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ON THE STABILITY OF EQUILIBRIA OF NONHOLONOMIC SYSTEMS

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1. Consider a mechanical system with generalized coordinates $x = (x^1, \ldots, x^n)$ on which constraints, linear in the velocities:

$$a_s \cdot \dot{x} = a_{si}(x)\dot{x}^i = 0, \qquad 1 \leq s \leq m, \quad m < n, \tag{1}$$

and nonintegrable in general, have been imposed. We shall assume that the covectors a_s are linearly independent at all points $x \in \mathbf{R}^n$. Let $2T = G\dot{x} \cdot \dot{x} = g_{ij}\dot{x}^i\dot{x}^j$ be twice the kinetic energy and V the potential of the force field. The dynamics of such a system is described by the Lagrange equations with multipliers of constraints:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = -\frac{\partial V}{\partial x} + \lambda^j a_j, \qquad a_s \cdot \dot{x} = 0, \quad 1 \leq s \leq m. \tag{2}$$

It is well known that critical points of the potential function are equilibrium positions. The converse is not, however, true. Whittaker [1] formulated the problem of finding conditions for stability of equilibria of nonholonomic systems and established the first results in this direction. Whittaker used the method of small oscillations to investigate the stability of equilibria coinciding with critical points of the potential. A history of the problem and a survey of results concerning nonholonomic systems is contained in [2]. We supplement Whittaker's results by Theorems 1 and 2 below.

We assume that the potential V and the coefficients a_{si} and g_{ij} are analytic functions of x. Suppose that $dV(0) = 0$ and let $V = V_k + V_{k+1} + \cdots$ be the Maclaurin series of V. Here V_s is a homogeneous form of degree s, so that $k \geq 2$. Let Π be the $(n-m)$-dimensional plane in $\mathbf{R}^n = \{x\}$ given by the equations

$$a_s(0) \cdot x = a_{si}(0)x^i = 0, \qquad 1 \leq s \leq m.$$

If W is a function, we let $\hat{W}$ denote its restriction to Π. It is clear that $\hat{V}_s \colon \Pi \to \mathbf{R}$ is a homogeneous form of degree s.

THEOREM 1. *If $\hat{V}_k$ does not have a local minimum at the point $x = 0 \in \Pi$, the equilibrium state $x = 0$, $\dot{x} = 0$ is unstable.*

THEOREM 2. *Suppose that the point $x = 0$ is a local maximum (not necessarily strict) of V, and $\hat{V} \not\equiv 0$. Then the equilibrium state $x = 0$, $\dot{x} = 0$ is unstable.*

When $k = 2$, Theorem 1 coincides with Whittaker's result [1]. Theorem 2 is a corollary of Theorem 1. It is worth emphasizing that Theorems 1 and 2 hold even in the case when the constraints (1) are integrable. If, in particular, there are no constraints (that is, $m = 0$ and $\hat{V}_k \equiv V_k$), then Theorem 1 coincides with the result of [3] and [4] devoted to the problem of obtaining a converse of the Lagrange-Dirichlet theorem.

Theorem 2 extends to nonholonomic systems Hagedorn's result [5] about the instability of an equilibrium position of a holonomic system (at which the potential has a local maximum), and it coincides with Hagedorn's result when $m = 0$. If the constraints

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are integrable, then the condition $\hat{V}_k \equiv 0$ can be dropped (see [4]). It is not clear whether Theorem 2 holds without this additional condition in the case of nonintegrable constraints. We note that Theorem 2 cannot be proved by the method of [5] using Maupertuis' variational principle, because this principle does not hold for nonholonomic systems.

2. The proof of Theorem 1 is based on the following observation [3]: if equations (2) have a solution $x(t)$ which asymptotically approaches the point $x = 0$, then the equilibrium state $x = 0$, $\dot{x} = 0$ is unstable. In fact, in view of reversibility, equations (2) admit a solution $x(-t)$ which tends to zero as $t \to -\infty$.

The existence of asymptotic solutions can be established using a general result pertaining to differential equations of the form

$$\ddot{x}^s = \Gamma^s_{ij}(x)\dot{x}^i\dot{x}^j + f^s(x), \qquad 1 \leq s \leq n. \tag{3}$$

Assume that the functions Γ^s_{ij} and $f^s(x)$ are analytic in a neighborhood of $x = 0$, and suppose that $f^s(0) = 0$, so that $x = 0$ is an equilibrium position. Expand $f^s(x)$ in a Maclaurin series: $f^s = f^s_{k-1} + f^s_k + \cdots$. We will suppose that $k \geq 3$. Consequently, the equilibrium $x = 0$ is degenerate.

We will assume that there exists $e \in \mathbf{R}^n$, $|e| = 1$, for which

$$f_{k-1}(e) = \kappa e, \qquad \kappa > 0, \tag{4}$$

where $f_{k-1} = (f^1_{k-1}, \dots, f^n_{k-1})$. It is not hard to show that the "simplified" equation $\ddot{x} = f_{k-1}(x)$ has a solution

$$x_0(t) = a/t^{1/p}, \qquad a = |a|e \in \mathbf{R}^n,\ p = (k-2)/2.$$

Introduce the $n \times n$ constant matrix

$$A = t^2 \frac{\partial f_{k-1}}{\partial x}(x_0(t)).$$

THEOREM 3. *Suppose that the matrix A satisfies one of the following conditions:*

(α) *The spectrum of A does not contain numbers of the form*

$$\mu_s = \frac{s}{p}\left(\frac{s}{p} + 1\right), \qquad s = 2, 3, \dots .$$

(β) *The spectrum of A contains only one number in the sequence $\{\mu_s\}$, the eigenvalues of A are real, and the eigenvectors form a basis of $\mathbf{R}^n$.*

Then equation (3) has a solution asymptotic to the point $x = 0$.

COROLLARY. *Under the conditions of Theorem 3, the equilibrium state $x = 0$, $\dot{x} = 0$ is unstable.*

Case (α) of Theorem 3 is proved in [3]. The asymptotic solution can be found as a series

$$\sum_{i=1}^{\infty} \frac{x_i}{t^{i/p}}, \qquad x_i \in \mathbf{R}^n,\ x_1 = a. \tag{5}$$

Case (β) of the theorem is actually contained in [4]. The asymptotic solution can be represented as a double series

$$\frac{1}{t^{1/p}} \sum_{\substack{i,j=0 \\ lj<i}}^{\infty} \frac{x_{ij}(\ln t)^j}{t^{i/p}}, \qquad x_{ij} \in \mathbf{R}^n,\ x_{00} = a. \tag{6}$$

Here the natural number l is the index of the unique μ_l in the spectrum of A. Series (5) and (6) converge for sufficiently large t.

3. To prove Theorem 1 we reduce equations (2) to the form (3). We can reduce the matrix $G(0)$ to the identity by a linear change of coordinates. Eliminating the Lagrange multipliers λ^i, we write equations (2) in these coordinates (again denoted by x) in the form (3). In addition, the functions $a_s \cdot \dot{x}$, $1 \leq s \leq m$, will be first integrals of the resulting system. It is not difficult to establish that

$$f_{k-1} = -V_k' + \lambda^i a_i(0), \tag{7}$$

where the multipliers $\lambda^1, \ldots, \lambda^m$ are determined by the linear system of equations

$$\lambda^i(a_s(0) \cdot a_i(0)) = a_s(0) \cdot V_k', \qquad 1 \leq s \leq m. \tag{8}$$

Let e be a unit vector on Π, the plane defined in §1, at which the restriction of V_k to Π attains its minimum. It is clear that

$$V_k'(e) = -\kappa e + \nu^j a_j(0), \qquad \kappa > 0.$$

In view of (7) and (8), $f_{k-1}(e) = \kappa e$. Thus, this establishes the existence of the required unit vector (see (4)).

It is easy to show that the vectors $a_1(0), \ldots, a_m(0)$ are eigenvectors of the matrix

$$A = t^2(-V_k'' + (\lambda^i)' a_i(0))(x_0(t))$$

with eigenvalue zero. The remaining eigenvectors lie in Π and are eigenvectors of the symmetric matrix $t^2 \hat{V}_k''(x_0(t))$. Its eigenvalues are computed in [3]. One of them is equal to $2(k-1)k/(k-2)^2$, and the rest are nonpositive. When k is odd, we have case (α) of Theorem 3, and when k is even, case (β). Consequently, by Theorem 3, there exists an asymptotic solution $x(t)$ of the equations (2) written in the form (3). Since $\dot{x}(t) \to 0$ as $t \to \infty$, the constants of the first integrals $a_s \cdot \dot{x}$ are zero. This proves Theorem 1.

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BIBLIOGRAPHY

1. E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, 4th ed., Cambridge Univ. Press, 1937; reprint, Dover, New York 1944.

2. A. V. Karapetyan and V. V. Rumyantsev, Itogi Nauki i Tekhniki: Obshchaya Mekh., vol. 6, VINITI, Moscow, 1983, pp. 3–128. (Russian)

3. V. V. Kozlov, Prikl. Mat. Mekh. **46** (1982), 573–577; English transl. in J. Appl. Math. Mech. **46** (1982).

4. V. V. Kozlov and V. P. Palamodov, Dokl. Akad. Nauk SSSR **263** (1982), 285–289; English transl. in Soviet Math. Dokl. **25** (1982).

5. Peter Hagedorn, Arch. Rational Mech. Anal. **42** (1971), 281–316; **47** (1972), 395.

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