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Phenomena of Nonintegrability in Hamiltonian Systems

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In the last ten or fifteen years mathematicians have again become interested in problems related to the integrability of equations in classical dynamics, which, as a rule, are Hamiltonian equations. New, completely integrable systems have been found (multidimensional analogs of classical problems among them), and various algebro-geometric constructions have been suggested which elucidate the causes for the existence of "hidden" conservation laws. It is also useful to consider the peculiarities of the behavior of phase trajectories of nonintegrable Hamiltonian systems and present strict proofs of their nonintegrability. This paper is dedicated to the analysis of various phenomena of qualitative nature which hinder the integration of Hamiltonian equations.

1. Let us first recall the definition of a Hamiltonian dynamic system. Assume that M^{2n} is an even-dimensional manifold (phase space), ω is a closed nondegenerate 2-form on M (symplectic structure), H is a real function on M (Hamiltonian). Since ω is nondegenerate, the function H can be associated with a unique vector field v_H which is defined by the equation $\omega(v_H, \cdot) = dH$.

This field generates a Hamiltonian system on M, i.e.,

$$\dot{x}(t) = v_H(x(t)), \qquad x\colon \mathbf{R}_t \to M. \tag{1}$$

In suitable local coordinates $x_1, \dots, x_n$, $y_1, \dots, y_n$ (known as canonical coordinates) the form ω reduces to the form $\sum dy_s \wedge dx_s$ (Darboux's theorem). In the canonical coordinates x, y the form of Hamiltonian equations (1) is more customary:

$$\dot{x}_s = -\partial H/\partial y_s, \quad \dot{y}_s = \partial H/\partial x_s, \qquad 1 \le s \le n. \tag{2}$$

We have often to consider nonautonomous Hamiltonian systems in which the Hamiltonian H explicitly depends on time.

If differential equations are not of the form (2), this does not yet mean that they are not Hamiltonian. By virtue of this remark an interesting problem arises concerning the identification of Hamiltonian dynamic systems with an invariant measure and first integrals (in the autonomous case). Here is a simple example. Let

$$\dot{x} = Ax, \qquad x \in \mathbf{R}^n, \tag{3}$$

be a linear system with constant coefficients, which possesses a quadratic integral $f = (Bx, x)$, $B^{\mathrm{T}} = B$. If the operators A and B are nondegenerate, then system (3) is Hamiltonian with a Hamilton's function f. In particular, n is even. In $\mathbf{R}^n$ the symplectic structure is defined by the formula

$$\omega(x', x'') = (BA^{-1}x', x'').$$

2. Dynamic systems (Hamiltonian systems in particular) are customarily classified into integrable and nonintegrable, various definitions of integrability being possible, each with a certain intrinsic theoretic interest. A system which is integrable in the sense of one definition may prove to be nonintegrable in the sense of another definition. I will give examples later on. It is customary to associate the concept of an integrable system with a sufficiently large number of independent integrals ("conservation laws"). Thus, for a complete integrability of Hamiltonian equations with n degrees of freedom (on M^{2n}), it is sufficient to know n independent integrals $F_1, \ldots, F_n$, which are pairwise in involution: Poisson's brackets $\{F_i, F_j\} = \omega(v_{F_i}, v_{F_j})$ are zero. It is well known that compact energy surfaces $H = h$ of a completely integrable Hamiltonian system are stratified into multidimensional tori with a quasiperiodic motion.

If we have a nonautonomous Hamiltonian system with a Hamiltonian $H\colon M^{2n} \times \mathbf{R}_t \to \mathbf{R}$, then, for it to be completely integrable, it is sufficient to have n independent integrals $F_s\colon M^{2n} \times \mathbf{R}_t \to \mathbf{R}$ $(s = 1, \ldots, n)$, which are in involution for all values of t. The case when the Hamiltonian H and the integrals $F_1, \ldots, F_n$ are periodic with respect to t with the same period p is the most important for applications. Then it is natural to take $M^{2n} \times \mathbf{T}^1 \{\operatorname{mod} p\}$ rather than $M^{2n} \times \mathbf{R}$ as an extended phase space. If the integral surfaces $\{(z, t) \in M^{2n} \times \mathbf{T}^1 : F_s(z, t) = c_s,\ 1 \leq s \leq n\}$ are compact, then they are $(n+1)$-dimensional tori with a quasiperiodic motion.

With due regard for the theorem on the straightening out of trajectories, it is reasonable to discuss the integrability of a dynamic system either in the neighborhood of the equilibrium position or in a sufficiently large region of a phase space where trajectories are recurrent.

Before investigating the integrability of specific systems, we must elaborate the concept of a set of independent integrals. We shall deal exclusively with analytic Hamiltonian systems. In that case it is natural to consider sets of analytic integrals which are independent at least at one point (then they are independent almost everywhere). We must bear in mind, however, that an analytic Hamiltonian system may possess integrals of the class C^r but at the same time not possess integrals of the class C^{r+1}. (We do not exclude the value $r = 0$: we consider a continuous function to be an integral when it is locally nonconstant and assumes constant values on each trajectory.) We shall consider the canonical Hamiltonian equations (2) with a Hamiltonian $H = \alpha y + f(x, t)$ as an example, where α is a real parameter, f is a 2π-periodic analytic function with respect to x and t [1]. Since the function H is periodic with respect to the variables x and

t, it is natural to take a direct product $\mathbf{R}\times\mathbf{T}^2 = \{y; x, t \bmod 2\pi\}$ as an extended phase space. We write equation (2) in the explicit form

$$\dot{x} = \alpha, \qquad \dot{y} = -\partial f/\partial x = -F(x,t). \tag{4}$$

We seek the integral of this system in the form $y + g(x,t)$, where $g\colon \mathbf{T}^2 \to \mathbf{R}$ is a smooth or analytic function which must satisfy the equation

$$\partial g/\partial t + \alpha \partial g/\partial x = F(x,t). \tag{5}$$

Equation (5) is well known in the theory of small denominators ([**2**], see also [**3**]). Assume that

$$F = \sum{}' F_{mn} e^{i(mx+nt)}, \qquad g = \sum{}' g_{mn} e^{i(mx+nt)}.$$

Then

$$g_{mn} = \frac{F_{mn}}{i(m\alpha + n)}.$$

For almost all α the numbers g_{mn} are Fourier coefficients of a certain analytic function. Now if the irrational numbers α can be sufficiently rapidly approximated by rational numbers, then equation (5) can have a solution of only finite smoothness or have no solutions. Generalizing these observations, we can show that for a certain $f \in C^\omega(\mathbf{T}^2)$ there are sets M_ω, $M_\infty, \dots, M_k, \dots, M_0, M_\varnothing$, dense everywhere in $\mathbf{R}$, such that for $\alpha \in M_\omega$ equations (4) have an analytic integral, for $\alpha \in M_\infty$ there is a smooth integral but there is no analytic integral, $\dots$, for $\alpha \in M_k$ there is an integral of the class C^k but there are no integrals of the class $C^{k+1}, \dots$, for $\alpha \in M_0$ equations (4) possess only a continuous invariant function, for $\alpha \in M_\varnothing$ there are even no continuous integrals. We can derive the density of the set $M_\varnothing$ in $\mathbf{R}$ from the result obtained by A. B. Krygin concerning the ergodicity of cylindrical cascades [**3**]. Note that if we consider equations (4) in $\mathbf{R}^3 = \{x, y, t\}$ rather than in $\mathbf{R}\times\mathbf{T}^2$, then this system turns out to be completely integrable. It should be emphasized that system (4) can be explicitly integrated by simple quadratures for all values of α, but its behavior as a whole depends considerably on the Diophantine properties of the number α.

3. When mathematicians realized the impossibility of solving equations of classical dynamics in a closed form, strict results appeared concerning their nonintegrability. The first of those results was, evidently, Liouville's theorem (1841) stating that the equation $\ddot{x} + tx = 0$ cannot be solved by quadratures (see [**4**]). In 1887 Bruns stated that there are no algebraic integrals in the problem of three bodies independent of the classical ones (see [**5**]). This theorem was generalized by Painlevé to the case when integrals are algebraic with respect to the velocities of three gravitating bodies [**6**]. These classical results are of no importance for dynamics, however, since they do not take into account the peculiarities of the behavior of phase trajectories. Equations of motion may happen to be completely integrable but do not have, say, integrals which are polynomial with respect to velocities. Here is a simple example [**7**]. The motion of a point charge along a "plane" torus $\mathbf{T}^2 = \{x, y \bmod 2\pi\}$ in a constant magnetic field is described by the equations

$$\ddot{x} + \Omega\dot{y} = 0, \quad \ddot{y} - \Omega\dot{x} = 0; \qquad \Omega = \text{const}. \tag{6}$$

They have an energy integral $\dot{x}^2 + \dot{y}^2 = h$. It can be shown [7] that equations (6) do not have an additional integral, polynomial with respect to velocities, with smooth and single-valued coefficients on $\mathbf{T}^2$. System (6) is completely integrable, however: the function $\sin(\dot{x} + \Omega y)$ is an additional integral, for instance. The integral $\dot{x} + \Omega y$ is linear with respect to velocities, but it is a multivalued function in the phase space $\mathbf{R}^2 \times \mathbf{T}^2$.

Poincaré was the first to pose a problem on the nonintegrability of the Hamiltonian equations as a whole and to get some results in this respect [8]. He investigated Hamiltonian differential equations of the following kind:

$$\begin{gathered} \dot{x}_s = -\partial H/\partial y_s, \quad \dot{y}_s = \partial H/\partial x_s, \qquad 1 \leq s \leq n, \\ H = H_0(x_1, \ldots, x_n) + \varepsilon H_1(x_1 \cdots x_n, y_1 \cdots y_n) + \cdots. \end{gathered} \tag{7}$$

The Hamiltonian H is a power series with respect to ε, and its coefficients are analytic functions in $\mathbf{R}^n \times \mathbf{T}^n = \{x; y \bmod 2\pi\}$. For $\varepsilon = 0$ we have a completely integrable system. Differential equations (7) are often encountered in applications, and therefore Poincaré considered the problem of their investigation to be the "basic problem of dynamics." Poincaré tried to find out whether equations (7) have first integrals $F(x, y, \varepsilon)$, which are analytic in $D \times \mathbf{T}^n \times (-\varepsilon_0, \varepsilon_0)$, where D is a domain in $\mathbf{R}^n = \{x\}$. It is shown in [1] that it is more expedient to consider a problem on the existence of formal integrals in the form of power series $\sum F_s(x, y)\varepsilon^s$ with coefficients analytic in the domain $D \times \mathbf{T}^n$. This problem is closely connected with the possibility of realizing the classical scheme of perturbation theory.

The problem of the existence of analytic integrals of system (7) for fixed values of the parameter $\varepsilon \neq 0$ is more complicated. One of the most popular problems of this kind is the investigation of the complete integrability of the Hamiltonian system near a stable equilibrium. The formal analysis of this problem dates back to Birkhoff [9] and Siegel, who presented strict proofs [10] (for the discussion of these problems, see [11]).

In the majority of the integrated problems of classical mechanics the known first integrals are extended to a complex domain of variation of the phase variables as single-valued holomorphic (or meromorphic) functions. In connection with this remark an interesting problem arises concerning the complete "complex" integrability of a holomorphic Hamiltonian system. In this case we must bear in mind that the absence of holomorphic integrals in a complex domain does not yet mean that the Hamiltonian system is not integrable in a real sense. Here is a simple example. The linear Hamiltonian system $\ddot{z} + (\alpha^2 + \beta\wp(t))z = 0$ possesses an analytic integral $f(\dot{z}, z, t)$, which is periodic with respect to t, in a real domain (Floquet-Lyapunov theorem). For almost all α and β, however, this system does not have a holomorphic integral in a complexified phase space (see [12]).

In "complex" completely integrable Hamiltonian systems the level surfaces of involute integrals often prove to be not simply real tori $\mathbf{T}^n$, but, being extended to a complex domain, to be Abelian manifolds $\mathbf{T}^{2n}$. In this case the general solution is expressed by the ϑ-functions of complex time. Systems possessing these properties are often said to be "algebraically integrable." When we seek the necessary conditions for algebraic integrability, we usually follow the method of Kovalevskaya, which she applied in 1888 to the dynamics of a rigid body. For the present-day state of these problems see [**13**, **14**, **15**] (see also the report made by P. Van Moerbeke at the International Congress of Mathematicians in Warsaw in 1982).

4. In recent years Poincaré's ideas have been further elaborated, and new phenomena in the behavior of Hamiltonian systems hindering their integrability have been discovered. This made it possible to present strict proofs of the nonintegrability of a number of significant problems of Hamiltonian mechanics (a heavy asymmetric top, a rigid body in ideal fluid, the problem of four-point vertices, etc.). What lies behind nonintegrability consists in the following. There are an infinite number of resonant tori filled with periodic trajectories in the phase space of an unperturbed completely integrable system. These tori become disintegrated when perturbation is added. The families of periodic solutions located on them yield pairs of nondegenerate periodic solutions. The first integrals are dependent on the trajectories of the nondegenerate periodic solutions. The disintegrated resonant tori accumulate, as a rule, by a pair of doubled separatrices (asymptotic surfaces) of the unperturbed problem. When a perturbation is added, the separatrices themselves split and, as a rule, intersect, forming a rather tangled network. The nondegenerate periodic solutions of the perturbed problem, being extended to the plane of complex time, are not single-valued functions, and their branching impedes the presence of holomorphic first integrals in the complexified phase space. For the necessary details see [**1**], which also contains the review of the achievements in this field covering the period up to 1983. In what follows we discuss new problems pertaining to the analysis of the phenomena of nonintegrability.

5. One of the problems of this kind consists in the investigation of perturbed integrable Hamiltonian systems, where at each stage of perturbation theory only a finite number of resonant invariant tori disintegrate. We consider as a model example Hamiltonian equations

$$\dot{x} = -H'_y, \quad \dot{y} = H'_x, \quad H = H_0(x) + \varepsilon H_1(y), \qquad x \in \mathbf{R}^n,\ y \in \mathbf{T}^n, \tag{8}$$

where $H_0 = \frac{1}{2}(Ax, x)$ is a nondegenerate quadratic form with respect to the variables x, and H_1 is a trigonometric polynomial:

$$H_1 = \sum_{m \in \mathbf{Z}^n} h_m e^{i(m,y)}, \qquad h_m = \text{const}. \tag{9}$$

Just as Poincaré did [8], we shall discuss the fact that system (8) has an additional integral as a formal series

$$\sum_{s\geq 0} F_s(x,y)\varepsilon^s$$

with single-valued analytic coefficients in $\mathbf{R}^n \times \mathbf{T}^n$. Since the Fourier series of the perturbation function (9) contains only a finite number of harmonics, Poincaré's results and their known generalizations cannot be applied to systems with a Hamiltonian (8).

We can treat Hamiltonian equations (8) as equations of motion of a mechanical system with a configurational space $\mathbf{T}^n$, kinetic energy H_0, and a small potential εH_1. It should be emphasized that a positive definiteness of the quadratic form H_0 is not presupposed.

Let us agree on some designations. Let $\xi, \eta \in \mathbf{R}^n$. We set $\langle \xi, \eta \rangle = (A\xi, \eta)$. We designate by $\mathfrak{M}$ a finite set of integer-valued vectors $m = (m_1, \dots, m_n)$ for which $h_m \neq 0$. Since $H_1 \neq \text{const}$, it follows that $\mathfrak{M}$ contains at least two elements. Assume that $i_1, i_2, \dots, i_n$ is any permutation of the indices $1, 2, \dots, n$. We set

$$\alpha_{i_1} = \max_{\mathfrak{M}} m_{i_1}, \alpha_{i_2} = \max_{\substack{\mathfrak{M} \\ m_{i_1}=\alpha_{i_1}}} m_{i_2}, \dots, \alpha_{i_n} = \max_{\substack{\mathfrak{M} \\ m_{i_1}=\alpha_{i_1} \\ \dots\dots \\ m_{i_{n-1}}=\alpha_{i_{n-1}}}} m_{i_n}. \tag{10}$$

We term the vector $\alpha = (\alpha_1, \dots, \alpha_n)$ the vertex $\mathfrak{M}$. Formulas (10) yield $n!$ vertices of the set $\mathfrak{M}$, but they are not all different. If we replace $\mathfrak{M}$ in formulas (10) by $\mathfrak{M}\backslash\{\alpha\}$, then we get an integer-valued vector β which is a vertex of the set $\mathfrak{M}\backslash\{\alpha\}$.

THEOREM 1. *We assume that the set $\mathfrak{M}$ has a vertex α such that:*

(i) *the vertices α and β are linearly independent,*

(ii) $m\langle \alpha, \alpha \rangle + 2\langle \alpha, \beta \rangle \neq 0$ *for all integers $m \geq 0$.*

Then the Hamiltonian system (8) *does not have a complete set of independent integrals representable as a power series $\sum F_s(x,y)\varepsilon^s$ with coefficients analytic in $\mathbf{R}^n \times \mathbf{T}^n$.*

This theorem was established by V. Kozlov and D. Treshchev. Note that when all the coefficients $h_m \neq 0$ in the Fourier expansion (9), the nonintegrability of equations (8) follows from the classical result of Poincaré [8]. For $n = 2$ we can assert still more: if in the expansion (9) there are an infinite number of vectors $m \in \mathbf{Z}^n$, which are pairwise independent, then equations (8) are nonintegrable either (see [1]). The proof of Theorem 1 is based on the analysis of the classical scheme of perturbation theory applied to the Hamiltonian equations (8).

There is the following corollary of Theorem 1.

COROLLARY. *If $n = 2$ and the functions H_0 and H_1 satisfy the hypothesis of Theorem* 1, *then the equations with the Hamiltonian $H_0 + H_1$ do not possess* *an additional integral in the form of a polynomial with respect to the momenta x with analytic and single-valued coefficients on $\mathbf{T}^2$.*

This statement is an addition to the classical results concerning the conditions of the existence of polynomial integrals with respect to momenta (see [6]). It should be emphasized that the potential H_1 need not be necessarily small here.

6. One more problem, which we shall discuss here, is connected with topological and geometric conditions for a complete integrability of Hamiltonian systems, which we come across in classical mechanics. Assume that M^n is a complete analytic Riemannian manifold and Σ^{2n-1} is a foliated space of unit tangent vectors. The Riemannian metric defines on Σ^{2n-1} a dynamic system which is a geodesic flow. From the viewpoint of mechanics a geodesic flow describes the motion of a particle along M^n by inertia with unit velocity. The famous principle of Mopertuis reduces the motion under the action of potential forces to a geodesic flow.

THEOREM 2 [**16**]. *If M is a compact two-dimensional surface of genus larger than one, then the geodesic flow on Σ does not possess a nonconstant analytic integral.*

The proof of this theorem is based on the analysis of the set of unstable periodic trajectories. Since the genus of M is greater than one, the Gaussian curvature is negative in the mean. If the curvature is negative everywhere, then the flow on Σ is Anosov's system [**17**]. In that case all periodic trajectories are unstable, they densely fill Σ everywhere, and the geodesic flow does not even possess a continuous integral. The hyperbolic behavior of phase trajectories lies at the basis of the proof of the nonintegrability of the restricted problem of three bodies advanced by Alekseev [**18**]. Note that a curvature negative in the mean is not always negative everywhere.

Theorem 2 has been generalized in different directions. Taimanov has proved the absence of a complete involute set of analytic integrals of a geodesic flow in a multidimensional case when one of the following additional conditions is satisfied [**19**]:

(1) $\dim M < \operatorname{rank} H_1(M, \mathbf{Z})$,

(2) the fundamental group $\pi_1(M)$ does not contain a commutative subgroup of a finite index.

The first condition was proved in [**16**] for $n = 2$, and was formulated as a hypothesis by the author in [**20**]. The second condition is a new one. It would be interesting to find other topological obstacles hindering complete integrability.

Another possible way of achieving generalization is to consider domains with a geodesically convex boundary. Assume that $\hat{M}$ is a compact submanifold with the boundary on the analytic surface M^2. Let $\hat{\Sigma}$ denote the set of all points of Σ which are taken by the projection $\pi\colon \mathbf{T}M^2 \to M^2$ into points of $\hat{M}$. We say that $\hat{M}$ is geodesically convex if for any two close points of the boundary $\partial\hat{M}$, the shortest geodesic of the Maupertuis metric joining the points lies entirely in $\hat{M}$.

THEOREM 3. *If* $\operatorname{rank} H_1(\hat{M}, \mathbf{Z}) > 2$ *and* $\hat{M}$ *is geodesically convex, then the dynamic system on* Σ *does not possess a nonconstant analytic first integral. Moreover, there is no analytic integral even in the neighborhood of the set* $\hat{\Sigma} \subset \Sigma$.

S. Bolotin has found that the condition $\operatorname{rank} H_1(M, \mathbf{Z}) > 2$ can be replaced by a weaker condition $\chi(M) < 0$, where χ is an Eulerian characteristic [**21**]. He has also found an interesting application of the generalized Theorem 3 to the problem of the motion of a point in the gravitational field of n fixed centers. Let $z_1, \dots, z_n$ be different points of a complex plane $\mathbf{C}$. The Hamiltonian of a plane problem of n centers has the form

$$H = \tfrac{1}{2}|p|^2 + V(z), \qquad (z, p) \in U \times \mathbf{C},$$

where $U = \mathbf{C} \backslash \{z_1, \dots, z_n\}$ is a configurational space, V is a gravitational potential of attraction of a moving particle z by the stationary points $z_1, \dots, z_n$, i.e.,

$$V(z) = -\sum_{i=1}^{n} \mu_i |z - z_i|^{-1}, \qquad \mu_i > 0.$$

THEOREM 4 [**21**]. *Assume* $n > 2$. *Then the equations of the problem dealing with* n *centers do not have an analytic integral on the surface* $\{(z, p) \in U \times \mathbf{C} \colon H(z, p) = h \geq 0\}$.

Note that the conditions $n = 1, 2$ correspond to Keplerian and Eulerian integrable problems. When we prove Theorem 4, we make use of the Levi-Civita regularization. Assume that M is the Riemannian surface of the function $\sqrt{(z - z_1) \cdots (z - z_n)}$ and $\pi \colon M \to \mathbf{C}$ is a projection. It turns out that the Levi-Civita regularization reduces a phase flow on the surface $H = h$ to a geodesic flow on M with some complete metric. If D is a disc in a complex plane $\mathbf{C}$ of a sufficiently large radius, then the set $\hat{M} = \pi^{-1}(D)$ is compact, geodesically convex, and homotopically equivalent to M. By the Riemann-Hurwitz formula $\chi(M) = 2 - n < 0$ for $n > 2$.

We can generalize Theorems 2 and 3 to an irreversible case when additional hyroscopic forces act on a system [**7**, **22**]. The origin of these forces differs; i.e., they appear, for instance, upon a transition to a rotating reference system and in the description of the motion of charged particles in a magnetic field. We consider as an example a plane-restricted problem of many bodies: n huge gravitating bodies which are in relative rest, rotating, as a rigid body, about their barycenter with a constant angular velocity, where an additional body of an infinitesimal mass moves under the action of gravitational forces in the plane of the circular orbits of huge bodies. One can show [**22**] that for $n > 2$ the equations of motion of this problem do not possess an analytic integral on the energy surface $H = h > 0$. This statement has not been proved for a restricted problem of three bodies (when $n = 2$). Weaker theorems can be found in [**8**, **23**, **24**]. Note that, in accordance with Chazy's hypothesis (see [**25**]), the problem of three bodies is completely integrable on the level surface of the energy integral for $H > 0$. This hypothesis is related to a more general idea: in the problem of scattering with a noncompact configurational space the data at infinity are candidates for integrals. However, the realization of this idea is hindered by some difficulties of principle connected with the domain of definition and smoothness of the "integral of scattering." One of these difficulties is the possibility of capture in the problems of many interacting particles.

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