Instability of equilibrium in a potential field

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1. We consider the system of Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}=0$$

with analytic Lagrangian $L(x, \dot{x})$: $U \times R^n \to R$, where U is a domain of R^n. For mechanical systems $L = K(x, \dot{x}) - \Pi(x)$, and for fixed $x \in U$ the function K is a positive definite quadratic form in the velocities $\dot{x} \in R^n$. In mechanics the domain U is called the position space ($n = \dim U$ is the number of degrees of freedom), and the functions K and Π are called the kinetic and potential energies, respectively. Analytic solutions $x(t)$: $R \to U$ of the Lagrange equations are called motions. Critical points of the potential energy, and only they, are points of equilibrium ($x(t) \equiv$ const). If in a position of equilibrium the potential energy has a strict local minimum, then this equilibrium is stable (Lagrange's theorem). This is valid, of course, not only for analytic systems. Up to now it is not known whether the condition of Lagrange's theorem is necessary for stability of equilibrium. The problem of inverting Lagrange's theorem on stability goes back to Lyapunov [1]. The following is an old conjecture.

Conjecture on instability: If a position of equilibrium is isolated and the potential energy in this position does not have a local minimum, then the equilibrium is unstable.

The proof of this assertion published by Chetaev [2] is faulty. Until recently it has been proved only in certain special cases: for example, when the equilibrium is a non-degenerate critical point of the potential energy (Lyapunov [1]), or when Π is a homogenous form (Chetaev [3]). Recently Palamodov proved the conjecture on instability for two-dimensional systems ($n = 2$) with "Euclidean" kinetic energy $K = \langle \dot{x}, \dot{x}\rangle/2$, where $\langle,\rangle$ is the usual scalar product in R^2. I prove this result for systems with arbitrary kinetic energy.

2. We consider an analytic function $u(x)$ ($|x| < r$, $u(0) = 0$), for which $x = 0$ is an isolated critical point. If $u(x)$ does not have a local minimum, then the domain $U^- = \{u(x) < 0\}$ is not empty and its closure contains the point $x = 0$.

Lemma *(Palamodov [4]). For some $\rho > 0$, in the domain $U_\rho^- = U^- \cap \{|x| < \rho\}$ there is a continuous vector field $v(x)$ having the following properties:*

1) $\langle v, u'_x\rangle < 0$;

2) *v has continuous first-order partial derivatives everywhere in U_ρ^- except at points lying on a finite set Γ of smooth curves;*

3) *the Jacobian v' satisfies the inequality*

$$\langle v'\xi, \xi\rangle \geqslant c\langle \xi, \xi\rangle, \qquad c > 0.$$

The idea of constructing such a vector field is contained in a paper by Chetaev [3]. Following [4], we introduce another vector field $w = v - \sigma u'_x$, $\sigma > 0$. For small σ

$$\langle w'\xi, \xi\rangle = \langle v'\xi, \xi\rangle - \sigma\langle u''\xi, \xi\rangle \geqslant \alpha\langle \xi, \xi\rangle, \qquad \alpha > 0.$$

Moreover $\quad \langle w, u'\rangle \leqslant -\sigma\langle u', u'\rangle = -\sigma u'^2.$

Theorem. *Almost all solutions $x(t)$ of the system $\ddot{x} = -u'_x$ lying on the zero energy level $\dot{x}^2/2 + u(x) = 0$ either*

A) *leave U_ρ^- in a finite time, or*

B) *tend to thepoint $x = 0$ with respect to some sequence $\{t_k\}$, where $t_k \to \infty$ as $k \to \infty$.*

Using the property of invertibility (the simultaneous existence of solutions $x(t)$ and $x(-t)$), we deduce the instability of equilibrium at $x = 0$. Palamodov has proved [4] that all solutions of the equation $\ddot{x} = -u'_x$ with negative energy leave U_ρ^- in a finite time.

Proof of the theorem. Almost all motions $x(t)$ have the following property: the set of values of $t \in R$ for which $x(t) \in \Gamma$ is of dimension zero. In fact, the excluded motions are at most countable.

We consider the continuous function $l(t) = \langle w(x(t)), \dot{x}(t)\rangle$. For almost all t

(1) $$\dot{l} = \langle w'\dot{x}, \dot{x}\rangle - \langle w, u'\rangle \geqslant \alpha\dot{x}^2 + \sigma u'^2.$$

We note that (4) and (5) may not be satisfied if $|\gamma| = ((n-2)^2 + 4\mu_0)^{1/2}$ or if we drop the requirement that the $a_{kj}(x)$ are continuous at $\mathcal{O}$. Observe that the condition $\mu(\rho) \geqslant \mu_0 = \text{const} > 0$ is satisfied if for the cone $K(\xi) = \{x : |(x, \xi)| \leqslant b\ |x|\}$, $\xi \in R^n$, $b = \text{const} > 0$, for any $\rho \leqslant \rho_1$ we have $(K(\xi) \cap S_\rho^0) \subset (R^n \setminus \Omega) \cap S_\rho^0$ for some $\xi(\rho)$, $|\xi| = 1$.

2. We consider an equation of the form

$$a_{kj}(x)\, u_{x_j x_k} + a_j(x)\, u_{x_j} + a(x)\, u = 0 \tag{6}$$

with the boundary condition (2) for $n = 2$, assuming that a_{kj}, a_j, and a are bounded measurable functions in Ω and that (3) is satisfied, $a \leqslant 0$ in Ω.

Theorem 3. *Suppose that* $u(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *satisfies* (6) *in* Ω, *that the Jordan curve* Γ_1 *belongs to* $\partial\Omega$ *and contains* $\mathcal{O}$ *and a point* P *on the circle* $|x| = \rho_1$, $0 < \rho_1 \leqslant 1$, *and that* $u = 0$ *on* $\Gamma = \partial\Omega \cap \{x : |x| < \rho_1\}$. *Then* $|u(x)| \leqslant C_2\, |x|^{\alpha}$, *where* $C_2, \alpha = \text{const} > 0$ *and* α *depends only on* λ_1 *and* λ_2.

3. Let us consider a system of the plane theory of elasticity. Let $u(x) = (u_1(x), u_2(x))$. We denote $\partial u_i/\partial x_h$ by $u_{i/h}$ and $(u_{1/h}, u_{2/h})$ by $u_{/h}$, $E(u) = u_{i/k}u_{i/k}$. We define a generalized solution of the system

$$(a_{ij}^{hk}(x)\, u_{j/k})_{/h} = f_i(x) \qquad (i = 1, 2) \tag{7}$$

with the boundary condition

$$u = 0 \quad \text{on} \quad \Gamma, \qquad \Gamma \subset \partial\Omega, \quad \mathcal{O} \in \Gamma, \tag{8}$$

as a vector-valued function u such that $u_1 \in H_1(\Omega, \Gamma)$, $u_2 \in H_1(\Omega, \Gamma)$, and for any vector-valued function $v \in H_1(\Omega, \partial\Omega)$ we have the integral identity

$$\int_\Omega a_{ij}^{hk} u_{j/k} v_{i/h}\, dx = -\int_\Omega f_i v_i\, dx.$$

We suppose that the coefficients a_{ij}^{hk} of (7) are bounded and measurable $a_{ij}^{hk} = a_{hj}^{ik} = a_{ik}^{hj} = a_{ji}^{kh}$ $(i, j, k, h = 1, 2)$ and for any symmetric matrix $\{\eta_{ij}\}$: $\lambda_1 \eta_{ih}\eta_{ih} \leqslant a_{ij}^{hk}\eta_{ih}\eta_{jk} \leqslant \lambda_2 \eta_{ih}\eta_{ih}$, $\lambda_1, \lambda_2 = \text{const} > 0$. Let $l(t)$ be the length of the longest arc on $\sigma_t = \Omega \cap \{x : |x| = t\}$ whose ends belong to $\partial\Omega$, $\Omega_t = \Omega \cap \{x : |x| < t\}$.

Theorem 4. *Suppose that* $u(x) = (u_1(x), u_2(x))$ *is a generalized solution of the problem* (7)–(8), $\Gamma = \partial\Omega \cap \partial\Omega_T$, *and the set* σ_t *is not empty for* $0 < t \leqslant T$, *where* $T = \text{const} > 0$. *Then*

$$\int_{\Omega_t} E(u)\, dx \leqslant (\Phi(t))^{-1} \int_{\Omega_T} E(u)\, dx, \qquad 0 < t \leqslant T,$$

where $\Phi(t)$ *is defined for* $0 < t \leqslant T$ *and satisfies the differential equation* $\Phi'(t) = -A(t)\Phi(t)$, $A(t) = \pi\lambda_1(\lambda_1 + 2\lambda_2)^{-1}(l(t))^{-1}$ *and the initial condition* $\Phi(T) = 1$.

Theorem 5. *Suppose that the conditions of Theorem* 4 *are satisfied and that the coefficients* $a_{ij}^{kh}(x)$ *of* (7) *are continuous in* Ω_T. *Suppose also that* $\partial\Omega$ *for* $|x| \leqslant T$ *is such that if* $x^0 \in \partial\Omega$, *then every circle* $|x - x^0| = \rho$ *with* $\rho < \frac{1}{2}|x^0|$ *for* $|x^0| < T/2$ *contains a point of* $\partial\Omega$. *Then*

$$|u(x)|^2 \leqslant C_3 (\Phi(3|x|/2))^{-1} \int_{\Omega_T} E(u)\, dx, \qquad C_3 = \text{const} > 0, \quad x \in \Omega_{T/2}, \tag{9}$$

where the functions $\Phi(t)$ *is defined in Theorem* 4.

It is easy to see that if $l(t) \leqslant \beta t$, $0 < \beta \leqslant 2\pi$, then it follows from (9) that $|u| \leqslant C_4\, |x|^{\theta}$, where $\theta = \pi\lambda_1(\beta(\lambda_1 + 2\lambda_2))^{-1}$.

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