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items 136,137,140 148,159,163,

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Two important basic ideas of two-dimensional complex analysis were discussed in our works dedicated to its discretization:

1. The Laplace operator is factorizable

$\Delta = -\partial\bar{\partial}$

We constructed such discretization of Complex Analysis that this property remains true. It is possible only for the equilateral triangle lattice.and more general for the triangulated surfaces with triangles colored in black and white. We do not have such property for the square lattice. This idea is a by-product of the theory of Completely Integrable Systems. The first order linear operators discretizing $\partial, \bar{\partial}$ play leading role here (see Fig 1).

2. Another (related) problem requiring first order operators is the following:

What is a right discrete analog of the first order operators necessary for the construction of discrete analogs of the DG (Yang-Mills) connections? We discuss this problem here for the group GL_n where we found discretization different from the standard one (invented by K.Wilson many years ago). Our idea does not work for the compact Lie groups. The Continuous DG Connection in R^m is an overdetermined system of linear equations $i = 1, ..., m$:

$$\partial_i \psi = A_i(x)\psi, \psi \in R^n$$

Curvature is an obstruction for the solution)

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

or $R = \sum R_{ij} dx^i \wedge dx^j$; substitutions $\psi = G(x)\phi, G \in GL_n$, lead to the Gauge transformations

$$A_i \to G^{-1} A_i G - G^{-1} \partial_i G$$

Holonomy Map is defined as a solution to this system along the path $\Omega(x_0) \to GL_n$

Discrete GL_n Connections and Triangle Equation.

Let K be a simplicial complex (n-manifold) with selected family of n-simplices X and set of coefficients $b_{T:P} \neq 0$ for every n-simplex $T \in X$ and its vertex $P \in T$. The Triangle Operator

$$Q^X\psi(T) = \sum_{P \in T} b_{T:P}\psi(P)$$

is defined on the space of functions of vertices. Its image belongs to the space of functions of the simplices $\sigma \in X$.

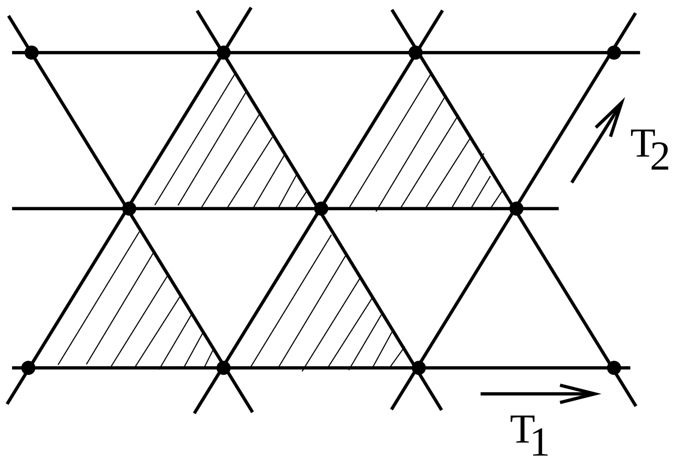
Three families X–"black", "white", "all"– will be especially considered: Let all n-simplices of K are colored into black and white colors. We have operators Q^b and Q^w where X is the set of all black (or white) simplices (see Fig 1 and Fig 2). The equilateral triangle lattice in R^2 is an important example (see Fig 1 below).

Another example is the case where X is simply set of all n-simplices $T \in K$. We call corresponding triangle equation $Q\psi = 0$ the "Discrete GL_n Connection". Remark: For $n = 2$ solutions to the equation $Q^b\psi = 0, b_{T:P} = 1$ are "The Discrete Holomorphic Functions". The solutions $Q^w\psi = 0$ are "The Discrete Anti-Holomorphic Functions". (see Fig 1). Here we have

$$Q^b = 1 + t_1 + t_2, Q^w = 1 + t_1^{-1} + t_2^{-1}$$

were t_1, t_2 are the basic shifts in the lattice.

Fig 1



We consider here only the family of all simplices. We call the ratio's $\mu_{ij}^T = b_{T:i}/b_{T:j}, T =< 0, 1, ..., n >$ "Coefficients of DG Connection. The

Abelian Gauge Transformations are defined by the pairs f, g of nonzero functions of simplices f_T and vertices g_P:

$$b_{T:P} \to f_T b_{T:P} g_P^{-1}, \psi \to g_P \psi$$

present the natural equivalence group for the discrete DG Connections. We have

$$\mu_{ij}^T \to g_j / g_i \mu_{ij}^T$$

and $\mu_{ii}^T = 1, \mu_{ij}^T \mu_{jl}^T \mu_{li}^T = 1$.

Following picture explains how Nonabelian Curvature appears for such "connections" (see Fig 3 for $n = 2$). For every vertex P we start from the vertex P_1 in its star $St(P)$. Knowing $\psi(P)$ and $\psi(P_1)$, we calculate step by step all $\psi(P_p)$ "along the circle" $P_1P_2...P_mP_1 \subset St(P)$. Contradiction might appear after returning to the original point P_1 in the form of non-unit triangle matrix

K_{P,P_1}. We call it "Curvature Operator".

Example: For $b_{T:P} = 1, n = 2$ " the zero curvature" property $K_P = 1$ simply means that even number of edges (triangles) enter P. This case is important for the Discrete Complex Analysis.

Fig 2

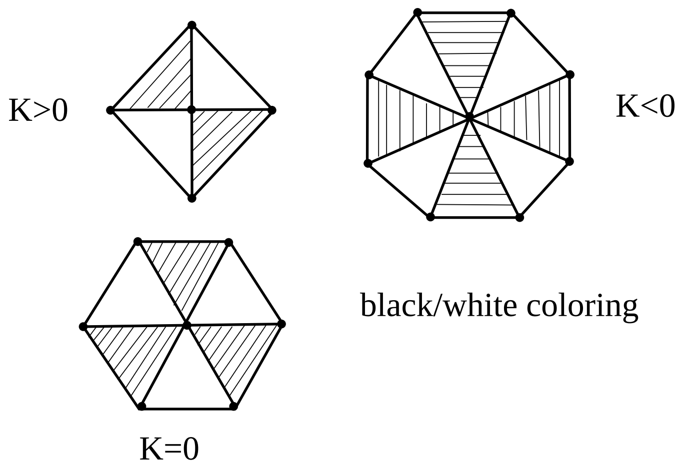
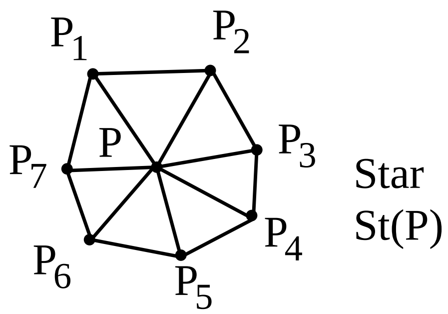


Fig 3



The Theory of Abelian and Non-abelian Holonomy and Nonabelian Curvarure was developed recently.

The Framed Abelian Holonomy.

Let oriented closed simplicial path $\gamma =< 0, 1, ..., N, 0 >$ be given equippe by the sequence of n-simplices $T_0, T_1, ..., T_N$ such that the edge $[i, i+1]$ belongs to the simplex T_i (see Fig). We call it Framed Path

$$\gamma =< 0, 1, ..., N, 0 | T_0, T_1, ...T_N >$$

The abelian holonomy along the closed path is $\mu(\gamma) = \Pi_i \, \mu_{i,i+1}^{T_i}$. We have semigroup of all closed framed paths $\Omega(M,0)$ and Abelian Holonomy Map

$$\mu : \Omega(M,0) \longrightarrow k^*$$

where k is our field.

Lemma 1:

1. The framed abelian holonomy map $\mu(\gamma)$ is invariant under all abelian gauge transformations for the closed framed paths.

2.Its image is completely determined by the quantities

$$\rho_{ij}^{TT'} = \mu_{ij}^{T}\mu_{ji}^{T'} = \mu(< iji|TT' >)$$

where the edge $[ij]$ belongs to the intersection $T \cap T'$, and by the image of the homology basis $\mu(\gamma_p)$ (with any fixed framing) generating the group $H_1(M, Z)$.

3.The reduced abelian holonomy

$$\mu_{red} : H_1(M, Z) \to k^* / \cup \mu(\rho_{ij}^{TT'})$$

maps homology classes into factor-group of k^* by the image of all local gauge invariant quantities $\rho_{ij}^{TT'}$ for all edges $[ij]$ and n-simplices T, T'. (It is enough to take here only the pairs T, T' meeting each other along the $n-1$-face.)

Lemma 2.

1.For $n \geq 2$ every basic element in the group $H_2(M, Z)$ provides relation on the quantities $\rho_{ij}^{TT'}$.

In particular, for $n = 2$ and oriented closed surfaces M we have

$$\prod_{[ij]\subset M} \rho_{ij}^{TT'} = 1$$

where $\partial T = [ij] + ..., \partial T' = [ji] + ...$, and orientation of T, T' is the same as of M. For $n > 2$ and $\Delta =< ijk >\subset T \cap T'$ we have relations $\rho_{ij}^{TT'} \rho_{jk}^{TT'} \rho_{ki}^{TT'} = 1$. For $\gamma =< 0, 1, ...m, 0|T_0 T_1 ... T_m T_0 >$ where $T_p \cap T_{p+1}$ is an $n-1$-face, we have $\Pi_{p=0}^{p=m} \rho_{ij}^{T_p T_{p+1}} = 1$. This set of relations is complete, but some of them may be dependent.

Theorem 1. All coefficients of the Discrete Connection are uniquely determined by the framed abelian holonomy μ. Every homomorphism μ satisfying to lemma 1, is a framed holonomy for some connection.

Proof: Uniqueness: I. Take any star $St([ij])$ of the edge. Solve in it the equation

$$\rho_{ij}^{TT'} = \tilde{\mu}_{ij}^{T}\tilde{\mu}_{ji}^{T'}$$

The chain ρ is a multiplicative 1-cocycle in the dual decomposition where T are the vertices. Therefore it is possible to solve this equation up to constant $\delta_{ij}\delta_{ji} = 1$, $\mu_{ij}^{T} = \delta_{ij}\tilde{\mu}_{ij}^{T}$.

II. Now we try to solve the equation $\mu^T_{ij}\mu^T_{jk}\mu^T_{ki} = 1$. Construct the coboundary $\tilde{\mu}^T_{ij}\tilde{\mu}^T_{jk}\tilde{\mu}^T_{ki} = \tilde{\mu}^T[ijk]$. The identity $\rho^{TT'}_{ij}\rho^{TT'}_{jk}\rho^{TT'}_{ki} = 1$ implies that $\tilde{\mu}^T[ijk]$ does not depend on T. So this quantity is a 2-cochain $\tilde{\mu}$. It is obviously closed. If it is exact, we have

$$\tilde{\mu}[ijk] = \delta^{-1}_{ij}\delta^{-1}_{jk}\delta^{-1}_{ki}$$

Our solution is:

$$\mu^T_{ij} = \tilde{\mu}^T_{ij}\delta_{ij}$$

This substitution solves our equation. Easy to see that this solution is unique up to the gauge transformation and multiplicative one-cocycle. Everything is fixed by the values of $\mu(\gamma_q)$ where γ_q form basis of the group $H_1(M, Z)$. So our connection is unique. Existence: It exists if all requirements of Lemma 1 are satisfied. We construct it by the procedure described above. For the

2-cocycle $\tilde{\mu}[ijk]$ we have:

$$\prod_{\Delta} \tilde{\mu}[\Delta] = \prod_{ij} \rho_{ij}^{TT'}$$

for $n = 2$. Here product is extended to all 2-simplices Δ, T, T' with right orientation, and $T \cap T' =]ij]$. For $n > 2$ similar relation appear for every 2-cocycle implying that the cocycle $\tilde{\mu}$ is always exact for DG Connections.

Nonabelian Curvature and Holonomy

Consider the Thick Path $T_1 T_2 ... T_m$ such that $T_i \cap T_{i+1}$ is an $n-1$-face $\Delta_i \neq \Delta_{i+1}$. We assume that the initial (''in'') face $\Delta_0 \subset T_1$ and final (''out'') face $\Delta_m \subset T_m$ are fixed. The ''in'' and ''out'' faces in every simplex T_q are exactly Δ_{q-1} and Δ_q correspondingly, $q = 1, 2, ...m$ (see Fig).The abstract combinatorial model of thick path is $\Delta \times R^+$ where $\Delta \times 0$ is identified with $\Delta_0 =< 0, 1, ..., n-1 >$ (see Fig). A word in the free semigroup of the length m should be given $A = a_0^{i_0} ... a_{n-1}^{i_{n-1}}$ where $\Sigma_q i_q = m, i_q \geq 0$. Starting from $\Delta_0 = \Delta \times 0$, we add new vertices step by step, moving along the word A from the left to the right. For every new factor a_q we add one vertex at the axis $q \times R^+$ higher than the previos vertices (see Fig). Finally we are going to have exactly i_q vertices at every angle line $q \times R^+$ for $t > 0$. Initial simplex Δ_0 and the word A completely determine all thick path $\gamma_{(\Delta_0, A)}$ in oriented manifolds M.

We call thick path κ closed if $\Delta_m = \Delta_0$ but this property depends on the manifold M and initial simplex

Δ_0. The Nonabelian Holonomy Homorphism is defined

$$R_{\Delta_0} : \Omega^{thick}(M, \Delta_0) \to GL_n(k)$$

solving the triangle equation along the thick path κ and making after that the permutation P which it induces for the vertices of the initial face Δ_0.

Lemma 3. The Holonomy map R_{Δ_0} is a homomorphim of the

semigroup of thick paths $\Omega^{thick}(M, \Delta$
into the group $GL_n(k)$.

The Nonabelian Curvature around the simplex $\sigma =< 01...n-2 >$ is a nonabelian holonomy $R_{\Delta_0}(\kappa)$ where $\Delta_0 =< \sigma, n-1 = [0] >$ and $\kappa = a_{n-1}^m$. Here m depends on σ in such a way that this thick path ”around σ” is minimal closed (see Fig). The simplices $T_p, p = 1, 2, ...m, m+1 \sim 0 (mod m)$, look like $< \sigma, [p-1], [p] >, p = 1, ..., m_\sigma \sim$

0, with last pair $[p], [p+1]$ on the axis a_{n-1} (see Fig)

$\Delta_p^{out} =< \sigma, [p] >, \Delta_p^{in} =< \sigma, [p-1] >$

Lemma 4. The nonabelian curvature map has a form $K_{\sigma,[p]} = A_{[p+m-1]} \ldots A_{[p]}$ where $A_{[p]}$ is a lower triangle n-matrix with $1, \ldots 1, \mu^{T_{p+1}}_{[p],[p+1]}$ along diagonal. The only nontrivial row is the last one. It is defined by the simplex $T_{[p+1]} =< 0, \ldots, n-2, [p], [p+1] >$: $-\psi_{[p+1]} =$

$$= \sum_{q=0}^{n-2} \mu^{T_{p+1}}_{0,[p+1]} \psi_q + \mu^{T_{p+1}}_{[p],[p+1]} \psi_{[p]}$$

$$A_p(\psi_q) = \psi_q, q = 0, 1, ..., n-2$$

So the diagonal part of curvature operator is $(1, ..., \mu_\sigma =$

$$= (-1)^m \prod_{s=p}^{p+m-1} \mu_{[s],[s+1]}^{T_{s+1}} = \det K_\sigma)$$

Its last row is $\alpha_{\sigma,0,[p]}, ..., \alpha_{\sigma,n-2,[p]}, \mu_\sigma$ where $q \in < 0, 1, ..., n-2 >, p = 1, 2, ..., m$.

We call $\alpha_{\sigma,q,[p]}, \mu_\sigma$ the Local Curvature Coefficients. The Connection is locally SL_n if $\mu_\sigma = 1$.

The Connection is locally Flat if $K_\sigma = 1$ as a matrix.

Lemma 5. The coefficients of Local Curvature are equal to

$$\alpha_{\sigma,q,[p]} = \sum_{k=0}^{k=m-1} (-1)^{k+1} \mu_{q,[p-k]}^{T_{p-k}} \times$$

$$\times \left(\prod_{s=0}^{k-1} \mu_{[p-s+1],[p-s]}^{T_{p-s}} \right)$$

Lemma 6. The α coefficients transform as 1-cochains, μ_σ is invariant under gauge transformations. The quantities $\alpha^*_{\sigma,q,[p]} =$

$\alpha_{\sigma,q,[p]}\mu_{[p],q}^{T_p}$ are gauge invariant, and $\alpha^*_{\sigma,q,[p+1]} =$

$$-\alpha^*_{\sigma,q,[p]}/\rho_{[p],q}^{T_pT_{p+1}} + 1 - \mu_\sigma$$

Theorem 2. The gauge invariant Curvature coefficients can be expressed through the abelian data:

$$\alpha^*_{\sigma,q,[p]} = \sum_{k=0}^{m-1} \prod_{j=0}^{k} (-1)^{k+1} \rho_{[p-j],q}^{T_{p-j+1}T_{p-j}}$$

Following formulas are true for all $q = 0, 1, ..., n-2 \in \sigma$:

$$\prod_{p=1}^{m} \rho_{q,[p]}^{T_p T_{p+1}} = (-1)^m \mu_\sigma$$

The inverse expressions are valid in generic case where all coefficients $\alpha^*, \alpha^* - 1 + \mu_\sigma$ are nonzero:

$\rho_{[p],[p+1]}^{T_p T_{p+1}} =$

$$= \alpha^*_{\sigma,q,[p]} / \{\alpha^*_{\sigma,q,[p+1]} - 1 + \mu_\sigma\}$$

For the locally SL_n connections we have $\mu_\sigma = 1$, so the conditions like $\alpha = 0$ do not depend on the initial point $[p] \in St(\sigma)$.

We did not developed yet the theory of characteristic classes i.e. topological invariants of connections. In continuous case they are $Tr(R \wedge ... \wedge R)$.