Topological Phenomena in Metals

Quasiperiodic Functions on the Plane: Topology and Dynamics

Magnetic Field as a Main Source of Topological Phenomena

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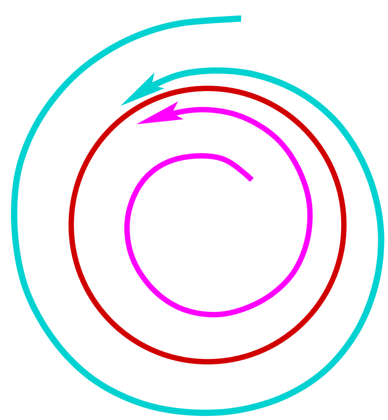
Introduction. Three new branches of Mathematic were started in the late XIX Century. Long period they were developed far from Theoretical Physics:

1. Dynamical Systems: Fig 1



2. Topology: Fig 2

3. Algebraic Geometry:

$$y^n + a_1(x)y^{n-1} + ... + a_n(x) = 0$$

Take famous equations KdV and SG:

$$u_t = 6uu_x + u_{xxx}, u_{\chi\eta} = \sin u(x,t)$$

The Inverse Scattering Transform discovered for KdV in 1967, describes Solitons and their interaction: How they are passing through each other.
Huge Family of the exact "Finite-Gap" solutions, periodic and quasiperiodic in x, were found in 1974 for KdV, solving the periodic problem. People use

them to describe the so-called "Non-linear WKB" Approximation. Riemann Surfaces play key role here. Theory of the Schrodinger Operators $L = -\Delta + u(x)$ with periodic potential also was improved for $n = 1, 2$. How to transform this area into something useful? Still few people can use it now.

Conclusion: Except the Spectral Theory of One- and Two-Dimensional Periodic Schrodinger Operators, The Algebraic Geometry and Nonstandard Algebraic Methods appear in the Exotic Classical and Quantum Systems only.

Quantum Field people also started to use Algebraic Geometry and New Algebra for the String Theory. In XIX Century Jacobi, Neumann, Clebsch, Kovalevskaya developed Riemann Surfaces studying some strange integrable cases. This connection was reestablished by the Modern Mathematical Physics. Pure Mathematics completely forgot it.

Topology: The Ancient Greeks already observed knots (see Fig 2). Topological observations have been made by

Euler out of pure curiosity. In XIX Century Electricity, Magnetism and Hydrodynamics pushed ahead topological ideas. Gauss, Maxwell, Kelvin made several topological observations: The Linking Number, The famous Integral Relation for the Curvature, The Critical Points of Functions, The Integral associated with Vorticity, Properties of Knots. Kelvin intended to classify atoms using knots (turned out to be a completely wrong idea). Betti (chemist) invented cycles and homology. Cauchy and Riemann invented a lot of topological ideas through the Complex Analysis. Topology of Riemann Surfaces was understood. Physicists and Chemists always enjoyed topological ideas. They played with topology without any applications already in XIX Century. Does not it remind you similar phenomena today?

Poincare founded Topology and Dynamics. Till 1970s they were developed mostly by pure mathematicians. Only for two-dimensional sphere and 2-torus qualitative dynamics is simple. For two-dimensional surfaces even Hamiltonian

Dynamics might be enormously complicated for genus ≥ 2 (Fig 3).

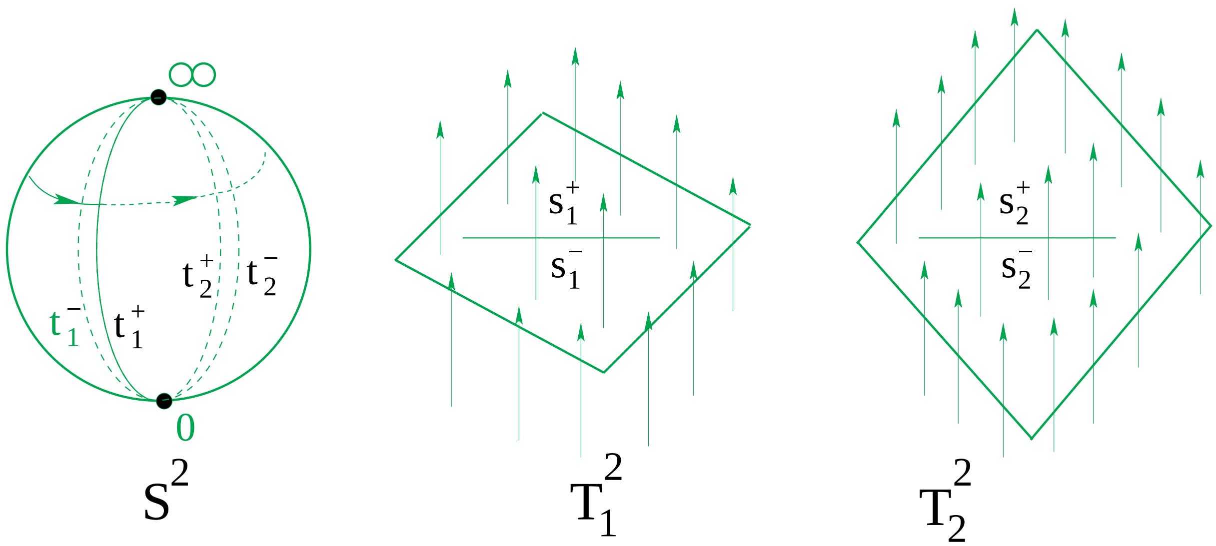


Fig 3: $g = 2$. Two tori T_1^2, T_2^2 with the different straight line flows are given. The equal obstacles s_1, s_2 are fixed in each of them $s_i \subset T_i^2$ Meeting the obstacle, the trajectory jumps to another torus.

Topology and Dynamics. Topological Phenomena in Physics.

General Relativity: Only the idea of manifold is needed. The Schwarzschild-Kruskal Manifold, 1916-1960; Dynamics of Cosmological Models (since late 1960s). Dirac Magnetic Monopole was not understood till 1970s.

Topological phenomena discovered by physicists in 1970s and 80s:

1.Yang-Mills Theory: t'Hooft-Polyakov Monopole and Instantons: Pontryagin Characteristic Classes, Fibre Bundles and Degree of Map are needed.

2.Singularities in the Condensed Matter Physics; Liquid Crystals, Superconductors (Superfluid Liquids): Homotopy Groups are needed.

3.Integral Quantum Hall Effect (2D metals): It corresponds to the Geometry and Topology in the Space of Quasimomenta. Fibre Bundles and Chern Classes are needed.

Normal Metals in the Magnetic Field: Topology and Dynamics in the space of Quasimomenta.

This area was intensively discussed by

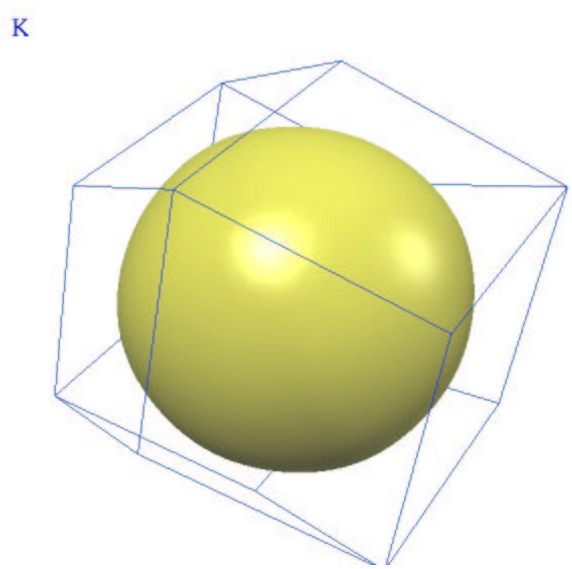
physicists around 1960. Some important dynamical observations have been made by the Russian School (I.Lifshitz, M.Azbel, M.Kaganov, V.Peschanski). However, it was too early: Nontrivial Dynamical Systems were practically unknown to quantum physicists. Some misleading mistakes were committed; nobody really understood these works. This subject was dropped. It attracted my attention in early 1980s. Nontrivial physical conclusions were found in the late 1990s only.

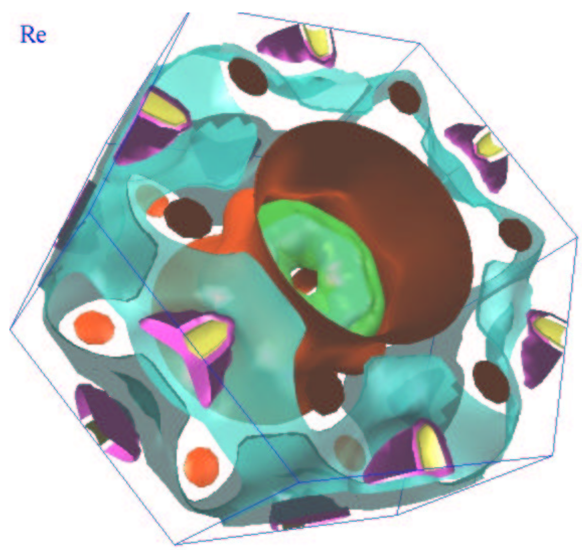
A.Zorich, S.Tsarev, I.Dynnikov, A.Maltsev, R.Deleo worked with me. Some Physicists in the Landau Institute and Maryland convinced me that it would be stupid to miss this subject if something new can be found here. Most people think that "everything is already done in this area".

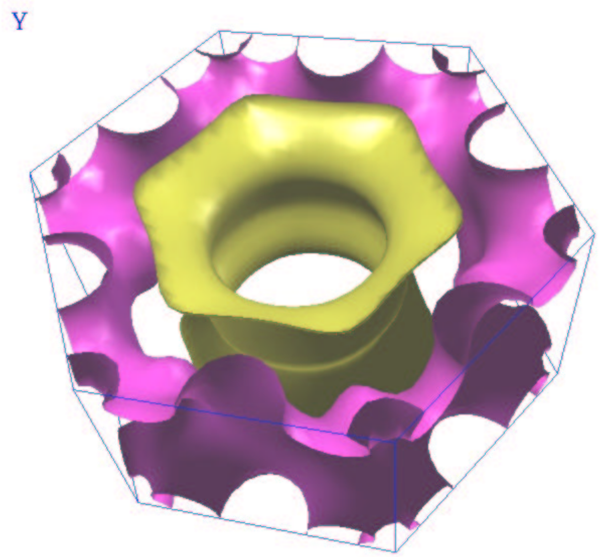
Let me remind some basic ideas.
The Space of Quasimomenta is a 3-torus T^3 defined by the reciprocal lattice, for the single crystal normal metal:

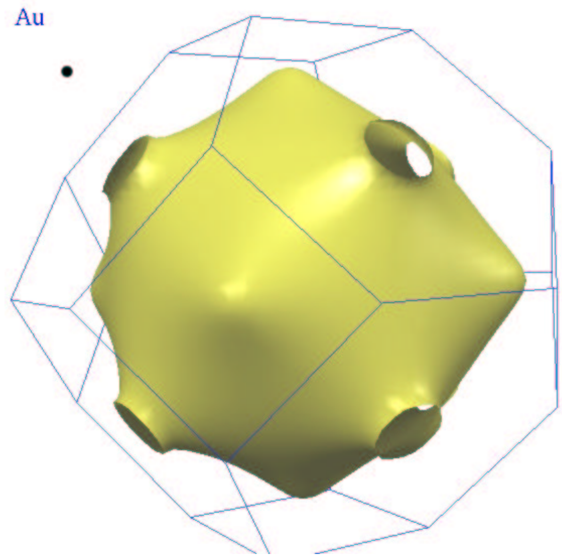
$$T^3 = R^3/Lattice$$

with coordinates $p = (p_1, p_2, p_3)$. The dispersion relation defines a Fermi Surface M^2 by the equation $\epsilon(p) = \epsilon_{Fermi}$. It is a 2-manifold M^2 located in the 3-torus T^3. It is a boundary of the domain $\epsilon \leq \epsilon_{Fermi}$. The Fermi surfaces of metals are very well known (Fig 4).









Some of them have nontrivial topology. For the low temperature only the electrons located nearby of Fermi Surface are essential for the electrical conductivity. Let me define A Topological Rank $r(M^2)$ of the (connected) Fermi Surface: there are 3 independent directions in R^3. Fermi surface in R^3 remains connected along some of them.

Their number is equal to $r \leq 3$ (see Fig 5):

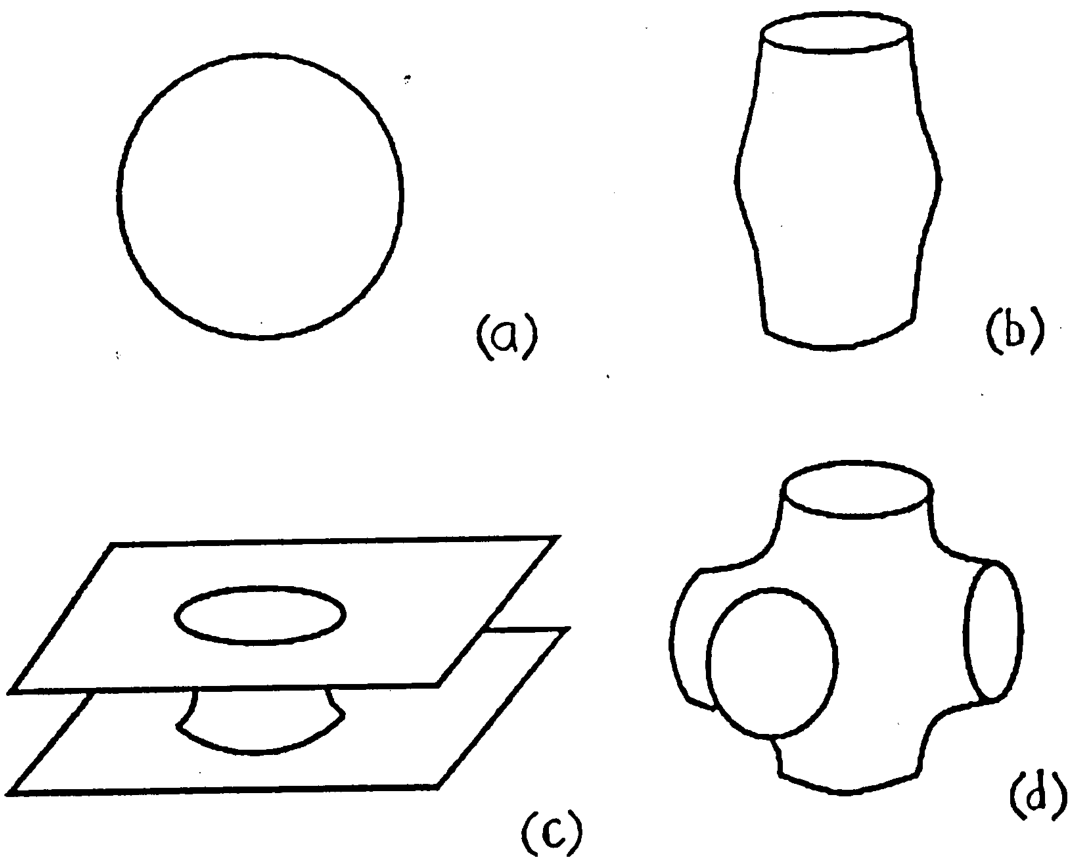


Fig 5,a: $r = 0$; Fig 5,b: $r = 1$;

Fig 5,c: $r = 2$; Fig 5,d: $r = 3$.

Topological rank is more important than the genus of Fermi Surface.
In the Magnetic Field electrons start to move. For the low temperature and broad range of magnetic fields

$$1t \leq B \leq 10^3 t$$

we can use "semiclassical" approximation for the motion of electrons along the Fermi Surface. The electron trajectories coincide with sections of the Fermi Surface by the planes orthogonal to magnetic field. It is Hamiltonian System on the Fermi Surface. The Geometric Strong Magnetic Field Limit claims that everything important for the electrical conductivity should follow from that system. It was basically understood in 1960s.

Useful Remark: One might say that the electron trajectories are the level curves of the Quasiperiodic Functions on the Plane with 3 independent frequencies. Another application of this idea will be presented.

It would be mistakable to think that these dynamical systems are trivial. We divide "typical" trajectories on the three types (see Fig 6, a,b,c).

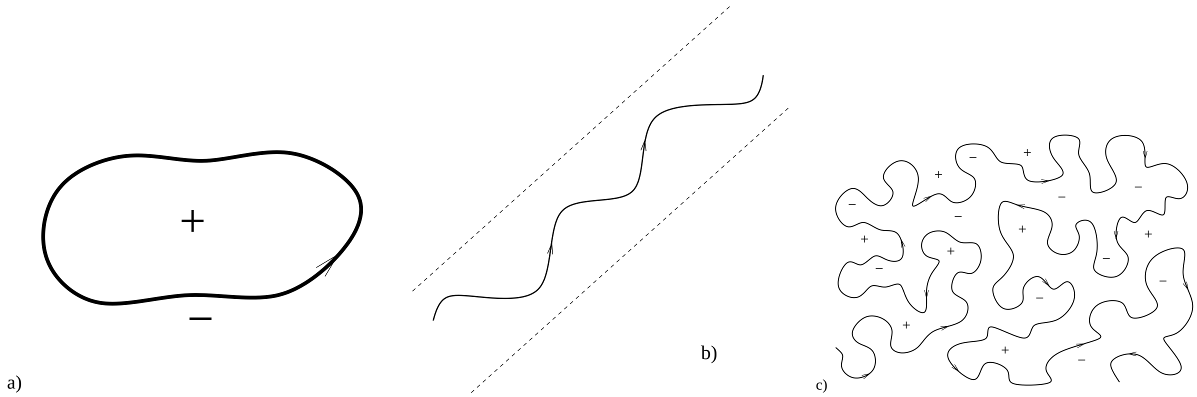


Fig 6,a: Compact type;

Fig 6,b: Stable Quasiperiodic;

Fig 6,c: Chaotic.

The contribution of the trajectories of the Compact Type is of the order B^{-2} or B^{-1} for some components of the magnetoconductivity in the plane orthogonal to B

The contribution of quasiperiodic trajectory is essentially known: in the same plane magnetoconductivity σ has a limit where Exactly One Eigenvalue Vanishes. Its direction coincides with the mean direction of trajectory in the space of quasimomenta. Simple examples of Stable Quasiperiodic trajectories were found many years ago.

What can one say about the contribution of the whole system? You have to combine all trajectories.

Our Results: 1. For the generic Fermi Surfaces and directions of Magnetic Field

B only Compact and Stable Quasiperiodic Trajectories survive.

2.The exceptional directions of magnetic field occupy a set in S^2 whose fractal dimension is no more than 1 (probably, even strictly less than 1). There are examples such that Chaotic Trajectories really appear. Their contribution to the magnetoconductivity σ decreases for all 3 directions including the direction of B.

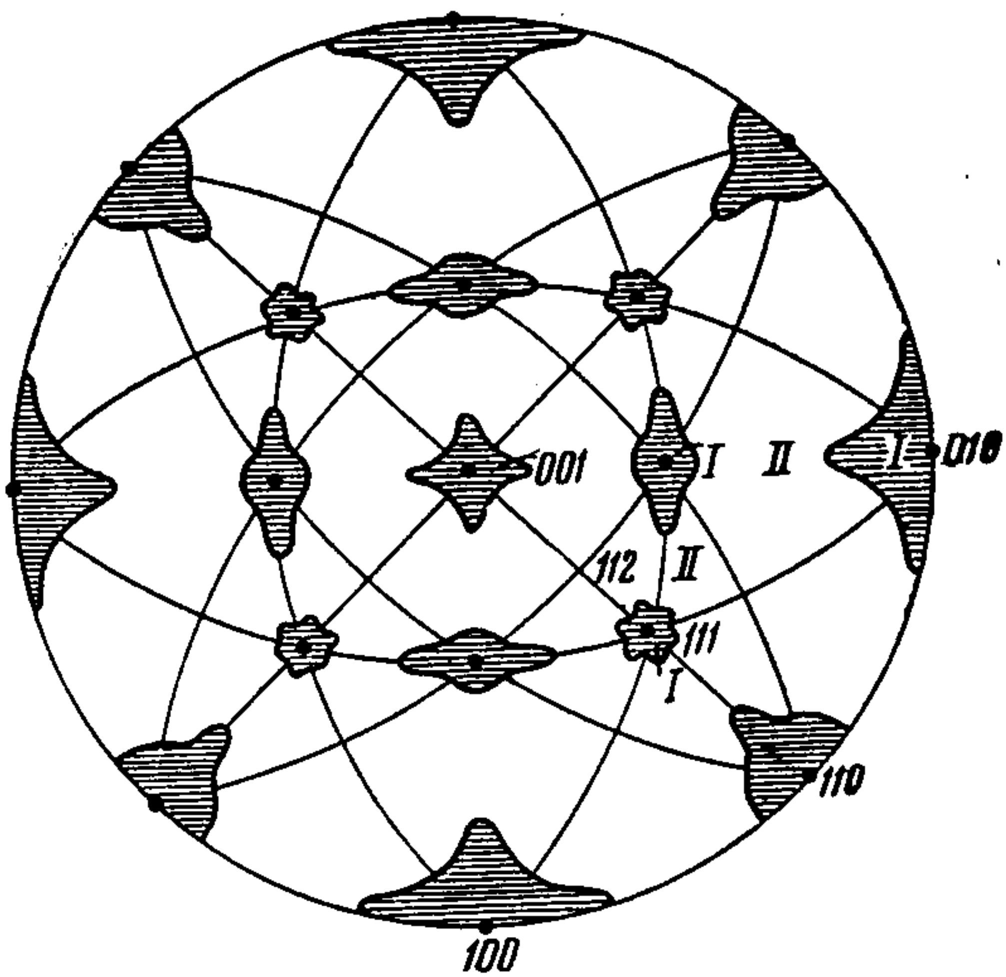
3.Topological Resonance:

I.All Stable Quasiperiodic Trajectories have the same mean direction. (A misleading mistake was done exactly here in the old works of Lifshitz group).

II. This direction can be obtained as intersection of the plane orthogonal to magnetic field with some Integral Plane (i.e. generated by 2 vectors of the reciprocal lattice) . It can be characterized by the 3 relatively prime integers

$$m = (m_1, m_2, m_3)$$

This plane remains unchanged under the small variations of B. Therefore these integer numbers can be easily extracted from observations. The sphere S^2 is essentially covered by these domains with area μ_m corresponding to the values of the numbers m and by the (largest) domain where everything is trivial $m = (0, 0, 0)$. The remaining part has a measure equal to zero.



Other applications: 2D Quasiperiodic Potentials.

Another application of theory of 2D quasiperiodic functions (pointed out by Maltsev) is given by the quasiperiodic potentials in high-mobility 2D electron structures in presence of orthogonal magnetic field $\mathbf{B}$. We consider a hight-mobility 2D electron structure (like in $AlGaAs-GaAs$ structures) at low temperatures ($T \leq 4.2K$). There are different ways to make an additional modulation potential (Holographic illumination, modulating bias, piezoeffect) affecting the electron motion in the plane. For weak modulation potentials the geometry of open level curves of potential $V_B^{eff}(\mathbf{r})$ plays the main role to the "drift" conductivity in the limit $\tau \to \infty$. 1D-modulated potentials were considered by Beenakker to explain the effect of "commensurability oscillations". Double-periodic potentials also were considered by physicists. Easy to make quasiperiodic modulating potentials using the superposition of several independent interference pictures; our dynamical results can be applied.