The splitting of separatrices and the generation of isolated periodic solutions in Hamiltonian systems with one-and-a-half degrees of freedom

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1. Consider the Hamiltonian equations

$$\dot{x} = H'_y, \quad \dot{y} = -H'_x; \quad (x, y) = z \in \mathbf{R}^2 \tag{1}$$

with Hamiltonian $H(z, t, \varepsilon) = H_0(z) + \varepsilon H_1(z, t) + o(\varepsilon)$. We assume that H is analytic in z, t, ε, and 2π-periodic in t. We further assume that the unperturbed system, with Hamiltonian H_0, has the following two properties:

1) the point $z = 0$ is a non-singular critical point of H_0 of index 1 (a saddle-point);

2) the set $\{z: H_0 = 0\} \setminus \{z: dH_0 = 0\}$ has a bounded connected component w whose closure is $w \cup \{0\}$.

Thus, $z = 0$ is an unstable equilibrium point of the unperturbed system, and the curve w is doubly-asymptotic to a trajectory of it (a loop of the separatrix). Let $z_a(t)$ be one of the doubly-asymptotic solutions of the unperturbed system (1). We define a 2π-periodic function by

$$J(\lambda) = \int_{-\infty}^{\infty} \{H_0, H_1\}(z_a(t+\lambda), t)\, dt,$$

where $\{\ ,\ \}$ is the Poisson bracket. Note that

$$\frac{d^n J}{d\lambda^n} = \int_{-\infty}^{\infty} \underbrace{\{H_0 \{\ldots \{H_0, H_1\}\} \ldots\}}_{n+1}\, dt. \tag{2}$$

Since the mean of J over a period is equal to zero, J must vanish at some point.

Let T_ε be a periodic map of the system (1). A point $\zeta \in \mathbf{R}^2$ is a periodic point of T_ε of period $m \in \mathbf{N}$ if $T_\varepsilon^m \zeta = \zeta$. The periodic points, and only these, are initial values (at $t = 0$) of periodic solutions of (1). A periodic point ζ is called *non-singular* if the eigenvalues of the map $z \mapsto T_\varepsilon^m z$, linearized in a neighbourhood of ζ, are different from 1. Note that the non-critical bounded equipotential lines of H_0 consist entirely of singular periodic, or of non-periodic, points of T_0.

Theorem. *Assume that a function J has a simple zero. Then there are infinitely many analytic functions $\zeta_n: (-\varepsilon_n, \varepsilon_n) \to \mathbf{R}^2, \varepsilon_n > 0$, such that:*

(1) $\zeta_n(\varepsilon)$ is a periodic point of T_ε for all $\varepsilon \in (-\varepsilon_n, \varepsilon_n)$, and is non-singular when $\varepsilon \neq 0$;

(2) $H_0(\zeta_n(0)) < 0$, and the distance of $\zeta_n(0)$ from a loop of w tends to zero as $n \to \infty$.

In general, the sequence ε_n converges to zero. Thus, for sufficiently small values of $\varepsilon \neq 0$ the theorem guarantees the existence of a large, but finite, number of distinct non-singular periodic solutions of (1). Note that, if all the zeros of J are simple, then the stable and unstable separatrices of the perturbed fixed hyperbolic point $z_\varepsilon = O(\varepsilon)$ of T_ε intersect transversely for small values of $\varepsilon \neq 0$, forming a rather tangled network (see [1], [2]). Using the methods of symbolic dynamics it can be proved that there are infinitely many distinct periodic points of T_ε in a neighbourhood of w (see [3]). But these periodic points are in no way connected with the points $\zeta_n(\varepsilon)$.

2. The proof of the theorem is based on an application of Poincaré's small parameter method. For this we pass from the variables $z = (x, y)$ to the canonical action variable I and angle variable φ mod 2π of an unperturbed integrable system in the domain of $\mathbf{R}^2$ given by the inequalities $-c < H_0(z) < 0$, where c is a small positive constant. In the new variables $H = F_0(I) + \varepsilon F_1(I, \varphi, t) + o(\varepsilon)$, where $H_0(z) = F_0(I)$, $H_1(z, t) = F_1(I, \varphi, t)$. The function H is understood to be 2π-periodic in φ and t. Let $\omega(I) = (F_0)'$ be the frequency of the oscillations in the unperturbed system. It is easy to see that $\omega \to 0$ when the equipotential line $H_0(z) = F_0(I)$ unboundedly approximates to a loop of w. Hence, there are infinitely many values I_n such that $\omega(I_n) = 1/n$. We assert that on the invariant curves $I = I_n$ for small values of ε is exactly where the generation takes place of the non-singular periodic points $\zeta_n(\varepsilon)$ of T_ε referred to in the theorem. To see this, one must check that the conditions of a well-known theorem of Poincaré hold (see [1], §§42, 74):

(A) $F_0''(I_n) \neq 0$, (B) the 2π-periodic function

$$f_n(\lambda) = \int_{-\pi n}^{\pi n} F_1\,(I_n,\ (t+\lambda)/n,\ t)\,dt$$

has a non-singular critical point.

It can be shown that (A) holds for sufficiently large values of n. The critical points of f_n coincide with the zeros of the function

$$f_n' = \int_{-\pi n}^{\pi n} \{H_0,\ H_1\}\,(z_n\,(t+\lambda),\ t)\,dt,$$

where $z_n(\cdot)$ is a $2\pi n$-periodic solution of the unperturbed problem. If $z_n(0) \to z_a(0)$ as $n \to \infty$, then the sequence $f_n'(\lambda)$ converges to $J(\lambda)$ uniformly in λ. By using (2) we can show that $f_n'' \to J'$ uniformly in λ. Since, by hypothesis, J has a simple zero, it follows that, for sufficiently large n, f_n satisfies (B), as required.

Note that the perturbed system cannot have non-singular long-period solutions of period $2\pi/\omega = 2\pi n/m$ with $m \neq 1$. More precisely, the existence of such solutions does not in general follow from a consideration of perturbations of the first order in ε. An example is the well-known problem of the planar oscillations of a satellite in a weakly-elliptical orbit [4]. The transversality of the intersections of the separatrices for small non-zero values of the eccentricities of the orbit was shown in [5].

3. The above theorem can be used to establish the existence of a family of non-singular periodic solutions in Hamiltonian systems, whose non-integrability is proved by using the method of splitting of separatrices. Several examples of such systems are given in [2].

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