v_m; \quad (9.n-4)$$
$$\dot{A}_{n-4} + i2mA_{n-5} = (n-3)\, im\, A_{n-3}v_m; \quad (9.n-5)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Here a point denotes differentiation with respect to t. In the derivation of equations (9), it was assumed that $m \neq 0$.

Let us at first consider the case when n is odd. Then, the subsystem of equations (9.n–2), (9.n–4), ..., (9.1) is closed:

$$\begin{aligned} A_{n-2} &= nv_m; \\ 2A_{n-4} &= (n-2)\, A_{n-2}v_m; \\ 3A_{n-6} &= (n-4)\, A_{n-4}v_m; \\ &\ldots\ldots\ldots \\ \frac{n-1}{2} A_1 &= 3A_3v_m. \end{aligned}$$

Whence

$$\left(\frac{n-1}{2}\right)!\, A_1 = n!!\, (v_m)^{(n-1)/2}. \quad (10)$$

The last equation of system (3)

$$(a_0)_t = a_1V_x \quad (11)$$

gives that $imA_1v_m = 0$. Since by assumption $m \neq 0$, then taking into account (10) we arrive at $v_m = 0$. Applying downwards induction in m, we obtain that V, generally, does not depend on x.

Let us now consider the case of even $n \geqslant 2$. Let us, at first, find A_{n-2}, A_{n-4}, ..., A_0 from Eqs. (9.n–2), (9.n–4), ..., (9.0):

$$\begin{aligned} A_{n-2} &= nv_m; \\ 2A_{n-4} &= (n-2)\, A_{n-2}v_m; \\ &\ldots\ldots\ldots \\ lA_{n-2l} &= (n-l+2)\, A_{n-l+2}v_m; \\ &\ldots\ldots\ldots \\ \frac{n}{2} A_0 &= 2A_2v_m. \end{aligned}$$

Whence

$$A_{n-2l} = \frac{n(n-2)\ldots(n-l+2)}{l!} (v_m)^l \quad (1 \leqslant l \leqslant n/2). \quad (12)$$

To find A_{n-3}, A_{n-5}, ... let us use equations (9.n–3), (9.n–5), ...:

$$\begin{aligned} A_{n-3} &= (n-1)\, a^0_{n-1}v_m - \dot{A}_{n-2}/im; \\ 2A_{n-5} &= (n-3)\, A_{n-3}v_m - \dot{A}_{n-4}/im; \\ &\ldots\ldots\ldots \\ \frac{n-2}{2} A_1 &= 3A_3v_m - \dot{A}_2/im. \end{aligned}$$

From these equations, with regard to the relations (12), we obtain that

$$\left(\frac{n-2}{2}\right)!\, A_1 = (n-1)!!\, a^0_{n-1}\, (v_m)^{(n-2)/2} + \frac{iF(n)}{m} v_m^{n/2-2}\dot{v}_m, \quad (13)$$

where F is a positive number (depending only on n).

For even n from Eq. (11) follows

$$\dot{A}_0 = imA_1v_m. \quad (14)$$

In (12) letting $\ell = n/2$, we obtain:

$$A_0 = \frac{n!!}{(n/2)!!} v_m^{n/2}. \tag{15}$$

It is possible to get from (13), (14), and (15), the following equation for v_m:

$$G(n)\,\dot{v}_m = ima_{n-1}^0 v_m, \tag{16}$$

where $G(\cdot)$, as well as F, assumes real values and $G > 0$. From the linear equation (16) flows the desired formula

$$v_m = c\exp(i\beta t), \quad \beta = ma_{n-1}^0/G(n).$$

In passing it was established that β depends linearly on a_{n-1}^0. It remains for us to prove that $G(n) = n$. For this, let us make the substitution $x \mapsto x - (a_{n-1}^0/n)\,t$. As a result, we obtain an equation of the form (1) with a potential

$$\hat{V}(x, t) = V(x - (a_{n-1}^0/n)\,t,\ t), \tag{17}$$

admitting an integral (2), in which $a_{n-1} = 0$. The function $\hat{V}$ is a trigonometric polynomial in x, where, in accord with (16), the coefficient for the principal harmonic equals some complex number c. It then follows from formula (17) that the principal Fourier coefficient of V equals $c\exp(ima_{n-1}^0/n)\,t$. Consequently, $G(n) = n$.

Theorem 2 is proved.

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