Henri Poincare and XXth Century Topology.

dedicated to the 150th birthday of Henri Poincare

and 100th birthday of Henri Cartan

S.P.Novikov

University of Maryland, College Park and Landau Institute, Moscow

110 years passed since Henri Poincare' published his memoir ,,Analysis Situs'' (AS, 1895). Topology under the name Analysis Situs appeared as a new branch of Mathematics.

I published my first work in 1959. Since 1970s I work in different areas of Mathematics and Mathematical Physics. However, I consider myself primarily as a topologist. During my lifetime I heard a lot of romantic tales about the first works of Poincare' and his predecessors partly based on the oral traditions of topologists.

I. What exactly Poincare' did in his works? This is the list: 1.Sur l' Analysis Situs, C.R., 1892, 115, 633-636; 2.Sur la generalisation d'un theoreme d'Euler relatif aux polyedres, C.R., 1893, 117, 144-145; 3.Analysis Situs,– J.Ecole Polytechniques, 2e ser, 1895 Cahier 1, 1-121; 4.Sur le nombres de Betti.–C.R., 1899, 128, 629-630; 5.Complement a' "l'Analysis Situs", Rendic. Circ. mat. Palermo, 1899, 13, 285-243; 6.Second complement a' "l'Analysis Situs",– Proc. London Math. Soc., 1900, 32, 277-308; 7.Sur "l'Analysis Situs",– C.R., 1901, 133, 707-709; 8.Sur la connexion des surfaces algebriques,– C.R., 1901, 133, 969-973; 9. Sur certaines surfaces algebriques; troiseme complement a' "l'Analysis Situs",-1902, Bull. Soc.Math. France, 30, 49-70; 10.Sur le cycles des surfaces algebriques; quatrieme complement a' "l'Analysis Situs",-J. math pures et appl., 5e ser, 1902,8, 169-214; 11.Cinquieme complement a "l'Analysis Situs",-Rendic. Circolo mat. Palermo, 1904, 18, 45-110; 12.Sur un theoreme de geometrie,-Rendicotti Circolo mat. Palermo, 1912, 33, 375-407.

II. How his ideas were reflected in the works of the XXth Century Topologists?

Prehistory:Ancient Greeks, knots and Alexander The Great: He took sword and unknotted it in his own way. Is not it exactly what our Nature is doing with DNA in the living creatures?

Two observations of Euler. 1.The Euler identity for the numbers of Vertices (V), Faces (F)and Edges (E) of the Convex Polytop in euclidean 3-space: $V - E + F = 2$

According to AS, french "admiral" extended this relation to the nonconvex case.

2. The Imbedding Problem: Three houses on the plane cannot be joined with three wells by paths not crossing each other.

Gauss invented the Linking Number for any pair of closed curves

$r_1(t), r_2(s)$ in the 3-space not crossing each other. He wrote this quantity analytically:

$$\{r_1, r_2\} = \oint_{r_1} \oint_{r_2} \frac{(dr_1 \times dr_2, r_1 - r_2)}{|r_1 - r_2|^3}$$

He found also famous global relation for the curvature K of the closed convex surface S:

$$\int \int_S K d\sigma = 4\pi$$

The idea of Linking Number had a background in the Electromagnetism. His pupil, Listing did something in 1840s and invented the word ”Topology”.

Maxwell: For the isolated island there is a relation between minima (Mi), maxima (Ma) and saddles (S) of the height function:

$$Mi + Ma - S = 1$$

Kelvin found an integral associated with vorticity:

$$I = \int\int\int_{R^3} (v, curl \quad v) d^3x$$

Whitehead about 1950 expressed Hopf Invariant for the homotopy groups of spheres in the form of Kelvin Integral. Kelvin intended to classify atoms through the topology of knots (turned out to be a completely wrong idea). His pupil Tait studied knots in the late XIX Century. Some of his observations were proved only in 1980s using Jones Polynomial (see in the book of L.Kaufmann "Knots and Physics").

Complex Analysis and Theory of Riemann Surfaces were the immediate predecessors of Poincare'. According to Poincare', Betti invented homology. We see that Physicists always loved topological ideas until they are clear and nonabstract. Does not it remind the same phenomenon today?

Topological works of Poincare': Their Contribution and immediate byproduct

Poincare' published 11 works in Topology; the central one is AS.

Two Comptes Rendus notes preceding AS were published in 1892 and 1893. Riemann and Betti are named as the Ideological Predecessors in AS: Riemann developed Analysis Situs for Riemann

Surfaces; Betti invented Cycles and Homology. According to my information, Betti was Chemist. It explains such terminology as Homology.

The contents of AS: No motivation by the applications was given. Following subjects were discussed in details: 1.Definition of Manifolds; 2.Cycles and Homology; 3. Intersection Index and Duality; 4.Differential Forms and Cycles; 5.Extension of the Euler Characteristics for Polyhedra; 6.Fundamental Group; 8.Manifolds and Discrete Groups; 9.Another Approach to Manifolds, Polytopes.

His definition of manifolds exactly describes the C^1–manifolds with nondegenerate imbedding in the euclidean space. Poincare' understood the idea of orientation and used it. As we know, these ideas could not be taken as an initial basis for the construction of Topology at that time. Only after the discovery of transversality and other tools for dealing with differentiable manifolds by Whitney in 1930s such program became realizable.

The definition of cycles and homology classes is taken from Betti.

Cycles are the linear combinations of closed orientable submanifolds with integral or rational coefficients. The homological equivalence of cycles is given by the submanifolds with boundary. This definition is wrong as we know: it is based on the non-local objects. After the proper corrections it leads to the "bordism" and 'cobordism" groups instead of homology. Atiyah invented this extraordinary homology theory in early 1960s. This nonlocal theory is much more complicated and surprisingly rich (see in my article S.Novikov, Methods of Algebraic Topology from the viewpoint of the Cobordism Theory, Izvestia, 1967, 31 n 4, 855-951). Let me point out that it could be constructed only after the Great Revolution in Topology (1935-1955). It was R.Thom who clarified relationship between cycles and submanifolds in early 1950s; his works used all new technology. No way is known to prove this kind of results elementary.

Poincare formulated the Duality Law for the complementary Betti numbers of the oriented closed manifolds $b_i = b_{n-i}$. He understood it as a byproduct of the nondegeneracy property of the intersection index between the cycles of complementary dimensions and formulated corollaries of that for the middle-dimensional homology. Within this approach one cannot prove these facts. Lefshetz developed these ideas very far in 1920s and 1930s but proofs were found only in 1950s in many cases.

Poincare described differential forms (without formalism of external product). He formulated what we call "DeRham Theorem". He never returned to differential forms in the latest topological works. This approach was developed in 1920s: E.Cartan invented convenient formalism of exterior product and constructed (co)homology ring. How to prove that they are isomorphic to the ordinary homology? It was solved by E.Cartan himself in the special cases and by DeRham in the general form in 1930s.

Universal extension of Euler Characteristics for all Polyhedra and invention of Fundamental Group were the impressive achievements of AS. There was no way for him in AS to calculate it except discrete groups. Combinatorial group theory did not existed at that time. One might say that this work started it. It was developed by Dehn in the first decade of the XXth Century who actively applied it in the theory of knots (I found no traces of knots in the works of Poincare).

It was finally clarified in 1950s and 60s by Papakiriakopulos, Haken and Waldhausen that fundamental group provides a complete set of invariants for knots, the theoretical algorithm was found to test unknotness. In 1930s Nielsen and Magnus made important contribution to the combinatorial group theory later developed by people in Algebra and Theory of Algorithms.

Poincare already began to work with polyhedra in AS but only in the next works this approach started to dominate. In the series of works after AS Poincare constructed the homology theory based on the chains in polyhedra. It leads to the exact treatment of Poincare duality, to the calculations of homology. Fundamental Group also was effectively developed through the combinatorial approach. Torsion numbers appeared.

Poincare finally established that full nonabelian fundamental group is needed for the characterization of the 3-sphere (The famous Poincare Conjecture). Let us mention that for 3-manifolds fundamental group combined with all homological quantities cannot characterize them completely: A new remarkable invariant was discovered by Reidemeister in 1930s leading to the classification of lens spaces. It is homotopy non-invariant quantity. Later its extension by Whitehead led to the theory of noncommutative determinant about 1940.

The development of Poincare duality by Lefshetz and Alexander led to the duality in the group theory (Pontryagin), to the idea of cohomology and its ring structure in 1930s (Kolmogorov and Alexander).

Following Conjecture was formulated in the work "Sur les lignes geodesiques des surfaces convexes,- Trans. Amer. Math. Soc.,- 1905, 6, 237-274: The number of nonselfintersecting closed geodesics on the convex 2-surface is 3 or more? New variational approach was found by Birkhof in 1913 and developed by Morse in 1920s. Finally this conjecture was proved by Liusternik-Schnirelman in 1930s for all Riemannian manifolds homeomorphic to the 2-sphere; The complete verification of the proof was finished only 60 years later). He also started topology of symplectic maps in his last mathematical work (1912). In the philosophical article published in 1912 Poincare wrote that the most important topological problem for him now is how to prove that our space is 3-dimensional. Is this quantity a topological invariant? Poincare considered Continuous Homeomorphism as a most fundamental equivalence relation for manifolds. For C^1 diffeomorphic manifolds dimensions are obviously equal. This problem was solved by Brauer in 1913 who invented the degree of map. In 1915 Alexander proved homotopy invariance of homology. His results were made rigorous in 1940s by Eilenberg and others who formalized singular homology theory. Cell complexes appeared in 1940s making rigorous the Morse Theory with the help of transversality technic.

Topology after Poincare'

I see the XXth Century Topology as a series of the following periods:

I. The Post-Poincare Period (1912-1940) Several outstanding people developed the ideas discussed by Poincare'. Their names appeared in the previous part. Let me add the name of H.Hopf who really discovered in 1930s the new deep homotopy topology associated with the homotopy groups of spheres. Hopf started many fundamental directions between 1920th and 1950th.

II. The Great Revolution in Topology (1935-1955) The Theory of Smooth Manifolds including the idea of Transversality and Analysis on Manifolds was created; The Fibre Bundles, Connections and Characteristic Classes were discovered, various Homotopy Obstruction Theories were constructed. The Categorial Homology Theories of Spaces, Sheafs and Fibre Bundles were developed leading to the great Homological Methods like Exact and Spectral Sequences, Cohomological Operations and other tools; Huge machinery for calculation of the Homotopy Groups of Spheres was constructed; the Cobordisms were essentially calculated; The Homological Algebra and Hopf Algebras were invented. Let me give list of the most important names: Whitney, Hopf, Pontryagin, Chern, Hodge, Steenrod, Whitehead, Eilenberg-McLane, Leray, Serre, H.Cartan, Thom, Borel. Milnor and Grothendick began at the end of that period inventing new great ideas. They started the next period when I worked. The names of main topological players are dropped here (see my new article S.Novikov, Topology of XX Century: The view from inside, Russian Math Surveys, 2004, vol 59,n 6).)

III. Byproduct of the Great Revolution (1955-1970); Topology as a Centrum of Interaction. During that period many fundamental problems of Topology were solved; Several applications of topological ideas in other areas of mathematics were found. (The subjects where I was involved myself are colored in Blue here. The problems whose solution really required the use of the new algebraic methods are marked at the beginning by the sign (!); The results obtained in the late

1960s and 1970s whose full proof is not written yet in the literature, are marked at the beginning by the sign (?).).

Manifolds, $n \neq 4$: (!)Nonstandard Differentiable Structures on the 7-Sphere were discovered and classified for $n > 4$; Nonsmoothable manifolds were found; The torsion of Pontryagin Classes was found to be noninvariant; Poincare

Conjecture and h-cobordism theorem were proved; (!)The classification of immersions and imbeddings of manifolds were obtained; (!)Classification Theory of multidimensional smooth manifolds was constructed; (!)Relationship between smooth and PL manifolds was understood; (!)Topological Invariance of the Periods of Pontryagin Classes along the cycles was proved;(!)The so-called Annulus Conjecture was proved;

(!)(?)Hauptvermutung was proved for manifolds without 2-torsion in the 3-homology; (!)Counterexamples to Hauptvermutung were found for the nonmanifolds first and later for manifolds; (!,?)Some sort of classification of Topological Manifolds was obtained. Several fundamental problems of the 3D Topology and knot theory were also solved in 1960s. Let us mention that in 1980s these people realized the great computational program leading to the proof of the famous 4-color problem. The existence and uniqueness of differentiable structure on 3-manifold was established in early 1950s by the elementary methods.New achievements of the 3D Topology made possible the development of the Hyperbolic Topology of 3-manifolds in 1970s. The technic of differential topology was extended in 1970s to the

4-manifolds. It led to the construction of the purely continuous homeomorphisms only: it was proved that homotopy equivalent simply connected 4-manifolds are homeomorphic. The discovery of nonstandard differential structures here belongs to the new era in Topology: it is a byproduct of the activity of physicists and achievements of the qualitative theory of the nonlinear PDEs.

Calculations: The stable homotopy groups of the classical Lie Groups were found through the calculus of variations; (!) Hopf Invariant Problem was solved, the nonexistence of the Divisible Algebras in higher dimensions was proved; (!)The Extraordinary Homology Theories were invented: the K-Theory brought new dimensions to the homological methods; (!)The Cobordism Theory was developed; it improved methods of studying stable homotopy groups of spheres and fixpoints of the compact groups acting on manifolds ; The Categorial "Localization Idea" was invented helping to solve some deep problems of the Homotopy Topology; Nontrivial finite-dimensional H-spaces were discovered.

Interaction with other areas of Mathematics: (!) Riemann-Roch

Theorem in Algebraic Geometry was proved as an application of the Cobordism Theory. Completely new approach was discovered leading to the so-called K-Theory; (!) The index problem for the Elliptic PD Operators was solved on the basis of Cobordisms and K-Theory; The revolutionary new understanding of the Topology of Multidimensional Dynamical Systems was invented by topologists; Main problems were solved for the codimension one foliations including the proof of existence of compact leaves on the 3-sphere; New areas of Algebra were created such as the Algebraic K-Theory, Theory of Hopf Algebras, The Homological Algebra.

IV. Dispersion of Multidimensional Topology, The Hyperbolic Topology; Discovery of Topological Phenomena in Physics (1970s)

V. Revival of Topology; Physicists in Topology (1980-2000)

Let me recommend my minibook published in Encyclopedia of Math Sciences: S.Novikov, Topology I, EMS, vol 12, Springer, 1997.

VI. New ideas: Is it possible to solve difficult problems in the theory of 3-manifolds analytically?