

The sufficiency of the condition follows from (4), and its necessity follows from the following lemma on the estimates of the Neumann function $N(x, y)$ in a semicylinder:

LEMMA 3. Let $y \in C$ and $N(x, y)$ be a solution of the following problem: $-\Delta_x N(x, y) = \partial(x - y) - \lambda(x)$ in C , $\partial N(x, y)/\partial \nu_x = 0$ on ∂C , where $\lambda \in \mathcal{C}_0^\infty(C)$, and $\int_C \lambda(x) dx = 1$. If $|x - y| > \varepsilon$, then

$$|x - y| > \varepsilon, \text{ mo } \begin{cases} |N(x - y)| \leq k \exp(-k_0(x_n - y_n)) & \text{for } x_n - y_n > \varepsilon, \\ |N(x, y)| \leq k(y_n - x_n) & \text{for } y_n - x_n > \varepsilon, \end{cases}$$

where ε is a small positive constant. The ratio of N to the fundamental solution of the Laplace operator in R^n is bounded above and is separated from zero in the zone $|x - y| \leq \varepsilon$.

Let, e.g., $F = \{x \in \partial C: x'/\delta(x_n) \in \sigma, x_n \geq 0\}$, where σ is a domain in R^{n-1} such that $\bar{\sigma} \subset \omega, \delta$ and δ is a decreasing function on $(0, \infty)$. Then the regularity criterion (5) can be expressed in

the form $\int_0^\infty \delta(s)^{n-3} s ds = \infty$ if $n > 3$, and $\int_0^\infty \frac{s ds}{|\log \delta(s)|} = \infty$ if $n = 3$.

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A CONJECTURE ON THE EXISTENCE OF ASYMPTOTIC MOTIONS IN CLASSICAL MECHANICS

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1. The motion of mechanical systems is described by the well-known Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \quad x \in R^n$$

with "natural" Lagrangian $L = k(\dot{x}, x) - u(x)$, where $k = \langle K(\dot{x})\dot{x} \rangle / 2$ is a positive definite quadratic form (the kinetic energy), and $u: R^n \rightarrow R$ is the potential energy of the given system. One can always choose the coordinates x to ensure that $K(0) = E$. Let $u'(0) = 0$. Then $\dot{x}(t) \equiv 0$ is an equilibrium solution to Lagrange's equations. We call a solution $x(t) \neq 0$ asymptotic if $x(t) \rightarrow 0$ for $t \rightarrow \infty$. The energy integral $k + u = \text{const}$ implies that $\dot{x}(t)$ tends to zero, too. If $u(x)$ has a local (not necessarily strict) minimum at the point $x = 0$, then Lagrange's equations have no asymptotic solutions (at the point $x = 0$). It seems plausible that the asymptotic solutions actually exist when $x = 0$ is not a local minimum of the analytic function $u(x)$. In the C^∞ case this does not happen. Since the equations of motions are reversible (i.e., $x(-t)$ is a solution whenever $x(t)$ is), the existence of asymptotic solutions implies that the equilibrium $x = 0$ is unstable. The converse is not true. A simple example is: $x = (x_1, x_2) \in R^2$, $L = \dot{x}_1^2 + \dot{x}_2^2 + x_1^2$.

THEOREM. Let $x = 0$ be a critical point of the analytic function $u(x)$, and suppose that it is not a local minimum. Then there exist asymptotic solutions (at $x = 0$) to the Lagrange equations, provided that one of the following conditions is satisfied:

- A) $u(x)$ is a quasihomogeneous function.
- B) $u(x)$ is a semiquasihomogeneous function.
- C) $n = 2$ and $x = 0$ is an isolated critical point of $u(x)$.

2. In order to prove the theorem, one needs the following:

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LEMMA. Let $x = 0$ be an isolated critical point of the smooth function $u(x)$ which is not a local minimum. Assume that in a domain $U_{\bar{\epsilon}} = \{x: |x| < \epsilon, u(x) < 0\}$ there exists a differentiable vector field $v(x)$ such that

- 1) $\langle v, u' \rangle \leq 0$ in $U_{\bar{\epsilon}}$,
- 2) $\langle v' \xi, \xi \rangle \geq c \xi^2 \quad \forall \xi \in \mathbb{R}^n$, and $\forall x \in U_{\bar{\epsilon}} (c = \text{const} > 0)$,

and

- 3) $|v(x)| = 0 (|x|)$ as $x \rightarrow 0$.

Then Lagrange's equations have asymptotic solutions.

To carry out the proof, consider the differentiable function $f(t) = \langle w(x), \partial k / \partial \dot{x} \rangle |_{x(t)}$, where $w = v - \sigma u'$. For small $\sigma > 0$ and $|x|$ one has the bound $\dot{f} \geq c_1 u^2 + \sigma u'^2$, $c_1 > 0$ (see [1]). Let the solution $x(t)$ lie on the zero energy level. Since $f(t)$ is bounded, $x(t) \in \bar{U}_{\bar{\epsilon}}$ and $f \geq \sigma u'^2$ imply that $x(t)$ either leaves the small domain $\bar{U}_{\bar{\epsilon}}$ in a finite time or tends asymptotically to the point $x = 0$. Suppose that the equations have no asymptotic solutions. Let $x_m \in U_{\bar{\epsilon}}$ and let $x_m \rightarrow 0$ as $m \rightarrow \infty$. The trajectories passing through the x_m 's leave the domain $\bar{U}_{\bar{\epsilon}}$, intersecting the sphere $|x| = \epsilon$ at certain points y_m and with certain velocities v_m . Consider the sequence of solutions $x_m(t)$ having initial conditions $x_m(0) = y_m$, $\dot{x}_m(0) = -v_m$. Then, given any $T > 0$, one has, beginning with some m , $x_m(t) \in \bar{U}_{\bar{\epsilon}}$ for all $0 \leq t \leq T$. The sequence of functions $x_m(t): [0, T] \rightarrow \bar{U}_{\bar{\epsilon}}$ is equicontinuous (because $|\dot{x}| \leq \kappa$, $\kappa > 0$, by the conservation of energy). By Arzela's theorem, there exists a subsequence $x_{mp}(t)$ which converges on $[0, T]$ to a continuous function $x^{(1)}(t)$. Beginning with some number p , the functions $x_{mp}(t): [0, 2T] \rightarrow \bar{U}_{\bar{\epsilon}}$ are defined, and from the sequence they form one can extract a subsequence which converges uniformly to a function $x^{(2)}(t)$. The functions $x^{(1)}(t)$ and $x^{(2)}(t)$ coincide on $[0, T]$. Continuing this process indefinitely, we exhibit a continuous limit function $x(t): [0, \infty) \rightarrow \bar{U}_{\bar{\epsilon}}$. The continuous dependence of solutions upon the initial conditions ensure that $x(t)$ is differentiable and satisfies Lagrange's equations. Since $x(t) \in \bar{U}_{\bar{\epsilon}}$ for all $t > 0$, $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

3. First let the critical point $x = 0$ be isolated. A polynomial $u(x)$ is said to be a quasi-homogeneous function of degree $s \in \mathbb{N}$ with exponents $\alpha_1, \dots, \alpha_n \in \mathbb{N}$ if for any $\lambda \in \mathbb{R}$ we have $u(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n) = \lambda^s u(x_1, \dots, x_n)$. In case A), one can set $v(x) = Dx$, where $D = \text{diag}(\alpha_1, \dots, \alpha_n)$. A function $u(x)$ is said to be semiquasihomogeneous if $u = u_0 + u_1$, where u_0 is a quasihomogeneous function of degree s having an isolated singularity, while $u_1 = o(|x|_x^s)$, where $|x|_x = \sum |x_i|^{1/\alpha_i}$ is the "quasihomogeneous" norm in \mathbb{R}^n . In case B), the field $v(x)$ is a certain perturbation of Dx (see [1]). Finally, in case C), the field v was constructed in V. P. Palamodov's work [2] (under the restriction $K(x) \equiv E$; in paper [3], a device permitting the circumvention of this difficulty was proposed). Although this field is not differentiable everywhere, this does not affect the existence of the asymptotic solution. If in case A) the critical point $x = 0$ is not isolated, one can proceed as follows and prove that an asymptotic solution exists: Consider the function $f(t) = \langle Dz(t), \dot{z}(t) \rangle$, where $z(t)$ is a solution whose total energy is zero. One can show that the bound $\dot{f} \geq -c_2 u$, $c_2 > 0$, holds in $\bar{U}_{\bar{\epsilon}}$, provided ϵ is small. Applying the arguments of Sec. 2, we exhibit a "limit" solution $x(t)$ such that $x(t) \in \bar{U}_{\bar{\epsilon}}$ for t large enough, and $x = 0$ lies in the closure of the trajectory $x(t)$. The equality $2\dot{f} = \dot{g} \sum \alpha_m x_m^2$, where $g = \ln \sum \alpha_m x_m^2$, is plain. Also, $f(t) < 0$ along the solution $x(t)$. Thus, $g(t)$ is monotonically decreasing and takes arbitrarily large negative values. Consequently, $g(t) \rightarrow -\infty$ as $t \rightarrow \infty$, and so $x(t) \rightarrow 0$.

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