INVARIANT MEASURES OF EULER–POINCARÉ EQUATIONS ON LIE ALGEBRAS

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UDC 517.9+531.01

1. Let G be a Lie group, g be its algebra. According to the Maupertuis principle we shall treat the equations of geodesics of the left-invariant metric on G as the equations of motion of a mechanical system with space of positions G and left-invariant Lagrangian. We write them in tensor notation. Let ω^i be the velocity of the system, $(I_{ij}\omega^i\omega^j)/2$ be the kinetic energy, I_{ij} the inertia tensor, $m_k = I_{ki}\omega^i$ be the moment of the system. The (Euler–Poincaré) equations of motion of the system have the following form [1, 2]:

$$\dot{m}_k = c_{ik}^j \omega^i m_j, \quad 1 \leqslant k \leqslant n, \tag{1}$$

where the c_{ik}^j are the structural constants. They are a system of equations on g (or g*) with quadratic right sides.

We consider the problem of the existence, for a system of Euler–Poincaré equations, of an absolutely continuous invariant measure (i.m.) $fd^n\omega$ with summable density f. If f is a positive function of class C^1, then the i.m. is called an integral invariant (i.i.).

<u>THEOREM 1.</u> The Euler–Poincaré equations have an i.i. if and only if the group G is unimodular.

We recall that unimodularity of a group means the existence of a bilaterally invariant measure. A criterion for unimodularity has the following form: $c_{ik}^k = 0$ for all i. One can represent this condition in invariant form: $\mathrm{tr}(\mathrm{ad}_\omega) = 0$ for all $\omega \in g$.

To prove the sufficiency of the condition of Theorem 1 we calculate the divergence of the right side of (1) as a system of g*. It is equal to $c_{ik}^k\omega^i$. Consequently, by Liouville's theorem, the phase flow of the system (1) preserves the measure $d^n\omega$. The necessity follows from the following assertion.

<u>Proposition 1.</u> A system of differential equations with homogeneous right sides has an i.i. if and only if its phase flow preserves the standard measure. Here the density of the i.i. is a first integral of it.

<u>Proof.</u> Let f > 0 be the density of an i.i. of the system $\dot{x} = v(x)$. With the help of the substitution $f = \exp(-w)$, Liouville's criterion $\mathrm{div}(fv) = 0$ can be represented in the form of the equation $\dot{w} = \mathrm{div}\, v$. The right side of this equation is a homogeneous form of degree m − 1 (m is the degree of homogeneity of the vector field v). Since $w \in C^1$, $\dot{w} = O(|x|^m)$. Consequently, $\dot{w} \equiv 0$ and $\mathrm{div}\, v \equiv 0$, which is what was required.

2. In the case of small dimension of g one can give more precise information about an i.m. of the system (1). If n = 2 and the algebra g is non-Abelian, then Eqs. (1) do not have i.m. with summable (not just smooth) density. For n = 3 the condition of Theorem 1 may not hold only for solvable algebras. One can describe the latter with the help of the following relations: $[e_1, e_2] = 0$, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, where e_1, e_2, e_3 is a basis in g and the matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is nondegenerate. We shall distinguish the cases when the eigenvalues of the matrix A are (a) real numbers of the same sign, (b) real numbers of different signs, (c) complex numbers with nonzero real part, (d) purely imaginary numbers.

<u>Proposition 2.</u> In case (d) the Euler–Poincaré equations have i.i., in case (b) there is no i.i., but there is an i.m. with density of any finite smoothness, in cases (a) and (c) there is no i.m. with summable density.

In case (d) the algebra g satisfies the condition of Theorem 1. It is easy to explain the mechanism of existence of i.m. of finite smoothness under condition (b) with the example of the equations $\dot{x} = 2x$, $\dot{y} = -y$, which have an i.m. with density $|x|^s|y|^{2s+1}$ for all s > 0. In

M. V. Lomonosov Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 22, No. 1, pp. 69-70, January-March, 1988. Original article submitted October 16, 1986.

 0016-2663/88/2201-0058$12.50

cases (a) and (c), on each ellipsoid of integral energy $m_i\omega^i > 0$ there is an asymptotically stable equilibrium position. We stress that the conditions for the existence of i.m. formulated above are determined only by the structure of the algebra g and are independent of the choice of left-invariant metric.

A nonunimodular group, as is known, always has a unimodular normal subgroup of codimension one [3]. Let $\{e_k\}$ be a basis in g, where the vectors $e_1,\ldots,e_{n-1}$ form a basis in the corresponding "unimodular" ideal of the algebra g, and the vector e_n is orthogonal to $e_1,\ldots, e_{n-1}$ in the metric I_{ij}.

Proposition 3. If all eigenvalues of the matrix $A = \|c_{nk}^s\|$ $(k, s < n)$ lie in the left (or right) half-plane, then Eqs. (1) do not have an i.m. with summable density.

This assertion follows from the fact that the equilibrium $m_s = 0$ $(s < n)$, $m_n = \text{const} \neq 0$ is asymptotically stable (as $t \to +\infty$ or $t \to -\infty$) on the corresponding surface of integral energy.

3. We consider the more general situation when one imposes on the mechanical system a left-invariant linear constraint

$$a_i\omega^i = 0, \tag{2}$$

where a is a constant element of g^*. An example is the Suslov problem from nonholonomic mechanics: a rigid body with a fixed point is rotated so that at each moment of time the projection of the angular velocity in some direction fixed in the body is equal to zero. According to the principles of the dynamics of (1) in this situation one substitutes more generally:

$$\dot{m}_k = c_{ik}^j\omega^i m_j + \lambda a_k, \quad a_i\omega^i = 0. \tag{3}$$

Here λ is the Lagrange multiplier. The equations of (3) define a dynamical system on the hypersurface (2). Below we formulate a criterion for the existence of an i.i. in the general case of a compact algebra g. The Killing metric lets one identify g and g^*. In accord with this stipulation we shall consider a a vector from g.

THEOREM 2. The nonholonomic equations (3) have an i.i. if and only if the vector a is an eigenvector for the operator $\mathrm{ad}_{I^{-1}a}$

$$[I^{-1}a, a] = \mu a, \quad \mu \in \mathbf{R}. \tag{4}$$

Condition (4) holds automatically if the hyperplane (2) is characteristic for the inertia operator I. In the Suslov problem [g = so(3)] the last condition is a criterion for the existence of an i.i.

The idea of the proof of Theorem 2 is the following. We choose a basis in g such that $a_1 = \ldots = a_{n-1} = 0$, $a_n = 1$. In this case the first $n - 1$ equations of the system (3) (in which we set $\omega^n = 0$) do not contain the factor λ and are a closed system of differential equations with respect to $\omega^1,\ldots,\omega^{n-1}$ with quadratic right sides. The last equation together with the constraint equation determine the factor λ. (4) is sufficient for preserving the standard measure $d\omega^1 \wedge \ldots \wedge d\omega^{n-1}$. The necessity of this condition for the existence of an i.i. follows from Proposition 1.

4. In general when several linear constraints are imposed on the system the hyperplane (2) is replaced by the plane of "possible velocities" π of lower dimension. It is easy to show that if the plane π is characteristic for the inertia operator, then the phase flow of nonholonomic equations preserves the standard measure on π.

In conclusion, we note that if the distribution of planes of possible velocities is right-invariant, then the equations of motion always have an i.i. [4].

LITERATURE CITED

1. H. Poincaré, C. R. Acad. Sci., Paris, 132, 369-371 (1901).
2. V. Arnold, Ann. l'Inst. Fourier, 16, 319-361 (1966).
3. N. Bourbaki, Integration, Addison-Wesley (1977).
4. A. P. Veselov and L. E. Veselova, Funkts. Anal. Prilozhen., 20, No. 4, 65-66 (1986).