S.P.Novikov — Math 437
Spring 2015
Differential forms

Textbook: D.Bachman, A geometric approach to differential forms, Birkhäuser.
Additional material: M.Spivak, Calculus on Manifolds.

Contents

Program

1.Coordinates. Cartesian coordinates in a domain $U \subset \mathbb{R}^n$. Important examples (polar, cylindrical, spherical, hyperbolic, complex).

2.Differential 1-forms and vector fields, Jacoby matrix, change of coordinates, pull-back for functions and 1-forms, restriction to curves and integration. Orientation.

3.Important examples, closed 1-forms, Newton–Leibnitz property for integration, exact and closed 1-forms, 1-dimensional (co)homology of domains $U \subset \mathbb{R}^2$. The differential of the angle function. Topology of planar domain, number of "cuts" necessary to make it simply connected.

4.Complex coordinates z, $\overline{z}$ in $\mathbb{R}^2$. Cauchy 1-forms $f(z)\,dz$, the Cauchy–Riemann equation $\partial f/\partial \overline{z} = 0$. What is the value of the integral $\oint_{|z|=1} z^n\,dz$? Is the form $f(z)\,dz$ closed if $\partial f/\partial \overline{z} = 0$?

5.What is $d(X)$ if X is a 1-form? Differential 2-forms in $\mathbb{R}^n$. Change of coordinates and restriction of 2-forms to 2-surfaces in $\mathbb{R}^n$. Case $n = 2$. Integration. Case of any k-forms in $\mathbb{R}^n$.

6.Product of 1-forms as a 2-form. Product of differential forms. Associativity and skew commutativity for k-forms in $U \subset \mathbb{R}^n$.

7.The De Rham operator d acting on differential forms, its definition and algebraic properties. De Rham operator and multipication of forms. De Rham Operator and pull-back operation. Closed k-forms and cohomology (definition for the domains in R^n). Important examples of the closed forms in R^2 and R^3, angle form in R^2; area form for S^2 in spherical coordinates and as a restriction of the closed 2-form in R^3.

8.Change of coordinates for k-forms in $u \subset \mathbb{R}^n$. Pull-back operation for k-forms for arbitrary smooth maps $\varphi : U \to V$.

9.Integration of n-forms in $\mathbb{R}^n$. Integration of k-forms along the oriented k-surfaces $U \to \mathbb{R}^n$, $U \subset \mathbb{R}^k$.

10.The Stokes formula for integration

$$\int \cdots \int_{D} \underset{k+1}{d\Omega} = \int \cdots \int_{\partial D} \underset{k}{\Omega},$$

cases $k = 0, 1$.

11.First pair of Maxwell Equations and De Rham operator d in the 3-space and in 4-space. Electromegnetic field as a 2-form in R^4. Electric and Magnetic fields in R^3.

12.Minicourse in Linear Algebra. Vectors and covectors. Inner products, Gram Matrices. Nondegenerate inner product. Classification of symmetric Inner Products. Euclidean and Minkowski cases. Volume element and Gram Matrix. Groups $GL_n(R)$ and $GL_n(C)$, O_n, SO_n. How many connected pieces do they have? Same question for group $O_{1,1}$.

13.Nondegenerate Symplectic Inner Product in R^{2n} and 2-form, its powers and volume element, Pfaffian (square root of determinant) of the skew symmetric matrix and powers of 2-form.

15. Riemanian and Pseudoriemanian metrics. Euclidean and Minkowski cases, metrics of eulideasn plane, 2-sphere and pseudosphere in the polar coordinates. Arc length of curves, definition of geodesics as (locally) minimal arc curves. Arc length of timelike curves in Minkowski space. Arc length of the "Fermat metric" in R^2. The Fermat principle for the propagation of light and "Snells law" in the water/air case.

15. Duality of forms in euclidean metric, the case of 3-space. The operators div and curl in terms of differential forms. Product of 1-forms and vector product in R^3. Vector product and commutator of skew symmetric matrices.

16.Differential forms and homotopy of mappings. Poincare' lemma.

17.Curves in the euclidean plane. Curvature and Gauss map. Curvature as a pull-back of the form $d\phi$. Integral of curvature along the closed curve and degree of Gauss map.

18.Surfaces in R^3. Riemannian metric and curvature form. Mean and Gauss curvature. Gauss curvature and Gauss map. Total (integral) of Gauss curvature for the convex body in R^3.

19. Hopf invariant for the maps $S^3 \to S^2$ and Kelvin-Whitehead integral along S^3.

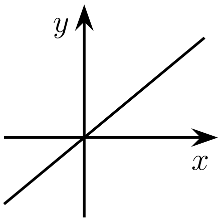
Homeworks

Homework 1

1. Calculate the Jacoby matrix for a shift, rotations, and affine transformations of $\mathbb{R}^n$.

2. Find the restriction of the 1-forms dx and dy to the lines

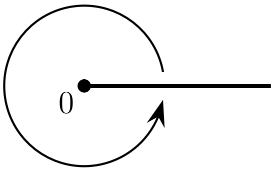
   a) $y = \lambda x$;



   b) $x^2 + y^2 = 1$.

3. Is $(\rho(x, y), \varphi(x, y))$ a pair of "Cartesian" type coordinates in $\mathbb{R}^2$? Is it true for $\mathbb{R}^2 \setminus \{0\}$?

4. In which domain (ρ, φ) may be treated as a pair of "Cartesian" coordinates? Is it ok for the domain $0 < \varphi < 2\pi$?



5. Is ρ a "good" Cartesian coordinate for the domain $\mathbb{R}^2 \setminus \{0\}$?

6. Find the restriction of the form

$$\frac{x\,dy - y\,dx}{x^2 + y^2}$$

   to the circle $x^2 + y^2 = 1$ (in the coordinates (ρ, φ)).

Homework 1 — Solutions

1. Shift: $x \mapsto x + b$, $\widehat{J} = \mathbb{1}$.

   Rotation: $x \mapsto Ax$, $\widehat{J} = A$.

   Affine transformation: $x \mapsto Ax + b$, $\widehat{J} = A$.

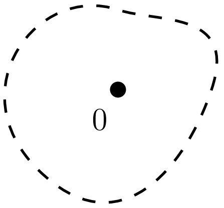
2. a) Use x as the parameter. $y = \lambda x$, $x = x$, $dx \to dx$, $dy \to \lambda\, dx$.

   b) $x^2 + y^2 = 1$, $x = \cos t$, $y = \sin t$, use t as the parameter.

$$\begin{aligned} dx &\to d(\cos t) = -\sin t\, dt, \\ dy &\to d(\sin t) = \cos t\, dt. \end{aligned}$$

3. (ρ, φ) is not a pair of Cartesian coordinates in $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{0\}$.

4. The domain $U \subset \mathbb{R}^2 \setminus \{0\}$ should *not* contain a closed path around 0.



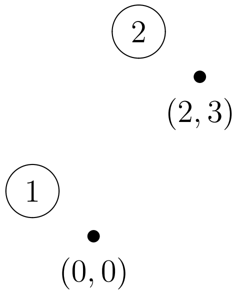
5. No, the level $\rho = \text{const} > 0$ is a circle $S^1 \not\subset \mathbb{R}^1$.

6. $$\frac{x\, dy - y\, dx}{\rho^2} = d\varphi.$$

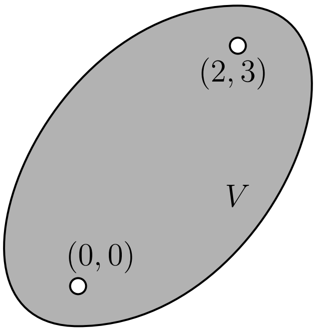
Homework 2

1. Calculate the change of coordinates $(\rho_1, \varphi_1) \to (\rho_2, \varphi_2)$,

$$\rho_1(\rho_2, \varphi_2), \varphi_1(\rho_2, \varphi_2) = ?$$



2. Prove that the 1-form $\alpha\, d\varphi_1 + \beta\, d\varphi_2$ is *closed* and *not exact* in a domain V like:



3. Find a domain in $S^2 \subset \mathbb{R}^3$ in which $d\theta$ is a well-defined 1-form.
4. Find a domain in S^2 in which $d\varphi$ is a well-defined 1-form.
5. The cylindrical coordinates (z, ρ, φ) in $\mathbb{R}^3$ are defined by $z = z$, $x = \rho \cos\varphi$, $y = \rho \sin\varphi$. In which domain $V \subset \mathbb{R}^3$ the form $d\varphi$ is a well-defined 1-form?

6. Calculate the integral

$$\oint\limits_{\gamma} (\alpha z^2 + \beta/z)\, dz, \quad z = x + iy, \quad \alpha, \beta = \mathrm{const}$$

along a closed contour γ.

Homework 2 — solutions

1.

$$\begin{aligned} x &= \rho_1 \cos\varphi_1, & x - 2 &= \rho_2 \cos\varphi_2, \\ y &= \rho_1 \sin\varphi_1, & y - 3 &= \rho_2 \sin\varphi_2. \end{aligned}$$

Conclusion:

$$\begin{aligned} \rho_1 \cos\varphi_1 &= 2 + \rho_2 \cos\varphi_2, \\ \rho_1 \sin\varphi_1 &= 3 + \rho_2 \sin\varphi_2. \end{aligned}$$

Solve these equations:

$$\begin{aligned} \rho_1^2 &= (2 + \rho_2 \cos\varphi_2)^2 + (3 + \rho_2 \sin\varphi_2)^2, \\ \cos\varphi_1 &= \frac{2 + \rho_2 \cos\varphi_2}{\rho_1}, \\ \sin\varphi_1 &= \frac{3 + \rho_2 \sin\varphi_2}{\rho_1}. \end{aligned}$$

2. $d(\alpha\, d\varphi_1 + \beta\, d\varphi_2) = 0$ (obvious).

$$\oint\limits_{C_1} \Omega = 2\pi\alpha, \quad \oint\limits_{C_2} \Omega = 2\pi\beta.$$

3. $d\theta$ is OK in the domain $S^2 \setminus$ poles.

4. $d\varphi$ is OK in $S^2 \setminus$ poles.

5. Cylindrical coordinates in $\mathbb{R}^3$. $d\varphi$ is OK in $\mathbb{R}^3 \setminus \mathrm{line}(x = 0, y = 0)$.

6.

$$\oint\limits_{C} \left(\alpha z^2 + \frac{\beta}{z}\right) dz = 2\pi i \beta$$

(Cauchy) $\oint_C z^n\, dz = 0$, $n \neq -1$.

Homework 3

1. Prove that the 1-form $x\,dy - y\,dx$ is invariant under all rotations of $\mathbb{R}^2$ around $(0,0)$.

2. Prove the same for $x\,dx + y\,dy$.

3. Introduce "hyperbolic coordinates" (χ, ψ): $x = \chi \cosh\psi$, $y = \chi \sinh\psi$. In which domain $U \subset \mathbb{R}^2$ are they defined?

4. Calculate the area 2-form $dx \wedge dy$ of $\mathbb{R}^2$ in the polar coordinates (ρ, φ): $dx \wedge dy = C\,d\rho \wedge d\varphi$, $C = ?$.

5. Calculate the area 2-form $dx \wedge dy$ in the complex coordinates z, $\overline{z}$, $dx \wedge dy = C\,dz \wedge d\overline{z}$, $C = ?$.

6. Calculate the volume 3-form of $\mathbb{R}^3$ in the spherical coordinates (r, φ, θ),

$$\begin{aligned} z &= r\cos\theta, \\ x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \end{aligned}$$

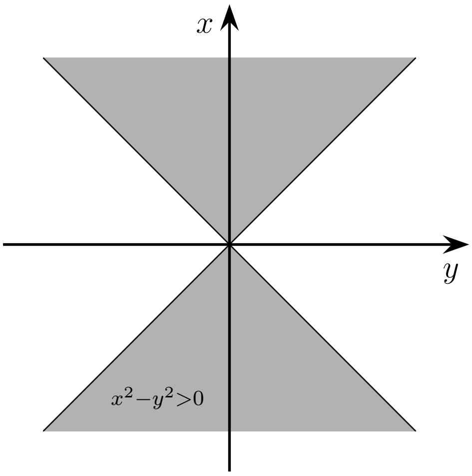
$\underbrace{dx \wedge dy \wedge dz}_{\text{the volume 3-form}} = (?)\,dr \wedge d\theta \wedge d\varphi.$

Homework 3 — Solutions

1. $x\,dy - y\,dx = d\varphi \cdot \rho^2$ — invariant under rotation.

2. $x\,dx + y\,dy = \rho\,d\rho$ — invariant under rotation: $\rho \mapsto \rho$, $\varphi \mapsto \varphi + \text{const}$, $d\rho \mapsto d\rho$, $d\varphi \mapsto d\varphi$.

   $x = \chi\cosh\psi$, $y = \chi\sinh\psi$, $x^2 - y^2 = \chi^2 > 0$



3. $dx \wedge dy = \rho\, d\rho \wedge d\varphi$.

4.

$$dx \wedge dy = \frac{dz \wedge d\overline{z}}{-2i}, \qquad \begin{cases} dz = dx + i\, dy, \\ d\overline{z} = dx - i\, dy. \end{cases}$$

5. $\mathbb{R}^3$: $dx \wedge dy \wedge dz = r^2 \sin\theta\, dr \wedge d\theta \wedge d\varphi$.

   For the unit sphere S^2 $(r = 1)$ we have Area $= \sin\theta\, d\theta \wedge d\varphi$, $0 \leqslant \theta \leqslant \pi$.

Homework 4

1. Prove that $d(f\,dz) = \dfrac{\partial f}{\partial \overline{z}}\,d\overline{z} \wedge dz$ in $\mathbb{R}^2$.

2. Calculate the area form in the hyperbolic coordinates $x = \chi \cosh \psi$, $y = \chi \sinh \psi$.

3. Introduce hyperbolic coordinates in a domain of $\mathbb{R}^3$

$$\begin{aligned} x &= \chi \sinh \theta \cos \varphi, \\ y &= \chi \sinh \theta \sin \varphi, \\ z &= \chi \cosh \theta. \end{aligned}$$

   Find a domain where these coordinates are well defined.

4. Find linear transformations of $\mathbb{R}^2$ such that $\langle A\eta, A\zeta\rangle = \langle \eta, \zeta\rangle$, where $\langle \eta, \zeta\rangle = \eta_1\zeta_1 - \eta_2\zeta_2$.

5. How many components does the group of such transformations have (see no. 4)?

6. Find the value of the integral

$$\oint\limits_{|z|=1} \frac{f(z)}{z^n}\,dz,$$

   where $f(z) = a_0 + a_1 z + \ldots + a_N z^N$.

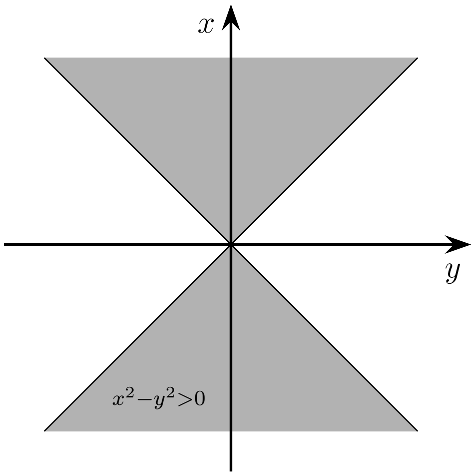
Homework 4 — Solutions

1. $df = f_x\,dx + f_y\,dy = f_z\,dz + f_{\overline{z}}\,d\overline{z}$ by definition.

   Here $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. For the form $f(z)\,dz$ we have $d(f\,dz) = f_{\overline{z}}\,d\overline{z} \wedge dz$ (calculation).

2-3. $x = \chi \cosh \psi$, $y = \chi \sinh \psi$,

$$\begin{aligned} dx \wedge dy &= (d\chi\, \cosh \psi + \chi \sinh \psi\, d\psi) \wedge (d\chi\, \sinh \psi + \chi \cosh \psi\, d\psi) \\ &= \chi\, d\chi \wedge d\psi(\cosh^2 \psi - \sinh^2 \psi) = \chi\, d\chi \wedge d\psi. \end{aligned}$$



4. Linear transformations A preserving inner product $\langle \eta, \zeta \rangle = \eta^1\zeta^1 - \eta^2\zeta^2$:
   a) hyperbolic rotations
   $$A = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix};$$
   b) time reflection
   $$T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T^2 = 1;$$
   c) space reflection
   $$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P^2 = 1.$$
5. Four components: $T^\alpha P^\beta A$, $\alpha, \beta = 0, 1$.
6. $$\oint\limits_{\substack{C \\ |z|=1}} \frac{f(z)}{z^n}\, dz = 2\pi i a_k, \quad k = n-1,$$
   $f(z) = a_0 + a_1 z + \ldots + a_N z^N,$
   $$\oint\limits_C \frac{dz}{z} = 2\pi i, \quad \oint\limits_C \frac{dz}{z^n} = 0, \ k \neq -1.$$

Homework 5

1. Find components of the group $GL_2(\mathbb{R})$.
2. Same for O_2.
3. Same for $O_{1,1}$ (preserving the quadratic form $ds^2 = (dx^0)^2 - (dx^1)^2$ or the inner product $\langle A\eta, A\zeta\rangle = \langle \eta, \zeta\rangle$, $\langle \eta, \zeta\rangle = \eta^0\zeta^0 - \eta^1\zeta^1$).
4. Prove that the Möbius band cannot be oriented, $M \subset \mathbb{R}^3$.

5. Prove that the following 2-form in $\mathbb{R}^3 \setminus \{0\}$

$$\Omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$$

coincides with the area form on the unit sphere S^2 $(x^2 + y^2 + z^2 = 1)$ and is invariant under rotations from SO_3.

6. Prove that the 2-form Ω/r^3 is closed in $\mathbb{R}^3 \setminus \{0\}$ and

$$\iint\limits_{S^2\colon x^2+y^2+z^2=1} \frac{\Omega}{r^3} \neq 0.$$

Homework 5 — Solutions

1. The eigenvalues are real. Choose a basis consisting of eigen vectors (or a Jordan basis).

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = A_0 = A, \quad A_t = \begin{pmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{pmatrix}$$

We come to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

For a Jordan cell we have

$$A_0 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \longrightarrow \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

Rotation family:

$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

2. O_2: $A = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$ and $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = P$, $\{A\}$ and $\{PA\}$ — the whole group O_2, two components.

3. $O_{1,1}$: $A = \begin{pmatrix}\cosh\psi & \sinh\psi\\ \sinh\psi & \cosh\psi\end{pmatrix}$, $P = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $T = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Components: $\{A\}$, $\{PA\}$, $\{TA\}$, $\{PTA\}$.

4. An orientation of the Möbius band does not exist because it has only a single side.

5-6.

$$\Omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$$

Substitute the spherical coordinates. We have

$$\Omega = r^3 \sin\theta\, d\theta \wedge d\varphi.$$

This is the area form on S^2, which is SO_3-invariant.

The form $A = \Omega/r^3$ is invariant under $x \mapsto \lambda x$, $y \mapsto \lambda y$, $z \mapsto \lambda z$. So, A comes from the map $\mathbb{R}^3 \setminus \{0\} \xrightarrow{\phi} S^2$, $(x, y, z) \mapsto (x/r, y/r, z/r) \in S^2$. So, $A = \phi^*(\text{Area})$. $d(\text{Area}) = 0$, hence $d(\phi^*\text{Area}) = 0$, Q.E.D.

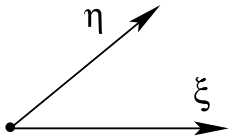
Homework 6

1. What is closed 1 - form? What is closed 2 - form? Are the following forms closed?

$$a)\ \frac{dz}{z^2 - a} \qquad b)\ x\,dy \wedge dz \qquad c)\ x\,dy \,+\, y\,dx$$

2. Prove that in $\mathbb{R}^2$ we have (orthonormal basis)

$$\langle \eta, \zeta \rangle \;=\; \eta_1\,\zeta_1 + \eta_2\,\zeta_2 \;=\; |\eta|\,|\zeta|\cos\varphi$$



3. Prove that symmetric **nondegenerate** inner product $\langle \eta, \zeta \rangle$ in $\mathbb{R}^2$ (bilinear) can be reduced to the form

$$\langle \eta, \zeta \rangle \;=\; \eta_1\,\zeta_1 \pm \eta_2\,\zeta_2$$

$+$ = euclidean space $\mathbb{R}^2$
$-$ = Minkovski space $\mathbb{R}^{1,1}$

4. Classify skew-symmetric inner products in $\mathbb{R}^2$, $\mathbb{R}^3$:

$$\langle \eta, \zeta \rangle \;=\; -\,\langle \zeta, \eta \rangle$$

Nondegeneracy

$$\forall \eta \;\; \exists \zeta \; : \quad \langle \eta, \zeta \rangle \;\neq\; 0$$

5. Gramm matrix of inner product G:

$$e_1, \ldots, e_n \;-\; \text{basis} \;, \quad g_{ij} \;=\; \langle e_i, e_j \rangle$$

Nondegeneracy: $\det g_{ij} \neq 0$ (prove).

6. New basis

$$\mathbf{e} \;=\; A\,\mathbf{e}' \qquad (e_i \;=\; a_i^j\,e_j')$$

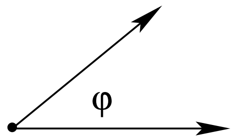
$$\langle e_i, e_j \rangle \;=\; g_{ij} \;=\; G \quad , \qquad G \;=\; A\,G'\,A^t$$

Prove!

Homework 6. Solutions.

2. High school

$$\langle \eta, \zeta \rangle \;=\; |\eta|\,|\zeta|\cos\varphi$$



Bilinearity - important

$$\langle \eta, \zeta \rangle \;=\; \eta_1\,\zeta_1 \,+\, \eta_2\,\zeta_2$$

3.

$$O_{1,1}\;:\quad ds^2 \;=\; (dx^0)^2 \,-\, (dx^1)^2 \qquad \mathbb{R}^{1,1}$$
$$\langle \eta, \zeta \rangle \;=\; \eta_0\,\zeta_0 \,\pm\, \eta_1\,\zeta_1$$

4. Skew-symmetric inner product

$$\langle \eta, \zeta \rangle \;=\; -\,\langle \zeta, \eta \rangle$$

Define a complement to any subspace. After appropriate normalization of basis vectors it becomes

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

5. Gramm matrix for inner product G:

$$\langle e_i, e_j \rangle \;=\; g_{ij} \quad , \qquad \mathbf{e} \;=\; A\,\mathbf{e}' \quad , \qquad G \;=\; A\,G'\,A^t$$

Corollary: $\det G \;=\; |\det A|^2\,\det \mathrm{G}'$

Homework 7

1. Prove that e^{At} is orthogonal if $A^t = -A$. Find differential equation for e^{At}.

2. Prove that

$$\det e^{At} \;=\; e^{\mathrm{Tr}(At)}$$

3. Prove that rotation in $\mathbb{R}^n$ splits into rotations in $n/2$ orthogonal planes for $n = 2k$. Use result of Problem 1 and skew-symmetric matrices.

4. Calculate $d\,(\Omega_{n-1})$ in $\mathbb{R}^n$ and its relation with operation

$$\mathrm{div}\;\zeta \;=\; \sum_{i=1}^{n} \frac{\partial \zeta_i}{\partial x^i}$$

for vector fields.

5. Calculate $d\Omega_1$ in $\mathbb{R}^3$ (1 - form) and its relation with operation $\mathrm{curl}\,(\zeta)$ for vector field ζ.

6. For 2×2 - matrix $B(t) \in O(2)$ in $\mathbb{R}^2$ prove equation (Frenét)

$$B(t)\;:\quad \left.\begin{matrix} e_1 & & e_1(t) \\ & \to & \\ e_2 & & e_2(t) \end{matrix}\right\} \;,\qquad \frac{de_1}{dt} \;=\; k\,e_2(t) \quad,\qquad \frac{de_2}{dt} \;=\; -\,k\,e_1(t)$$

Homework 7. Solutions.

1. $A^t \;=\; -\,A\,,\;\; e^A \in O_n\,?$

Proof:

$$\left(e^A\right)^t \;=\; e^{A^t} \;=\; e^{-A} \;=\; \left(e^A\right)^{-1}$$

OK.

Differential equation for $e^{At} \;=\; \psi$:

$$\dot{\psi} \;=\; A\,\psi \;=\; \psi\,A$$

2.

$$\det e^{At} \;=\; e^{\mathrm{Tr}\,At} \qquad ?$$

Proof: Diagonal matrices

$$A = \begin{pmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{pmatrix} \quad , \qquad e^{At} = \begin{pmatrix} e^{a_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{a_n t} \end{pmatrix}$$

$$\det e^{At} = e^{a_1 t} \cdot \dots \cdot e^{a_n t} = e^{a_1 t + \dots + a_n t} = e^{(\operatorname{Tr} A)\, t}$$

Set of the "diagonalizable" matrices is dense in space of matrices $n \times n$ over $\mathbb{C}$. So the identity

$$\det e^{At} = e^{\operatorname{Tr} At}$$

is true everywhere (it is polynomial).

OK.

3. **Rotations in** $\mathbb{R}^{2n}$: $A = e^B$, $B^t = -B$:

There exist orthonormal coordinates such that (this statement was not proved in this course)

$$B = \left(\begin{array}{cc|c|cc} 0 & a_1 & \dots & 0 & 0 \\ -a_1 & 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \dots & 0 & a_n \\ 0 & 0 & \dots & -a_n & 0 \end{array}\right) \quad , \qquad e^B = \left(\begin{array}{c|c|c} A_1 & \dots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \dots & A_n \end{array}\right) \quad ,$$

$$A_j = \exp \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix} \qquad - \qquad 2 \times 2 \text{ orthogonal matrix}$$

4. $\quad d\,\Omega_{n-1}$ in $\mathbb{R}^n$, $\quad \Omega_{n-1} = \sum_{i=1}^{n} a_i \, dx^1 \wedge \dots \wedge d\hat{x}^i \wedge \dots \wedge dx^n$

$$d\,\Omega_{n-1} = \sum \frac{\partial a_i}{\partial x^i} \, (-1)^i \, dx^1 \wedge \dots \wedge dx^n$$

OK.

5. $$\Omega \;=\; a\,dx \,+\, b\,dy \,+\, c\,dz$$

$$d\,\Omega \;=\; a_y\,dy{\wedge}dx + a_z\,dz{\wedge}dx + b_x\,dx{\wedge}dy + b_z\,dz{\wedge}dy + c_x\,dx{\wedge}dz + c_y\,dy{\wedge}dz \;=$$
$$=\; (b_x - a_y)\;dx \wedge dy \;+\; (c_y - b_z)\;dy \wedge dz \;+\; (c_x - a_z)\;dx \wedge dz$$

$$* \;:\; \Lambda^1\left(\mathbb{R}^3\right) \;\to\; \Lambda^2\left(\mathbb{R}^3\right)$$

$$*\,d\,\Omega \;=\; (c_y - b_z)\;dx \;-\; (c_x - a_z)\;dy \;+\; (b_x - a_y)\;dz \;=$$
$$=\; \text{curl}\;(a,\,b,\,c) \;=\; (c_y - b_z,\, a_z - c_x,\, b_x - a_y)$$

6. Frenét:

$$\begin{matrix} e_1 \\ e_3 \end{matrix} \;\to\; \begin{matrix} e_1(t) \\ e_2(t) \end{matrix}$$

$$\begin{array}{rcl} a) \quad \dot{e}_1 \;\perp\; e_1(t) & : & \langle e_1, e_1\rangle \;=\; 1 \quad\Rightarrow\quad \dot{e}_1 \;\perp\; e_1 \\ \dot{e}_1 \;=\; k\,n & & \text{(definition)} \\ \dot{e}_2 \;=\; -\,k\,n & & \text{(skew symmetry)} \end{array}$$

$$A \;=\; \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \qquad \left|\; \begin{array}{l} A(t) \;=\; 1 \,+\, B\,t \,+\, \dots \\ B \;=\; \dot{A}|_{t=0} \;=\; -\,B \end{array} \right.$$

Homework 8

1. Find 0 - cohomology group $H^0(GL_n(\mathbb{R}))$ and $H^0(GL_n(\mathbb{C}))$. $H^0(O_n) = ?$ $H^0(O_{n,1}) = ?$

2. Prove that all 1 - forms in S^2 invariant under rotations SO_3, are trivial (equal to 0) (Also true for S^n and SO_{n+1}).

3. Prove that for every rotation $A \in SO_3$ acting on S^2, the form closed $A^*\Omega$ and Ω are cohomologous (i.e. $A^*\Omega - \Omega = du$).

4. Prove that all **closed** forms in

$$\mathbb{T}^2 \; : \; (x^1, x^2) \simeq (x^1 + 2\pi, x^2) \simeq (x^1, x^2 + 2\pi)$$

are cohomologous to their shifts

$$h^* \Omega - \Omega = du \qquad h = h_{(a,b)}$$

$$h \; : \; (x^1, x^2) \to (x^1 + a, x^2 + b)$$

$$h^* \Omega = \Omega(x^1 - a, x^2 - b)$$

5. Prove that forms dx^1, dx^2 are linearly independent in $H^1(\mathbb{T}^2)$ and 2 - form $dx^1 \wedge dx^2 \neq du$ (nontrivial) in $\mathbb{T}^2$.

Homework 8. Solutions.

1. $H^0(GL_n(\mathbb{R})) = ?$, $H^0(O_n) = ?$, $H^0(O_{n,1}) = ?$

$$U = \bigcup_{i=1}^{N} U_i \quad , \qquad U_i \cap U_j = \emptyset$$

$$f \; : \; U \to \mathbb{R} \quad , \qquad d\,f = 0 \quad \leftrightarrow \quad f \text{ is locally constant}$$

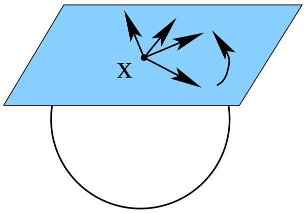
$$\text{So} \quad H^0(U) = \mathbb{R}^N \qquad (\text{number of pieces})$$

$GL_n(\mathbb{R})$ – 2 pieces, $O_{n,1}$ – 4 pieces, O_n – 2 pieces.

2. 1 - form ω in $\Lambda^1(S^2)$:

Rotations $A : S^2 \to S^2 \quad (\mathbb{R}^3 \to \mathbb{R}^3)$.

Let $x \in S^2$, $A(x) = x \Rightarrow A^\alpha(x) = x$



A^α – rotations in tangent plane.

No vectors in $\mathbb{R}^2$ invariant under **all** rotations. So, for any invariant 1 - form we have

$$\Lambda^1_{\text{inv}}(S^2) = 0$$

3. Let $A \in SO_3$, $A : S^2 \to S^2$.

$\exists A_t$ such that $A_0 = A$, $A_1 = I$ (Homotopic! $A_t \sim A_0 = I$)

We have for closed forms $A_t^* \Omega - \Omega = du_t$ (every 2-form in the sphere S^2 is closed, homotopy $F(x,t) = A_t$ is given).

OK.

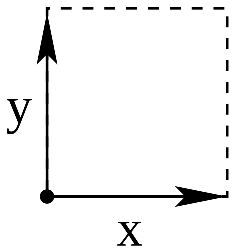
4. Like in the Problem 3, we have for shifts $A : \mathbb{T}^2 \to \mathbb{T}^2$ that there exists homotopy

$$A_t : \mathbb{T}^2 \to \mathbb{T}^2 \times R \qquad \text{such that} \qquad A_0 = A, \quad A_1 = I$$

So $\Omega(x, y) - \Omega(x-a, y-b) = du$, Ω is k - form in $\mathbb{T}^2$, $d\Omega = 0$.

5. dx^1 and dx^2 in $\mathbb{T}^2$ (closed 1 - forms).

$$\begin{array}{c|c} x = x^1 & (x, y) \simeq (x + 2\pi m, y + 2\pi n) \\ y = x^2 & m, n \in \mathbb{Z} \end{array}$$



$$\oint_0^{2\pi} dx^1 \;=\; 2\pi \quad , \quad \oint_0^{2\pi} dx^1 \;=\; 0$$

$$(x - \text{cycle}) \qquad\qquad (y - \text{cycle})$$

$$\oint_0^{2\pi} dx^2 \;=\; 0 \quad , \quad \oint_0^{2\pi} dx^2 \;=\; 2\pi$$

$$(x - \text{cycle}) \qquad\qquad (y - \text{cycle})$$

OK.

5a. $dx^1 \wedge dx^2$ − closed 2 - form

$$\iint_{\mathbb{T}^2} dx^1 \wedge dx^2 \;=\; 4\pi^2 \;\neq\; 0$$

So $dx^1 \wedge dx^2 \neq 0$ in $H^2(\mathbb{T}^2)$.

Homework 9

1. Calculate restriction of euclidean metric to $S^2 \subset \mathbb{R}^3$ given as

$$z^2 \;=\; 1 - x^2 - y^2 \;, \qquad z \;=\; \sqrt{1 - x^2 - y^2} \;, \qquad u \;=\; x \;,\; v \;=\; y$$

2. Calculate in the spherical coordinates restriction of the 2-form in R^3

$$\Omega \quad = \quad r^{-3}\,(x\,dy \wedge dz \;-\; y\,dx \wedge dz \;+\; z\,dx \wedge dy)$$

to the sphere

$$z \;=\; \sqrt{1 - x^2 - y^2} \;, \qquad u \;=\; x \;,\; v \;=\; y$$

and prove that it is an area form on the sphere

3. Prove that for the closed curve with nonzero tangent vector $C \subset \mathbb{R}^2$ we have

$$\oint_C k\,ds \quad = \quad 2\pi(\text{integer})$$

4. Consider 2 vectors $\eta, \zeta \in \mathbb{R}^3$, x, y, z − orthogonal coordinates. Prove that vector product $\eta \times \zeta$ is "dual" to $AB - BA$ (skew symmetric matrices)

$$A \;=\; *\,\eta \quad , \qquad B \;=\; *\,\zeta$$

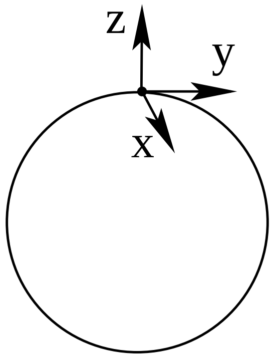
$$\eta \quad = \quad \begin{array}{ccc} 1 & 2 & 3 \\ (a, & b, & c) \end{array} \quad , \qquad *\,\eta \;=\; \begin{array}{cc} \begin{matrix} \;1\; & \;\;2\;\; & \;\;3 \end{matrix} & \\ \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

$$\begin{array}{ccccc} 1 & \to & (23) & = & -\,(32) \\ 2 & \to & (31) & = & -\,(13) \\ 3 & \to & (12) & = & -\,(21) \end{array}$$

$$*\;(\eta \times \zeta) \quad = \quad [*\,\eta\,,\, *\,\zeta] \quad = \quad AB \;-\; BA \qquad\qquad (?)$$

Homework 9. Solutions.

1. $z^2 \ = \ 1 \ - \ x^2 \ - \ y^2 \qquad (S^2) \ ,$



$$z \ = \ \sqrt{1-x^2-y^2}$$

$$ds^2 \quad = \quad dx^2 \ + \ dy^2 \ + \ dz^2 \quad =$$

$$= \quad dx^2 \ + \ dy^2 \ + \ \left(\frac{x\,dx \ + \ y\,dy}{\sqrt{1-x^2-y^2}} \right)^2$$

$$g_{11} \ = \ 1 + \frac{x^2}{1-x^2-y^2} \ , \quad g_{22} \ = \ 1 + \frac{y^2}{1-x^2-y^2} \ , \quad g_{12} \ = \ \frac{xy}{1-x^2-y^2}$$

$$g_{ij}|_{(0,0)} \ = \ \delta_{ij}$$

2. Area form:

$$d^2\sigma \quad = \quad \sqrt{\det g_{ij}} \ dx \wedge dy$$

$$\det g_{ij} \ = \ \left(1 + \frac{x^2}{1-x^2-y^2}\right)\left(1 + \frac{y^2}{1-x^2-y^2}\right) - \frac{x^2y^2}{(1-x^2-y^2)^2} \ = \ \frac{1}{1-x^2-y^2}$$

$$dx \ , \quad dy \ , \quad dz \ = \ - \frac{x\,dx \ + \ y\,dy}{\sqrt{1-x^2-y^2}} \quad , \qquad r^2 \ = \ x^2 \ + \ y^2 \ + \ z^2$$

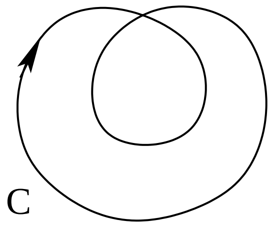
$$(x\,dy \wedge dz \ - \ y\,dx \wedge dz \ + \ z\,dx \wedge dy)|_{r=1} \quad = \quad \Omega\,|_{r=1} \quad =$$

$$= \quad (x\,dy \ - \ y\,dx) \ \wedge \ \frac{-\,x\,dx - y\,dy}{\sqrt{1-x^2-y^2}} \ + \ \sqrt{1-x^2-y^2} \ dx \wedge dy \quad =$$

$$= \quad dx \wedge dy \left(\frac{y^2 \ + \ x^2}{\sqrt{1-x^2-y^2}} \ + \ \frac{1 \ - \ x^2 \ + \ y^2}{\sqrt{1-x^2-y^2}} \right) \quad =$$

$$= \quad \frac{1}{\sqrt{1-x^2-y^2}} \ dx \wedge dy \quad = \quad \frac{dx \wedge dy}{z}$$

3. $\oint_C k\,ds \;=\; 2\pi m$ (number of rotations of $\tau(s)$) , $m \in \mathbb{Z}$

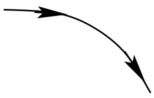


$$k\,ds \;=\; G^*(d\varphi) \quad , \qquad G \;:\; C \to S^1$$

$$G\,(s+T) \;=\; G(s) \;+\; 2\pi m \quad , \qquad m \in \mathbb{Z}$$

$$\vec{x}(s) \;=\; \left\{ \begin{array}{l} x(s+T) \;=\; x(s) \\ y(s+T) \;=\; y(s) \end{array} \right| \qquad \tau \;=\; \text{velocity } \dot{\vec{x}}(s)$$

Assumption $\tau \neq 0$, $|\tau| \;=\; |\dot{\vec{x}}| \;=\; 1$



$$k\,\Delta s \;=\; \Delta\varphi$$

4. $\eta\,,\,\zeta \;\in\; \mathbb{R}^3$, $(\eta \times \zeta)_{12} \;=\; \eta_1\zeta_2 \;-\; \zeta_1\eta_2 \;=\; *\,(\eta \times \zeta)_3$

$$(12) \;\to\; 3 \quad , \qquad (13) \;\to\; -2 \quad , \qquad (23) \;\to\; 1$$

Orthonormal basis. Product of 1 - forms

$$\underset{\eta}{\eta_i\,dx^i} \;\wedge\; \underset{\zeta}{\zeta_j\,dx^j} \;=\; \textstyle\sum_{ij}\,\eta_i\,\zeta_j\,dx^i \wedge dx^j$$

$$*\,(\eta \wedge \zeta) \;=\; (\eta \times \zeta) \qquad \text{(vector product)}$$

$$\begin{aligned} *\,\eta \;&=\; \eta_1\;dx^2 \wedge dx^3 \;-\; \eta_2\;dx^1 \wedge dx^3 \;+\; \eta_3\;dx^1 \wedge dx^2 \\ *\,\zeta \;&=\; \zeta_1\;dx^2 \wedge dx^3 \;-\; \zeta_2\;dx^1 \wedge dx^3 \;+\; \zeta_3\;dx^1 \wedge dx^2 \end{aligned}$$

$$*\,\eta \;=\; \begin{pmatrix} 0 & \eta_3 & -\eta_2 \\ -\eta_3 & 0 & \eta_1 \\ \eta_2 & -\eta_1 & 0 \end{pmatrix} \;=\; A \quad , \qquad *\,\zeta \;=\; \begin{pmatrix} 0 & \zeta_3 & -\zeta_2 \\ -\zeta_3 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & 0 \end{pmatrix} \;=\; B$$

Calculate AB and BA:

$$AB = \begin{pmatrix} -\eta_3\zeta_3 - \eta_2\zeta_2 & \eta_2\zeta_1 & \eta_3\zeta_1 \\ \eta_1\zeta_2 & -\eta_3\zeta_3 - \eta_1\zeta_1 & \eta_3\zeta_2 \\ \eta_1\zeta_3 & \eta_2\zeta_3 & -\eta_2\zeta_2 - \eta_1\zeta_1 \end{pmatrix}$$

$$BA = \begin{pmatrix} -\zeta_3\eta_3 - \zeta_2\eta_2 & \zeta_2\eta_1 & \zeta_3\eta_1 \\ \zeta_1\eta_2 & -\zeta_3\eta_3 - \zeta_1\eta_1 & \zeta_3\eta_2 \\ \zeta_1\eta_3 & \zeta_2\eta_3 & -\zeta_2\eta_2 - \zeta_1\eta_1 \end{pmatrix}$$

$$AB - BA = \begin{pmatrix} 0 & \eta_2\zeta_1 - \zeta_2\eta_1 & \eta_3\zeta_1 - \zeta_3\eta_1 \\ \eta_1\zeta_2 - \zeta_1\eta_2 & 0 & \eta_3\zeta_2 - \zeta_3\eta_2 \\ \eta_1\zeta_3 - \zeta_1\eta_3 & \eta_2\zeta_3 - \zeta_2\eta_3 & 0 \end{pmatrix}$$

(skew symmetric).

Conclusion: $AB - BA = *(\eta \times \zeta)$.

$$-\operatorname{Tr}(AB) = 2\langle \eta , \zeta \rangle \qquad \textbf{Inner Product!}$$

Homework 10

1. Calculate Riemannian metric

$$-\,ds^2 \;=\; \big(dx_0^2 \;-\; dx^2 \;-\; dy^2\big)\big|_{M^2\subset\mathbb{R}^{2,1}}$$

for the surface M^2 in Minkowski space $\mathbb{R}^{2,1}$ with indefinite metric

$$dx_0^2 \;-\; dx^2 \;-\; dy^2 \quad , \qquad M^2 \;=\; \big\{x_0^2 - x^2 - y^2 \;=\; 1\big\}$$

Prove that this metric **is positive**.
Express it in terms of **"pseudospherical"** coordinates:

$$x_0 \;=\; \rho\,\cosh\chi \quad , \qquad x_1 \;=\; \rho\,\sinh\chi\,\cos\varphi \quad , \qquad x_2 \;=\; \rho\,\sinh\chi\,\sin\varphi$$

$$x_1 = x \;,\; x_2 = y \quad , \qquad \rho^2 \;=\; x_0^2 \;-\; x^2 \;-\; y^2 \;=\; 1$$

$$-\,ds^2 \;=\; g_{ij}\;dx'^i\,dx'^j \quad , \qquad g_{ij} \;=\; g_{ij}(\theta,\varphi) \;, \qquad x'_1 \;=\; \theta \;,\; x'_2 \;=\; \varphi$$

2. Find volume element in $\mathbb{R}^{2,1}$ in coordinates $(\rho,\,\chi,\,\varphi)$, $d^3\sigma \;=\;?$ In which area of $\mathbb{R}^{2,1}$ these coordinates are OK?

3. Let metric be given in $U \subset \mathbb{R}^2$ as

$$ds^2 \;=\; dz\,d\bar z \,\cdot\, g\,(|z|^2)$$

Find area element $d^2\sigma$ for such metric, $g\,(|z|^2) \;>\; 0$. Write it in the form $(?) \cdot dz \wedge d\bar z$.

4. Let 2 - sphere be given in the "Riemann" form

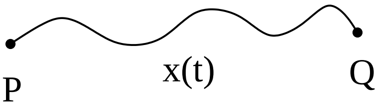
$$z \quad (\text{in } \mathbb{C} \;=\; S^2 \setminus \infty) \quad , \qquad w \quad (\text{in } \mathbb{C} \;=\; S^2 \setminus 0)$$

with $z \;=\; 1/w$ for $z \neq 0$, $w \neq 0$. Calculate metric

$$ds^2 \;=\; \frac{dz\,d\bar z}{(1 + |z|^2)^2}$$

in the local coordinate $w \;=\; 1/z$, $z \;=\; dx \;+\; i\,dy$.

5. **Geodesics:**= line with extremal (minimal) length among the paths joining P and $Q \in M$



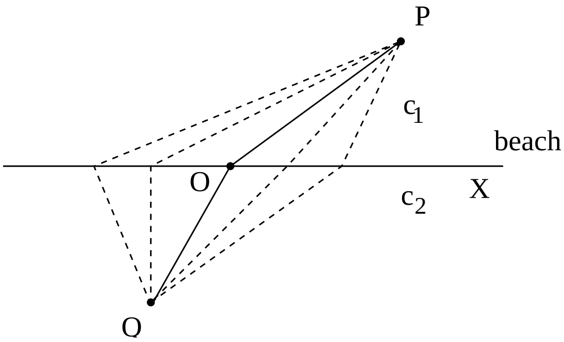
$$l(\gamma) = \int_P^Q |\dot{x}(t)|\, dt \quad , \qquad ds^2 = g_{ij}(x)\, dx^i\, dx^j$$

Fermat Principle: Let $c(x) > 0 =$ speed of light at the point x.

$$\text{“time”} = \int_P^Q \frac{|dx|}{c\,(x(t))} \quad , \qquad |dx| = |\dot{x}|\, dt$$

Minimal time principle for the propagation of light:
Find **geodesics** of metric

$$ds^2 = \frac{dx^2 + dy^2}{c^2(x)}$$



(they are straight lines for $y > 0$ and $y < 0$).

$$c = c_1 \quad (\text{Air}) \quad y > 0 \quad , \qquad c = c_2 \quad (\text{Water}) \quad y < 0$$

$$\int_P^Q \frac{dx}{c(x)} = \int_0^P \frac{dx}{c(x)} + \int_0^Q \frac{dx}{c(x)}$$

Homework 10. Solutions.

1. $$-\, ds^2 = \left((dx^0)^2 - dx^2 - dy^2\right)\Big|_{M^2} \quad , \qquad x^0 = t$$

$$M^2_{\rho=1} \;:\quad t^2 - x^2 - y^2 \;=\; 1 \quad , \qquad \rho^2 \;=\; t^2 - x^2 - y^2$$

$$t \;=\; \cosh\chi \quad , \qquad x \;=\; \sinh\chi\cos\varphi \quad , \qquad y \;=\; \sinh\chi\sin\varphi$$

$(\chi,\,\varphi)$ − coordinates:

$$dt \;=\; \sinh\chi\; d\chi \quad , \qquad dx \;=\; \cosh\chi\cos\varphi\; d\chi \;-\; \sinh\chi\sin\varphi\; d\varphi \quad ,$$

$$dy \;=\; \cosh\chi\sin\varphi\; d\chi \;+\; \sinh\chi\cos\varphi\; d\varphi$$

$$dt^2 \;-\; dx^2 \;-\; dy^2 \;=\; d\chi^2\left(\sinh^2\chi \;-\; \cosh^2\chi\right) \;-$$

$$-\; d\varphi^2\left(\sinh^2\chi\sin^2\varphi \;+\; \sinh^2\chi\cos^2\varphi\right) \;=\; -\left(d\chi^2 \;+\; \sinh^2\!\chi\; d\varphi^2\right)$$

- hyperbolic (Lobachevski) plane. (Sphere metric is $d\theta^2 \;+\; \sin^2\!\theta\; d\varphi^2$).

$$ds^2 \;=\; g_{ij}\; dx^i\, dx^j \quad , \qquad d^n\sigma \;=\; \sqrt{\det g_{ij}}\;\; dx^1\wedge\,\ldots\,\wedge dx^n$$

Gramm matrices are $(g_{ij} \;=\; g_{ji})$ following.

S^2	L^2	$\mathbb{R}^2$
$d\theta^2 \;+\; \sin^2\theta\; d\varphi^2$	$d\chi^2 \;+\; \sinh^2\chi\; d\varphi^2$	$d\rho^2 \;+\; \rho^2\, d\varphi^2$

2. **Area elements.**

$$ds^2 \;=\; d\chi^2 \;+\; \sinh^2\chi\; d\varphi^2 \quad , \qquad L^2 \;:\qquad G \;=\; \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\chi \end{pmatrix}$$

$$d^2\sigma \;=\; \sinh\chi\; d\chi\wedge d\varphi$$

$$ds^2 \;=\; d\theta^2 \;+\; \sin^2\theta\; d\varphi^2 \quad , \qquad S^2 \;:\qquad G \;=\; \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$d^2\sigma \;=\; \sin\theta\; d\theta\wedge d\varphi$$

$$ds^2 \ = \ d\rho^2 \ + \ \rho^2 \, d\varphi^2 \quad , \qquad \mathbb{R}^2 \ : \qquad G \ = \ \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

$$d^2\sigma \ = \ \rho \ d\rho \wedge d\varphi$$

Volume element in $\mathbb{R}^{2,1}$.
Coordinates ρ, χ, φ (correction), $\rho^2 \ = \ t^2 \ - \ x^2 \ - \ y^2$.

$$t \ = \ \rho \cosh\chi \quad , \qquad x \ = \ \rho \sinh\chi \, \cos\varphi \quad , \qquad y \ = \ \rho \sinh\chi \, \sin\varphi$$

$d^3\sigma \ = \ ?$

$$dt^2 \ - \ dx^2 \ - \ dy^2 \ =$$

$$= \ d\rho^2 \ \left[\cosh^2\chi \ - \ \sin^2\chi \, \cos^2\varphi \ - \ \sinh^2\chi \, \cos^2\varphi\right] \ - \ d\chi^2 \ - \ \rho^2 \sinh^2\chi \, d\varphi^2 \ =$$

$$= \ d\rho^2 \ - \ \rho^2 (d\chi^2 \ - \ \sinh^2\chi \, d\varphi^2)$$

$$G \ = \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sinh^2\chi \end{pmatrix} \quad , \qquad \sqrt{\det G} \ = \ \rho^2 \sinh\chi$$

$$d^3\sigma \ = \ \rho^2 \sinh\chi \ d\rho \wedge d\chi \wedge d\varphi$$

3. $$ds^2 \ = \ dz \, d\bar{z} \cdot g \quad , \qquad g \ = \ g(|z|^2) \quad , \qquad dz \, d\bar{z} \ = \ dx^2 \ + \ dy^2$$

$$G \ = \ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \quad , \qquad \sqrt{\det G} \ = \ g$$

$$d^2\sigma \ = \ g \ dx \wedge dy \ = \ \frac{1}{2i} \, g \ dz \wedge d\bar{z}$$

4. $$ds^2 \ = \ \frac{dz \, d\bar{z}}{(1 + z\bar{z})^2} \qquad\qquad (S^2 \ - \ \text{round sphere})$$

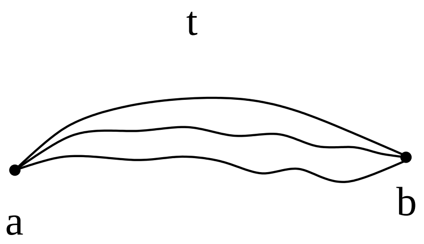
$$z \;=\; \frac{1}{w} \quad , \qquad \bar{z} \;=\; \frac{1}{\bar{z}}$$

$$ds^2 \;=\; \frac{d(1/w)\;\, d(1/\bar{w})}{(1\;+\;1/w\bar{w})^2} \;=\; \frac{dw\, d\bar{w}\;\cdot\;1/w^2\bar{w}^2}{(1+w\bar{w})^2/(w\bar{w})^2} \;=\; \frac{dw\, d\bar{w}}{(1+w\bar{w})^2}$$

Same!

5. $$ds^2 \;=\; \frac{dx^2+dy^2}{c^2(x,y)} \quad , \qquad x(t)\;,\;\; y(t) \;\;-\;\; \text{curve}$$

Arc length:

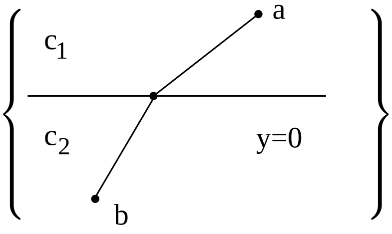


$$l(\gamma) \;=\; \int_a^b \frac{\sqrt{\dot{x}^2+\dot{y}^2}\; dt}{c\,(x(t),\, y(t))}$$

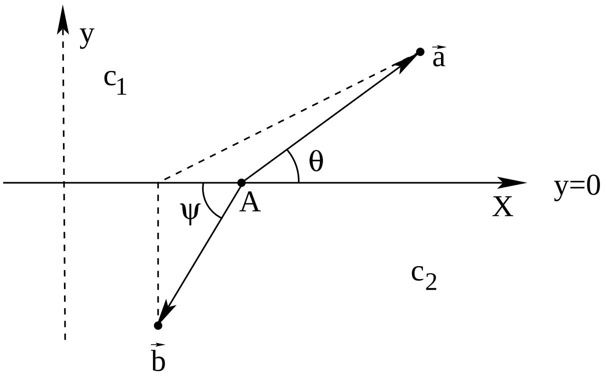
"Geodesics" = i.e. curve with minimal arc length

Example: Fermát Principle Let

$c(x,y) =$



Find geodesic joining a and b.



$$\vec{a} \;=\; (x_1,\, y_1) \quad , \qquad \vec{b} \;=\; (x_2,\, y_2)$$

Time from $\vec{a}$ to $\vec{A}$ $=$ $|\vec{a} - \vec{A}|/c_1$ $=$ T_1

Time from $\vec{A}$ to $\vec{b}$ $=$ $|\vec{b} - \vec{A}|/c_2$ $=$ T_2

Total time $=$ $T_1 + T_2$ (depends on $\vec{A}$), $\vec{A} = (A,\, 0)$.

$$T_1 + T_2 \;=\; \sqrt{(x_1 - A)^2 + y_1^2}\;/\,c_1 \;+\; \sqrt{(x_2 - A)^2 + y_2^2}\;/\,c_2 \;=\; T(A)$$

$$\min_A T(A) \;=\; ?$$

$$\frac{dT}{dA} \;=\; 0 \quad ?$$

$$0 \;=\; \underset{\displaystyle \sin\theta}{\underset{\uparrow}{\frac{1}{c_1}\,\frac{(A\,-\,x_1)}{\sqrt{(x_1 - A)^2 + y_1^2}}}} \;+\; \underset{\displaystyle -\sin\psi}{\underset{\uparrow}{\frac{1}{c_2}\,\frac{(A\,-\,x_2)}{\sqrt{(x_2 - A)^2 + y_2^2}}}}$$

$A - x_1 < 0\,, \qquad A - x_2 > 0\,.$

$$\frac{c_2}{c_1} \;=\; \frac{\sin\psi}{\sin\theta}$$

Midterm Tests.

First Midterm Test.

1. Differential 0-forms and 1-forms, define pull-back map, change of coordinated, restriction to curve, integration. Newton-Leibnitz formula for integration. Calculate differential 1-form dz/z in polar coordinates

2. Formulate necessary and sufficient condition for 1-form $X = udx + vdy$ in the planar domain $U \subset R^2$ to be closed. Find condition for closed form to be exact in terms of contour integration? Is the form $dz/(z^2 - 1)$ exact in the domain $U \subset R^2$ where 2 points ± 1 removed?

   Define 2-forms and De Rham differential operator d from 1-forms to 2-forms for the planar domains.

3. Define exterior (grassmanian) product of any number of 1-forms. Formulate main properties of this product. Calculate product $X_1 \wedge X_2 \wedge X_3$ of 3 forms in R^3:

   $$X_1 = 2ydx + 3dy + 6dz, , X_2 = 6xdx + 5dy + 12dz, , X_3 = (y + 3x)dx + 4dy + 9dz$$

Second Midterm Test.

1. Differential 1-forms and change of coordinates. Differential 1- and 2-forms in R^2, R^3, define the exterior product of 1-forms and De Rham Operator d for 1-forms. What is a closed form? What is an exact form? Calculate 2-form $d[(xdy - ydx)/(x^2 + y^2)^s]$. Prove that it is closed for $s = 1$ only in $R^2 minus 0$.

2. By definition, Electromagnetic Field F is a differential 2-form in R^4 or pair (E, B) of 1- and 2-forms in R^3 depending on time (i.e. the 4-space is presented as $R^3 \times R$). Formulate the 1st pair of Faraday Laws in R^3 (Non-relativistic form) and in $R^4 = (x^0 = ct, x^1, x^2, x^3)$ (Relativistic form) in terms of differential forms and De Rham Operator. Apply Stokes Formula to express the integral of Electric Field along the closed contour C in R^3 through the magnetic field inside of contour (its time derivative).

3. Prove that for every symmetric indefinite nondegenerate inner product in R^2 there exists a basis e, e' such that Gramm Matrix has a ("light-like") form

   $(e, e) = (e', e') = 0, (e, e') = 1$

Second Midterm Test. Solutions.

I. First problem is collection of definitions and calculation of $d\Omega$ for some specific 1 - form Ω .

Definitions:

1. 1 - form in $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, ... :

$$\sum_i u_i(x)\, dx^i$$

Change of coordinates $x(x')$ (pull-back, restriction, ...)

$$f(x) \quad \rightarrow \quad f\,(x(x')) \quad ,$$

$$dx^i \quad \rightarrow \quad \frac{\partial x^i}{\partial x'^k}\, dx'^k \quad ,$$

matrix form (Jacoby matrix):

$$d\vec{x} \quad \rightarrow \quad \left(\frac{\partial x}{\partial x'}\right) d\vec{x}'$$

2. 2 - form in $\mathbb{R}^2$, $\mathbb{R}^3$, $\ldots$:

$$\Omega \quad = \quad \sum_{ij} a_{ij}(x)\, dx^i \wedge dx^j$$

De - Rham operator

$$d \;:\; \Omega \;\rightarrow\; d\,\Omega$$

Multiplication of forms $\wedge$:

$$\big(f\, dx^i \wedge dx^j\big) \;\wedge\; \big(g\, dx^k \wedge dx^l\big) \quad = \quad fg\; dx^1 \wedge dx^j \wedge dx^k \wedge dx^l$$

Operator d:

$$df \;=\; \frac{\partial f}{\partial x^i}\, dx^i \quad , \qquad d\,\big(f\; dx^i \wedge \ldots \wedge dx^j\big) \;=\; df \wedge dx^i \wedge \ldots \wedge dx^j$$

3. Calculate $d\Omega$ for

$$\Omega \quad = \quad \big(x^2 + y^2\big)^{-s}\; (x\, dy \,-\, y\, dx)$$

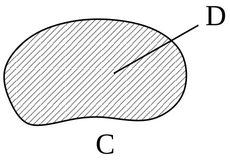
Exact form: $\Omega = d\omega$.

Closed form: $d\Omega = 0$.

Properties

$$\omega \;-\; \text{exact} \quad \Rightarrow \quad \oint_C \omega \;=\; 0 \quad \text{(closed contour)}$$

$$\omega \;-\; \text{closed} \quad \Rightarrow \quad \oint_C \omega \;=\; 0 \qquad C \subset \mathbb{R}^2$$



if domain D is simply connected
(ω is defined and smooth in D).

II. Second Problem:

Electromagnetic field $=$ 2 - form F in $\mathbb{R}^4$ (space-time)

$$\mathbb{R}^4 \; = \; (x^0 = ct,\, x,\, y,\, z)$$

First part of Maxwell Equations (Faraday Law)

$$d\,F \; = \; 0$$

$$3 \; - \; \text{dimensional form in} \quad \underset{(x,y,z)}{\mathbb{R}^3} \; \times \; \underset{(t)}{\mathbb{R}} \quad :$$

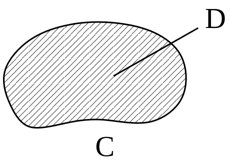
$$E \; = \; E_x\,dx \; + \; E_y\,dy \; + \; E_z\,dz$$

$$F \;\; = \;\; E \wedge dx^0 \; + \; B \quad , \qquad x^0 \; = \; ct$$

$$B \; = \; \sum_{i,j=1,2,3} b_{ij}\; dx^i \wedge dx^j$$

$$d_{(4)}\,E \; = \; 0 \qquad \Leftrightarrow \qquad \text{“Pair”} : \qquad \begin{cases} a) \;\; d_{(3)}\,B \; = \; 0 \;, \\ b) \;\; d_{(3)}\,E \; + \; \partial B/\partial x^0 \; = \; 0 \end{cases}$$

($d_{(4)} -$ in $\mathbb{R}^4$, $d_{(3)} -$ in $\mathbb{R}^3$).



Stokes:

$$\oint_C E \;\; = \;\; \frac{1}{c}\,\frac{\partial}{\partial t}\int_D B$$

Second pair of Maxwell Equations:

$$\begin{array}{ccccc} d\,(*F) & = & J^{(4)} & & (\text{“}4\ -\ \text{current”}) \\ & & & & \\ \uparrow & & \uparrow & & \\ \text{involves} & & \text{non} - \text{geometrical} & & \\ \text{inner} & & \text{term} & & \\ \text{product} & & & & \end{array}$$

This law is nongeometrical
(interaction with matter)

First pair $d\,F \;=\; 0$ **is purely geometrical.**

III. Third problem: $\mathbb{R}^2$ with indefinite inner product (symmetric). Find basis e, e' with Gramm Matrix

$$(e\,,\,e') \;=\; 1\;, \quad (e\,,\,e) \;=\; (e'\,,\,e') \;=\; 0 \quad , \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;=\; G$$

Proof. Start with basis

$$e_1\;,\;\; e_2\;, \qquad G' \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Put

$$\tilde{e} \;=\; e_1 \;+\; e_2\;, \qquad \tilde{e}' \;=\; e_1 \;-\; e_2$$

$$(\tilde{e}\,,\,\tilde{e}) \;=\; (\tilde{e}'\,,\,\tilde{e}') \;=\; 0\;, \qquad (\tilde{e}\,,\,\tilde{e}') \;=\; 2$$

Put

$$e \;=\; \frac{\tilde{e}}{\sqrt{2}}\;, \qquad e' \;=\; \frac{\tilde{e}'}{\sqrt{2}}$$

Lectures 1–30

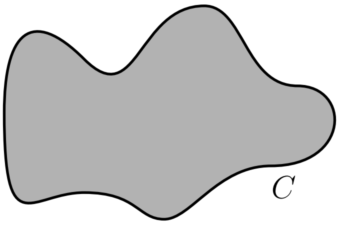
Introductory lecture: History of integration and calculus

1. XVII–XVIII centuries:

$$\int_a^x f(y)\,dy = g(x), \quad g'(x) = f(x), \ldots$$

2. XIX century: idea of a differential 1-form (Green, Cauchy, …) and first topological ideas (1820)

$$0 = \oint_C f(z)\,dz$$



$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ inside } C \quad \longrightarrow \quad \text{complex analytic function}$$

Integration of differential forms is the best possible type of integration:

(i) It is well-defined without use of any other structures (Riemannian metric, …).

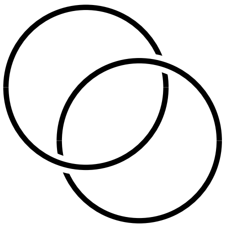
(ii) It is invariant under changes of coordinates both in the space (manifold) and in the body. Admissible changes of coordinates form a very broad class (even not one-to-one along the body).

(iii) They can be differentiated (d). This differential is invariant.

(iv) They have remarkable "Stokes Property" connecting (body) and (its boundary). It connects them with topology — "De Rham Cohomology" known since XIX century.

(v) They can be described in the beautiful “differential” language as well as in the classical vector (tensor) language.

3. Electromagnetism, Faraday laws, Gauss formula and differential forms, topological quantities, Maxwell equations (mid XIX century)

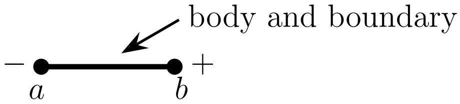


(linking number)

4. Vector (tensor) calculus (in the 3-space), Maxwell, Kelvin, Stokes, Poincaré.

   Stokes-type formulas:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$



   (simplest Newton–Leibnitz formula).

5. Poincaré, differential forms and topology (vector (tensor) calculus in multidimensional spaces) (1895). Hamiltonian systems, Riemann surfaces.

6. E.Cartan and differential forms (1920s). Discovery of differential forms in the modern notations. Topology.

7. De Rham Theorem (1930s). Proof of the Cartan–Poincaré conjecture.

8. Differential forms in modern mathematics. Complex geometry.

Differential forms are special tensors (“contravariant”, skew symmetric). Why are they especially important?

1. They provide a universal integration and differentiation (independent of any other structure like Riemannian metric and so on).

2. They present the best bridge between Analysis and Topology. Cauchy started to use these ideas in early XIX century.

3. They describe the laws of Electromagnetism (Faraday laws and Maxwell equations).

4. In the special case of Complex Geometry everything important can be written in terms of differential forms including Riemannian metric ("Kähler metric").

5. The differential notations of Cartan are extremely convenient.

Lecture 2. Differential 1-forms

Question: What is a differential 0-form?
Answer: Differential 0-forms are functions $f(x)$, 0-bodies are points, 0-chains are collections of points and numbers:

$$Q = \Big[\underset{(P_1,n_1)}{\bullet} \ \underset{(P_2,n_2)}{\bullet} \quad \dots \quad \underset{(P_k,n_k)}{\bullet} \Big], \quad n_j \in \mathbb{Z},$$

$$Q = \text{“0-chain”} = (P_1, n_1; P_2, n_2; \dots ; P_k, n_k).$$

The “value” (or the “integral”) of the 0-form f along a 0-chain Q is

$$\langle f, Q \rangle = \sum_{j=1}^{k} n_j f(P_j).$$

Example. Q is a single point P with $n_1 = \pm 1$,

$$\underset{P_1, n_1 = \pm 1}{\bullet} \qquad \langle f, Q \rangle = \pm f(Q).$$

What is a differential 1-form?

In the space with coordinates $(x^1, \dots, x^n)$, $\mathbb{R}^2 = \{(x, y)\}$, $\mathbb{R}^1 = \{(x)\}$, $\mathbb{R}^3 = \{(x, y, z)\}$ $(n = 1, 2, 3)$, ($\underline{\text{co}}$)-“vector field” $X = (\eta_1, \dots, \eta_n)$, $\eta_j = \eta_j(x^1, \dots, x^n)$.
$n = 1$: $\eta(x)$ has a single component;
$n = 2$: $(\eta_1(x, y), \eta_2(x, y))$, $x^1 = x$, $x^2 = y$.

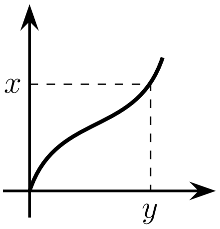
A differential 1-form is

$$X = \sum_{j=1}^{n} \eta_j \, dx^j.$$

$n = 1$: $X = \eta(x)\, dx$;
$n = 2$: $X = \eta_1(x, y)\, dx + \eta_2(x, y)\, dy$;
$n = 3$: $X = \eta_1(x, y, z)\, dx + \eta_2(x, y, z)\, dy + \eta_3(x, y, z)\, dz$, $x^1 = x$, $x^2 = y$, $x^3 = z$.
Question: How to integrate a differential 1-form?
Question: How to write it down in any other system of coordinates?

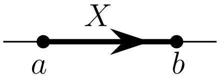
$$\eta(x)\, dx = \eta(x(y)) \frac{dx}{dy}\, dy, \quad x = x(y).$$

$x \longleftrightarrow y$

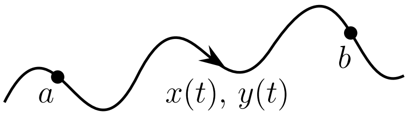


What is a one-dimensional “oriented” body?

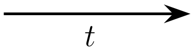
$n = 1$



$n = 2$

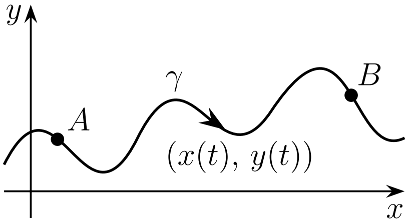


Oriented line



$n = 1.$

Oriented *parametrized* curve (body)



$t = a$ (point A)
$t = b$ (point B)

Integration of a 1-form along an “oriented” curve (“body”) γ (case $n = 2$)

$$\int_A^B \big(\eta_1(x,y)\,dx + \eta_2(x,y)\,dy\big) = I$$

along the curve γ.

Definition. (see $n = 2$)

Step 1. Restrict 1-form to the body. Substitute $x = x(t)$, $y = y(t)$, $a \leqslant t \leqslant b$

$$I = \int_a^b \Big(\eta_1(x(t), y(t))\frac{dx}{dt} + \eta_2(x(t), y(t))\frac{dy}{dt}\Big)dt = \int_a^b \varphi(t)\,dt.$$

Step 2. Calculate the ordinary integral along the line. "Parametrized curve" $x(t), y(t)$: $\mathbb{R}^1 \to \mathbb{R}^2$ (or $\mathbb{R}^1 \to \mathbb{R}^n$).

Vector notation: $X = (\eta_1, \eta_2)$. *Restriction* to the curve $\gamma = (x(t), y(t))$, $\dot{\gamma} = (dx/dt, dy/dt)$ for $n = 2$:

$$\langle X, \dot{\gamma} \rangle = \eta_1(x(t), y(t)) \frac{dx}{dt} + \eta_2(x(t), y(t)) \frac{dy}{dt}.$$

Integral along the curve γ reduces to the ordinary integral

$$I = \int_A^B \langle X, \dot{\gamma} \rangle \, dt,$$

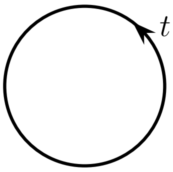
where $\langle \, , \rangle$ is the "inner product" in orthonormal coordinates.

We assume here that the space is Euclidean. In fact, our integral does *not* depend on the Euclidean geometry.

Examples:
$n = 1$. $X = \eta(x)\, dx$, $x = t$

$$I = \int_a^b \eta(x)\, dx.$$

$n = 2$. $x = \cos t$, $y = \sin t$ (circle)



$$X = \frac{x\, dy - y\, dx}{x^2 + y^2},$$

a very interesting differential 1-form (angle). It will be considered later.

Calculate the restriction to $x^2 + y^2 = \rho^2$.

Examples from physics:

1. The speed of particles is *not* a differential 1-form. One cannot integrate it without Euclidean geometry.

2. Electric field $E = (E_1, E_2, E_3)$ *is* a differential 1-form ($n = 3$).
3. Magnetic field is *not* a differential 1-form (it is in fact a 2-form).
4. The gradient of a function *is* a differential 1-form (n is arbitrary):

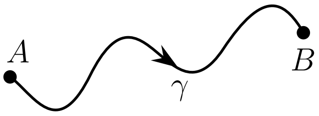
$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz + \dots$$

Integration (see the proof below).

Theorem 1.

$$\int\limits_A^B df = f(B) - f(A)$$

(along any smooth path $\gamma : \mathbb{R}^1 \to \mathbb{R}^n$ joining A and B)



We will prove it later.

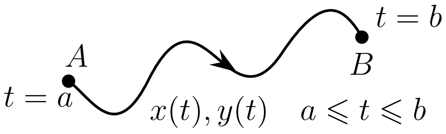
What is the "de Rham" d-operator? It is $d : f \to df$.

Proof of Theorem 1. Let $n = 1$. We have $A = a$, $B = b$,

$$\int\limits_a^b f'(x)\,dx = f(b) - f(a), \quad df = f'(x)\,dx.$$

Let $n > 1$. We have

$$df = f_x\,dx + f_y\,dy + \dots, \quad f_x = \frac{\partial f}{\partial x}, \dots$$



$$\int\limits_A^B df = \int\limits_a^b \Big(f_x(x(t), y(t)) \frac{dx}{dt}\, dt + f_y(x(t), y(t)) \frac{dy}{dt}\, dt \Big) \qquad \text{(let } g(t) = f(x(t), y(t))$$

$$= \int\limits_a^b dg(t) = g(b) - g(a) = f(B) - f(A).$$

The theorem is proved.

Lecture 3

0-forms = functions $f(x)$, $I = f(P)$.
1-forms = $X = \sum_i \eta_i(x)\, dx^i$ $(x = x^1, \dots, x^n)$.

The integral of a 1-form X along 1-body (parametrized curve) $\gamma = \{x(t) = (x^1(t), \dots, x^n(t)),\ a \leqslant t \leqslant b\}$ is defined by formula:

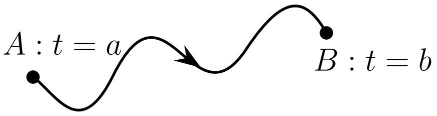
$$I = \int\limits_a^b \sum_i \eta_i(x(t)) \frac{dx^i}{dt}\, dt = \int\limits_\gamma X = \int \langle X, \dot\gamma \rangle\, dt, \quad \dot\gamma = (\dot x^1(t), \dots, \dot x^n(t)),$$

$$X = \eta_1(x(t)), \dots, \eta_n(x(t)), \quad \langle X, \dot\gamma \rangle = \sum_i \eta_i \dot x^i \quad (\text{“inner product”}).$$

Example. $X = df = \sum_i \frac{\partial f}{\partial x^i}\, dx^i$.

$$I = \int\limits_A^B df = f(B) - f(A)$$

(any path γ).



Change of coordinates

1. Along the curve γ

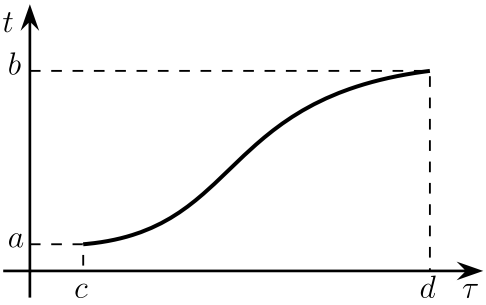
$$\gamma = \{x(t)\}, \quad t = t(\tau),\ dt/d\tau > 0.$$

Claim:

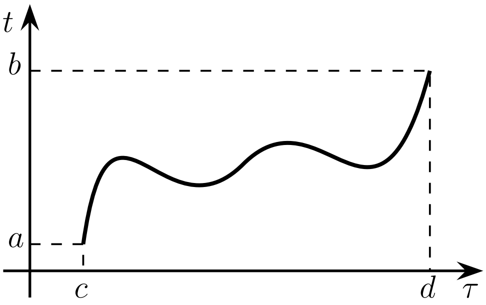
$$\int\limits_{\gamma(t)} X = \int\limits_{\gamma(t(\tau))} X$$

because on the line we have

$$\int\limits_a^b f(t)\, dt = \int\limits_c^d f(t(\tau))\, d\tau \cdot \frac{dt}{d\tau}$$



But it is also true for "non-monotonic" changes (sometimes we may have $dt/d\tau < 0$):



2. Change of coordinates in the space $x \leftrightarrow x'$

$$\begin{gathered} x^1(x'^1, \ldots, x'^n), \\ x^2(x'^1, \ldots, x'^n), \\ \cdots \\ x^n(x'^1, \ldots, x'^n), \end{gathered}$$

Let $n = 1$.

$$\eta(x)\,dx = \eta'(x')\,dx',$$

$$dx' \cdot \frac{dx}{dx'} = dx,$$

$$dx(x(x')) = \frac{dx}{dx'}\,dx'$$

one-to-one change!

Definition.

$$\eta(x)\,dx = \eta'(x(x'))\,dx'$$

is a definition of *the same* 1-form in the new coordinate x'. η and η' represent the same 1-form, so

$$\int\limits_a^b \eta(x(t))\,dt = \int\limits_a^b \eta'(x(x'(t))\,dt$$

because it is the same integral.

$$\eta' = \eta \cdot \frac{dx}{dx'}, \quad \text{for } n = 1,$$

in the same point $x(x')$, $x' \leftrightarrow x$.

For every $n \geqslant 1$ we define the same change of coordinates

$$\sum_i \eta'_i(x)\,dx'^i = \sum_{j,k} \eta_j \frac{\partial x^j}{\partial x'^i}\,dx'^i,$$

$$\eta'_i = \eta_j \frac{\partial x^j}{\partial x'^i} \qquad \Big[\underbrace{\overrightarrow{\eta}' = \overrightarrow{\eta} \cdot \frac{\partial x}{\partial x'}}_{\text{matrix multiplication}}\Big].$$

$n = 2$.

$$\eta_1(x,y)\,dx + \eta_2(x,y)\,dy = \eta'_1\,dx' + \eta'_2\,dy'$$

$x(x',y')$; $y(x',y')$ $\leftrightarrow$ one-to-one change.

$$\begin{aligned} dx &\to \frac{\partial x}{\partial x'}\,dx' + \frac{\partial x}{\partial y'}\,dy', \\ dy &\to \frac{\partial y}{\partial x'}\,dx' + \frac{\partial y}{\partial y'}\,dy'. \end{aligned}$$

Jacoby matrix

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial x'} & \dfrac{\partial x}{\partial y'} \\ \dfrac{\partial y}{\partial x'} & \dfrac{\partial y}{\partial y'} \end{pmatrix}.$$

Requirement: $\det J \neq 0$.

Change is one-to-one, invertible as a smooth map $x'(x,y)$, $y'(x,y)$

$$\begin{pmatrix} \dfrac{\partial x'}{\partial x} & \dfrac{\partial x'}{\partial y} \\ \dfrac{\partial y'}{\partial x} & \dfrac{\partial y'}{\partial y} \end{pmatrix} = J^{-1}.$$

$\overrightarrow{x} = (x^1, \dots, x^n)$, $\overrightarrow{x}' = (x'^1, \dots, x'^n)$. $\overrightarrow{x}(\overrightarrow{x}')$, $\overrightarrow{x}'(\overrightarrow{x})$

$$\overrightarrow{x}(\overrightarrow{x}'(\overrightarrow{x})) \equiv \overrightarrow{x}.$$

$$\frac{\partial x^i}{\partial x'^j}\frac{\partial x'^j}{\partial x^k} = \delta^i_k = \underset{\text{the unit matrix}}{\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}}.$$

$$\frac{\partial x}{\partial x'}\frac{\partial x'}{\partial x} \equiv \mathbb{1} \qquad \text{(matrix multiplication)}.$$

$$\varphi(x(y)) \quad \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial y} \qquad \text{(matrix multiplication)}.$$

Remark 1. Change of coordinates for ordinary vectors (like speed of particles). $\gamma(t) : x^1(t), \dots, x^n(t)$. Speed: $\dot{\gamma}(t) = (\dot{x}^1, \dots, \dot{x}^n)$. New coordinates $x(x')$.
Speed:

$$\frac{dx}{dt} = \frac{\partial x}{\partial x'}\frac{dx'}{dt} \qquad \left(\frac{\partial x^i}{\partial x'^j}\right) = J$$

$$\underset{(x)}{\dot{\gamma}} = J\underset{(x')}{\dot{\gamma}'} \quad \Rightarrow \quad \underset{(x')}{\dot{\gamma}'} = J^{-1}\underset{(x)}{\dot{\gamma}}$$

1-form:

$$\eta\, dx = \eta'\, dx', \quad dx = \frac{\partial x}{\partial x'}\, dx', \quad \eta\frac{\partial x}{\partial x'}\, dx' = \eta'\, dx',$$

so we have

$$\eta\frac{\partial x}{\partial x'} = \eta' \quad \text{or} \quad \eta J^\top = \eta' \quad \left(\eta_i \frac{\partial x^i}{\partial x'^j} = \eta'_j\right)$$

It is *not* the same law: the equality

$$J^{-1} = J^\top \;\Leftrightarrow\; J \in \underset{\text{(orthogonal)}}{O_n}\ !$$

is true for Orthogonal matrices only.)

Remark 2. For 1-forms this formula makes sense even if the change is *not* on-to-one

$$\eta' = J\eta \quad x = x(x'(t))$$

$x = (x^1, \dots, x^n)$, $x' = (x'^1, \dots, x'^n)$.

May be even $\det J = 0$ in some points (even everywhere). This formula makes sense even if $\dim x \neq \dim x'$, $\gamma : \underset{(x')}{\mathbb{R}^k} \to \underset{(x)}{\mathbb{R}^n}$

$$X = \sum \eta_i(x)\, dx^i, \qquad \gamma^* X = \sum_{i,j} \eta_i(x(t)) \frac{\partial x^i}{\partial t^j}\, dt^j, \qquad t = (t^1, \dots, t^k).$$

Example. Let $k = 1$, $n = 2$,

$$\gamma : \underset{t}{\mathbb{R}^1} \longrightarrow \underset{(x,y)}{\mathbb{R}^2}, \qquad x(t), y(t)$$

1-form $\eta_1\, dx + \eta_2\, dy = X$ in $\mathbb{R}^2$.

"Restriction" of a 1-form to $\gamma = \mathbb{R}^1$ is also a partial case of this formula

$$\gamma^* X = \eta_1(x(t), y(t)) \frac{dx}{dt}\, dt + \eta_2(x(t), y(t)) \frac{dy}{dt}\, dt.$$

Lecture 4

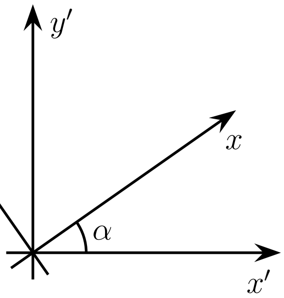
Coordinates in $\mathbb{R}^n$. Let $(x^1, \ldots, x^n)$ — Cartesian coordinates: $\overrightarrow{x} = (x^1, \ldots, x^n)$. $\overrightarrow{x}_1 \neq \overrightarrow{x}_2 \Leftrightarrow$ the points are distinct. Let $U \subset \mathbb{R}^n$ be an open domain with the same cartesian coordinates.

Examples (changes of coordinates).

a) Shift $\overrightarrow{x} = \overrightarrow{x}' + \overrightarrow{a}$ ($\overrightarrow{a}$ is a vector),

$$x'^i \to x'^i + a^i = x^i.$$

b) Rotation (let $n = 2$). $(x^1, x^2) = (x, y)$



$$\begin{aligned} x &= x' \cos\alpha + y' \sin\alpha, \\ y &= -x' \sin\alpha + y' \cos\alpha \end{aligned} \quad \Bigg| \quad (x', y') \to (x, y)$$

c) Reflection $x \to -x$, $y \to y$, $x' = -x$, $y' = y$.

d) Linear change

$$\begin{aligned} x &= ax' + by' \\ y &= cx' + dy' \end{aligned} \quad \Bigg| \quad \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

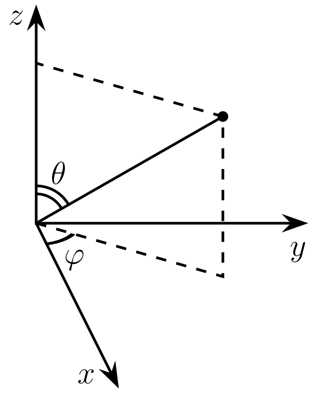
e) Affine change

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x' \\ y' \end{pmatrix} + \overrightarrow{\beta} \quad \Bigg| \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Polar coordinates ($n = 2$). $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $(x', y') = (\rho, \varphi)$, $\rho^2 = x^2 + y^2$, $x/\rho = \cos\varphi$, $y/\rho = \sin\varphi$.

Spherical coordinates ($n = 3$).

$$\begin{aligned} z &= r\cos\theta, \\ x &= r\sin\theta\cos\varphi, \qquad (r, \theta, \varphi) \\ y &= r\sin\theta\cos\varphi. \end{aligned}$$



"Formal" complex coordinates

$$\begin{aligned} z &= (x + iy), \\ \overline{z} &= (x - iy). \end{aligned} \qquad (z, \overline{z}), \quad i^2 = -1.$$

Let $n = 2$. A differential 1-form is $u(x, y)\,dx + v(x, y)\,dy$. Change of coordinates $(x, y) \leftrightarrow (x', y')$:

$$\begin{aligned} u(x, y) &\to u(x(x', y'), y(x', y')), \\ v(x, y) &\to v(x(x', y'), y(x', y')). \end{aligned}$$

For functions:

$$f(x, y) \to f(x(x', y'), y(x', y'))$$

— same functions in the new coordinates.

For 1-forms:

$$u\,dx + v\,dy = u'\,dx' + v'\,dy' \ ?$$

$$dx \to \frac{\partial x}{\partial x'}\,dx' + \frac{\partial x}{\partial y'}\,dy',$$

$$dy \to \frac{\partial y}{\partial x'}\,dx' + \frac{\partial y}{\partial y'}\,dy'.$$

For every function $f(x, y)$ we have:

$$df \to \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \to \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'}\,dx' + \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'}\,dy' + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'}\,dx' + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'}\,dy'.$$

Conclusion. In every coordinates we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = \frac{\partial f}{\partial x'}\,dx' + \frac{\partial f}{\partial y'}\,dy'.$$

(For all $n \geqslant 1$: $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\,dx^i$.)

Operator d: functions → 1-forms, does *not* depend on coordinates. It commutes with all C^∞-maps $\varphi : \mathbb{R}^k \to \mathbb{R}^n$ $(x'^1, \dots, x'^k) \overset{\varphi}{\to} (x^1, \dots, x^n)$, $\overrightarrow{x} = \varphi(\overrightarrow{x}')$.

The rule is:

$$\begin{aligned} \varphi^* &: f(\overrightarrow{x}) \to f(\overrightarrow{x}(\overrightarrow{x}')), \\ \varphi^* &: df(\overrightarrow{x}) \to df(\overrightarrow{x}(\overrightarrow{x}')), \\ & dx^i \to \sum_j \frac{\partial x^i}{\partial x'^j}\,dx'^j. \end{aligned}$$

Terminology: we call it"induced map" (or pull-back map) φ^* acting on functions and forms.

Examples.
Shifts $\overrightarrow{x} = \overrightarrow{x}' + \overrightarrow{a}$.
Rotations $\overrightarrow{x} = A\overrightarrow{x}'$, $A^\top = A^{-1}$ (orthogonal maps) $AA^\top = 1$.
General linear maps (A is *not* orthogonal).
Polar coordinates $(n = 2)$ $(x', y') = (\rho, \varphi)$, $\rho^2 = x^2 + y^2$.

Remark. Properties of Cartesian coordinates in any open $U \subset \mathbb{R}^n$ $(x^1, \dots, x^n)$:

a) every coordinate domain $x^j = c$ is ??? open domain in $\mathbb{R}^{n-1}$;

b) $\overrightarrow{x}_1 \neq \overrightarrow{x}_2 \Rightarrow$ points are distinct.

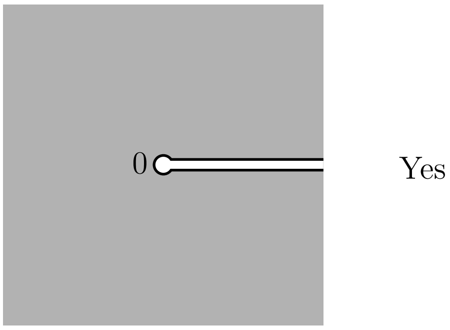
Lecture 5

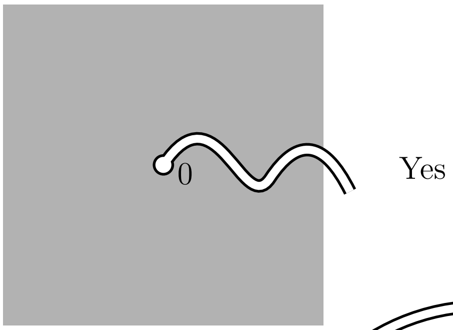
Coordinates in the plane $\mathbb{R}^2$:
Cartesian (x, y), $x \in \mathbb{R}$, $y \in \mathbb{R}$,
Polar (ρ, φ), $x = \rho \cos \varphi$, $y = \rho \sin \varphi$.

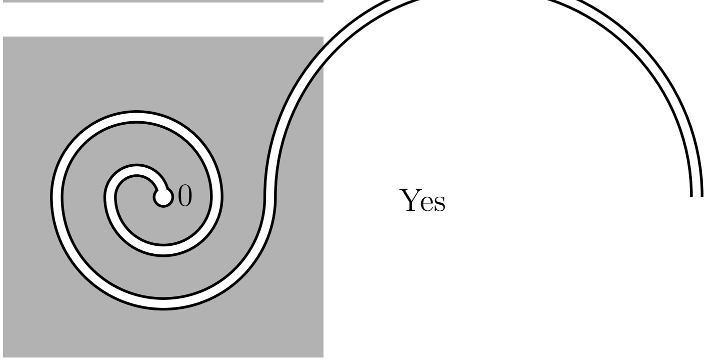
Changes: shifts, linear transformations, affine transformations well- defined for all points in $\mathbb{R}^2$ (or $\mathbb{R}^n$). We have also $\rho \geqslant 0$ in $\mathbb{R}^2$.

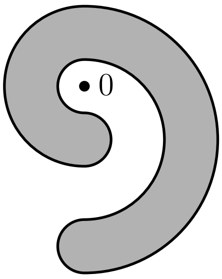
Polar coordinates are well-defined in domains *not* containing a closed path surrounding $0 \in \mathbb{R}^2$.

Examples.

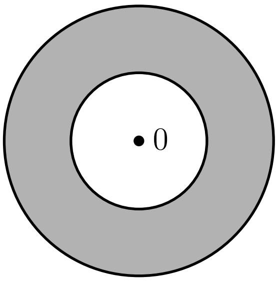








Yes



No

How does $d\varphi$ look in the coordinates (x, y)?

$$d\rho = d\sqrt{x^2+y^2} = \frac{2x\,dx + 2y\,dy}{2\sqrt{x^2+y^2}} = \frac{x\,dx + y\,dy}{\rho}.$$

$d\varphi =?$

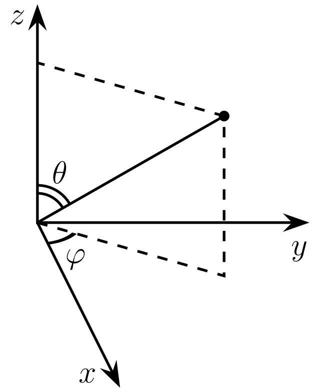
$$\begin{aligned} x &= \rho\cos\varphi & \Rightarrow \quad dx &= d\rho\cos\varphi - \rho\sin\varphi\,d\varphi, \\ y &= \rho\sin\varphi & \Rightarrow \quad dy &= d\rho\sin\varphi + \rho\cos\varphi\,d\varphi. \end{aligned}$$

$$\begin{aligned} x\,dy &= \rho\cos\varphi(d\rho\sin\varphi + \rho\cos\varphi\,d\varphi), \\ y\,dx &= \rho\sin\varphi(d\rho\cos\varphi - \rho\sin\varphi\,d\varphi), \\ x\,dy - y\,dx &= \rho^2\cos^2\varphi\,d\varphi + \rho^2\sin^2\varphi\,d\varphi \\ &= \rho^2\,d\varphi. \end{aligned}$$

$$\frac{x\,dy - y\,dx}{\rho^2} = d\varphi.$$

Spherical coordinates in $\mathbb{R}^3$:

$$\begin{aligned} z &= r\cos\theta, \\ x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\cos\varphi, \\ r^2 &= x^2+y^2+z^2. \end{aligned} \qquad (r,\theta,\varphi)$$

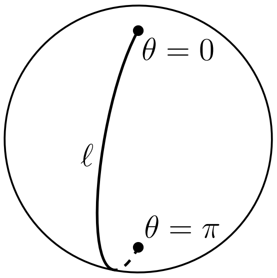


$r = 1$ is the unit sphere S^2. (θ,φ) are coordinates in S^2.

$$\begin{cases} 0 \leqslant \theta \leqslant \pi, \\ 0 \leqslant \varphi \leqslant 2\pi \end{cases}$$

Are (θ,φ) Cartesian coordinates in S^2?

In which open domains $U \subset S^2$ are they ok?



$$S^2 - \ell = U$$

Complex coordinates in $\mathbb{R}^2$: $x + iy = z$, $x - iy = \overline{z}$, $(z,\overline{z})$.

The differential of a function:

$$df(x,y) = f_x\,dx + f_y\,dy.$$

Make “change”

$$dx = \frac{1}{2}(dz + d\overline{z}), \quad dy = \frac{1}{2i}(dz - d\overline{z}).$$

$$\begin{vmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial \overline{z}}{\partial x} & \dfrac{\partial \overline{z}}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i \neq 0.$$

“Jacobian” $\neq 0$.

Let $A\,dx + B\,dy = X$ be a differential 1-form. We can write???

$$\frac{1}{2}A(dz + d\overline{z}) + \frac{1}{2i}B(dz - d\overline{z}) = \frac{1}{2}(A - iB)\,dz + \frac{1}{2}(A + iB)\,d\overline{z}.$$

Let $f = u(x,y) + iv(x,y)$ be a function.

$$\begin{aligned} df &= du + i\,dv = u_x\,dx + u_y\,dy + i(v_x\,dx + v_y\,dy) = (u_x + iv_x)\,dx + (u_y + iv_y)\,dy \\ &= A\,dx + B\,dy = \frac{1}{2}(A - iB)\,dz + \frac{1}{2}(A + iB)\,d\overline{z}. \end{aligned}$$

Definition.

$$\frac{A - iB}{2} = \frac{\partial f}{\partial z}, \quad \frac{A + iB}{2} = \frac{\partial f}{\partial \overline{z}},$$

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \overline{z}}\,d\overline{z},$$

$$\frac{\partial f}{\partial \overline{z}} = (u_x + iv_x) + i(u_y + iv_y) = (u_x - v_y) + i(v_x + u_y).$$

Condition (“analyticity”):

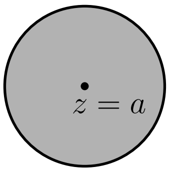
$$\frac{\partial f}{\partial \overline{z}} = 0 \quad \Rightarrow \quad u_x = v_y,\ v_x = -u_y.$$

Example. $f(z) = z^n$, $df = nz^{n-1}\,dz$.

Taylor series:

$$\sum_{n \geqslant 0} c_n(z - a)^n.$$

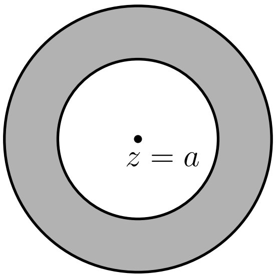
It is convergent in some disc (Abel)



Laurent series:

$$\sum_{n \in \mathbb{Z}} c_n(z - a)^n.$$

It is convergent in an annulus



The expression like $f(z, \overline{z})\,dz + g(z, \overline{z})\,d\overline{z}$ defines arbitrary complex-values differential 1-form in $\mathbb{R}^2$.

Lecture 6

Coordinates:

Polar (ρ, φ)

$\mathbb{R}^2$

$\rho \neq 0$?

which domains $U \subset \mathbb{R}^2 \setminus \{0\}$?

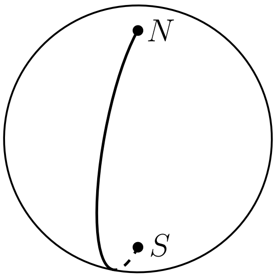
$d\varphi$ is a closed 1-form in $R^2 \setminus \{0\}$.

Spherical (r, θ, φ)

$\mathbb{R}^3 \supset S^2$ $(r = 1)$,

$r \neq 0$.

Sphere $S^2 \setminus \{N, S\}$

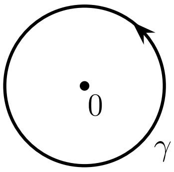


$\theta \neq 0, \pi$, φ?

$d\varphi$ is a *closed* 1-form in $S^2 \setminus \{N, S\}$.

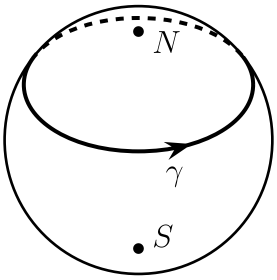
Claim. a) For any closed path γ around 0 in $\mathbb{R}^2 \setminus \{0\}$ we have

$$\oint_\gamma d\varphi = 2\pi.$$



b) For $S^2 \setminus \{N, S\}$ and γ around N, S we have

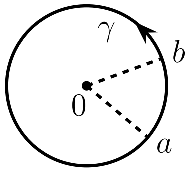
$$\oint_\gamma d\varphi = 2\pi$$

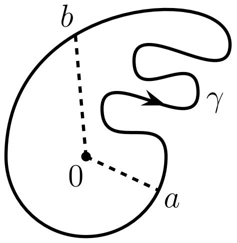


Proof of a).

$$\int_a^b d\varphi = \Delta\varphi \qquad \text{(along } \gamma)$$

(no matter which path if γ does *not* cross 0).





$\Delta\varphi$ does not depend on the path

Proof of b): Same.

Complex coordinates in $\mathbb{R}^2$:

$$\begin{array}{c|c|c} z = x + iy, & dz = dx + i\,dy, & dx = \dfrac{1}{2}(dz + d\overline{z}), \\ \overline{z} = x - iy, & d\overline{z} = dx - i\,dy, & dy = \dfrac{1}{2i}(dz - d\overline{z}). \end{array}$$

Let $f = u(x, y) + iv(x, y)$ and $f(x, y) \in \mathbb{C}$.

$$df = f_x\,dx + f_y\,dy = A\,dz + Bd\overline{z}.$$

Definition 1. We call $A = \partial f/\partial z$ abd $B = \partial f/\partial \overline{z}$ partial derivatives along the complex directions.

$$\left.\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \\ \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \end{aligned}\right| \quad \text{operators } \partial_z = \partial,\ \overline{\partial} = \partial_{\overline{z}}$$

For every 1-form $X = U\,dx + V\,dy$ we can write $X = \widetilde{U}\,dz + \widetilde{V}\,d\overline{z}$.

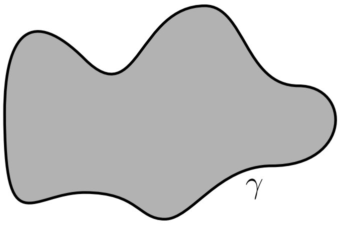
Definition 2. Complex analytic function $f = u + iv$ is such that:

$$\partial f/\partial \overline{z} = 0 \quad \Leftrightarrow \quad u_x + v_y = 0,\ v_x - u_y = 0.$$

Examples: $z = \rho e^{i\varphi}$.
$f = 1, z, z^2, \dots$ (any polynomial in z);
$f = P(z)/Q(z)$—a rational function (except “poles” $Q = 0$), $f = 1/z, 1/z^2, \dots$;
$f = \log z$—multivalued, $\log z = \log|z| + i \arg z = \log(\rho) + i\varphi$;
$f = e^z$, …

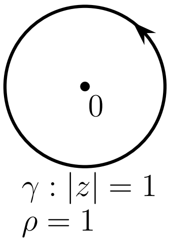
Important observation (topology):

$$\oint_\gamma f(z)\,dz = 0 \quad \text{if } \frac{\partial f}{\partial \overline{z}} \equiv 0 \text{ inside } \gamma.$$



the contour γ is “simply connected” (no holes inside).

Explanation. Let $f(z) = z^n$. We have



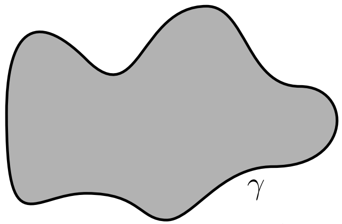
$$\oint_\gamma dz = 0, \quad \oint_\gamma z\,dz = 0, \quad \dots, \quad \oint_\gamma z^n\,dz = 0.$$

$$z = x + iy = \rho(\cos\varphi + i \sin\varphi) = \rho e^{i\varphi}.$$

Let γ be $\{\rho = 1\}$.

$$\oint_\gamma dz = ? \quad \oint_\gamma z\,dz = \oint_\gamma dw = 0, \dots ? \quad dw/dz = z.$$

The proof will appear later. It follows from the fact that for every smooth function $f(x, y)$ we have $\oint df \equiv 0$.

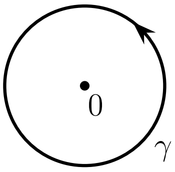


What is a *closed* 1-form?

Definition. a) A 1-form $X = u\,dx + v\,dy$ is closed iff for every point $P \in \mathbb{R}^2$ there exists a small neighbourhood $P \in U \subset \mathbb{R}^2$ such that $X = u\,dx + v\,dy = df$ in U. Same definition for all $n > 2$.

b) “An exact 1-form” X is such that $X = df$ in the whole domain V where it is defined.

Examples. a) The domain V is $\mathbb{R}^2 \setminus \{0\}$, $X = d\varphi$. It is closed but not exact because φ is *multivalued*. The obstruction is $\oint_\gamma d\varphi = 2\pi \neq 0$.



b) The domain V is $S^2 \setminus \{N, S\}$, $X = d\varphi$ is closed, but not exact.

c) The domain $V \subset \mathbb{R}^2 \setminus \{0\}$, where φ is one-valued smooth function, $d\varphi$ is exact in V.

Lecture 7

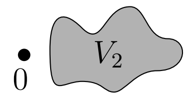
Our program and comparison with Bachman:

1. We assume that multivariable calculus is known (partial derivatives, differential of a function).
2. Special coordinates (polar, spherical, complex $z = x + iy$). Complex coordinates are missing in Bachman.
3. Theory of differential 1-forms. This theory is briefly done in Bachman. Something appears later only as partial cases of n-forms. This theory (including elementary topology) is fundamental especially complex 1-forms in $\mathbb{R}^2$ and Riemann surfaces (Cauchy, Riemann, Poincaré).
4. Contravariant property of the category of functions and forms under C^∞-maps $f : \mathbb{R}^m \to \mathbb{R}^n$ appears only in the Chapter "Manifold" in Bachman, it is called a "pull-back map" f^*

From 1-forms to 2-forms:
Bachman discusses "product" of 1-forms. We will discuss it later, and start from Calculus:
Problem. Let a differential 1-form be given in $\mathbb{R}^2$ (or $\mathbb{R}^n$): $X = u\,dx + v\,dy$ (in some domain V).
a) Is it locally the differential of some function f:

$$(x, y) \in U_\varepsilon \subset V, \quad X = df, \quad u = \frac{\partial f}{\partial x}, \; v = \frac{\partial f}{\partial y}?$$

b) Is it globally differential $X = df$ of any one-valued smooth function in the whole domain V?

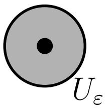
Examples. 1) $X = \lambda\,dx + \mu\,dy$, $f = \lambda x + \mu y$, $\lambda, \mu = \text{const}$.
2) $X = d\varphi = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$.
a) $V_1 = \mathbb{R}^2 \setminus \{0\}$, $d\varphi$ is *not* df for a 1-valued f in V;
b) $V_2 =$ 

, $X = df$, $f = \varphi$.

Closed 1-forms = locally differentials. Exact 1-forms = globally differentials. Closed forms modulo exact ones = Cohomology group $H^1(V, \mathbb{R})$ (The first "de Rham" cohomology).

What do we know from Calculus? $X = u\,dx + v\,dy$ ($(u,v) =$ "a covector field" in $V \subset \mathbb{R}^2$),

$$u = \frac{\partial f}{\partial x},\ v = \frac{\partial f}{\partial y} \text{ (locally) in } U_\varepsilon \subset V.$$



Claim. Nearby every point from V ($n = 2$) the condition

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

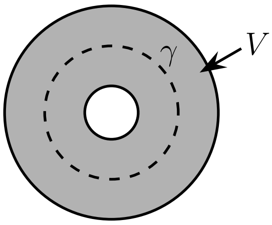
is necessary and sufficient for the "local" existence of f, $\nabla f = (u,v)$, or $df = X$.

$n \geqslant 2$: $X = \sum_i u_i\,dx^i$:

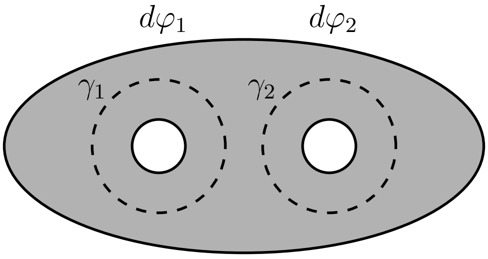
$$\frac{\partial u_i}{\partial x^j} = \frac{\partial u_j}{\partial x^i}.$$

"Global obstruction" for the existence of a single-valued $f =$ Topology.

Example.



$$\oint_\gamma d\varphi = 2\pi.$$



$$\oint_{\gamma_1} d\varphi_1 = 2\pi, \qquad \oint_{\gamma_1} d\varphi_2 = 0,$$
$$\oint_{\gamma_2} d\varphi_1 = 0, \qquad \oint_{\gamma_2} d\varphi_2 = 2\pi.$$

"The cohomology with real coefficients" $H^1(V,\mathbb{R})$ is generated by $d\varphi_1$, $d\varphi_2$ as a linear space over the field R. Similar answer we have for $H^1(V,C)$ using the complex-valued 1-forms.

Consider the formal expression

$$dX = \Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big)\, dx \wedge dy \quad (n=2),$$

$$dX = \sum_{i<j}\Big(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\Big)\, dx^i \wedge dx^j \quad (n \geqslant 2).$$

Claim. $dX = 0$ if and only if X is locally exact (i.e. "closed" 1-form).

Formal expressions like $a\, dx \wedge dy$,

$$\sum_{i<j} a_{ij}(x)\, dx^i \wedge dx^j \quad (n \geqslant 2)$$

are called "differential 2-forms".

We extend a_{ij} to all pairs (i,j) by the requirement:

$$a_{ij} = -a_{ji}, \quad a_{jj} = 0.$$

The operator $d :$ 1-forms $\to$ 2-forms will be defined.

The product of 1-forms $X \wedge Y$ will also be defined by the rule

$$\left.\begin{aligned} X &= u\, dx + v\, dy, \\ Y &= w\, dx + t\, dy, \end{aligned}\ \right| \quad X \wedge Y = (ut - vw)\, dx \wedge dy.$$

Lecture 8

From 1-forms to 2-forms: let a differential 1-form $X = u\,dx + v\,dy$ be given. It is exact if $X = df$, i.e. $f_x = u$, $f_y = v$. It is closed if *locally* there exists a function f such that $f_x = u$, $f_y = v$.

Example. $X = d\varphi$ in $\mathbb{R}^2 \setminus \{0\}$.

$$d\varphi = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Lemma 1. A differential 1-form

$$X = \sum_{i=1}^{n} u_i\,dx^i$$

in a domain $U \subset \mathbb{R}^n$ is closed iff the following equations are satisfied:

$$\frac{\partial u_i}{\partial x^j} = \frac{\partial u_j}{\partial x^i}.$$

(Calculus.)

Introduce a 2-form

$$dX = \Omega = \sum_{i\leqslant j} \Big(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\Big)\,dx^i \wedge dx^j,$$

where $dx^i \wedge dx^j = -dx^j \wedge dx^i$ by definition.

More generally,

$$\Omega = \sum_{i<j} a_{ij}(x)\,dx^i \wedge dx^j.$$

We have an operator

$$d : (\text{1-forms}) \to (\text{2-forms})$$

such that

$$d\Big(\sum_i u_i\,dx^i\Big) = \sum_{i<j} \Big(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\Big)\,dx^i \wedge dx^j.$$

We also have

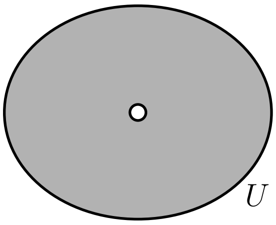
$$d : (0\text{-forms}) \to (1\text{-forms}),$$
$$df(x) = \sum_i \frac{\partial f}{\partial x^i}\, dx^i.$$

Properties of the operator d:

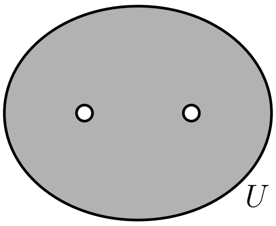
1. $d \cdot d \equiv 0$;
2. df = exact 1-forms, dX = exact 2-forms;
3. $\ker d$: closed 1-forms $dX = 0$. Closed 0-forms = constant functions $df = 0$;
4. Cohomology $H^1(U, \mathbb{R})$ = closed 1-forms modulo exact ones.

Examples:

a) Simply connected domain $U \subset \mathbb{R}^2$: $H^1 = 0$;

b) 

1 hole, $H^1 = \mathbb{R}$;



2 holes, $H^1 = \mathbb{R}^2$.

Define now multiplication of 1-forms $X \wedge Y$. It is bilinear and has a form for $n = 2$

$$\left.\begin{array}{r} X = u\,dx + v\,dy, \\ Y = w\,dx + t\,dy, \end{array}\right| \quad X \wedge Y = ?$$

$X \wedge Y = ut\,dx \wedge dy + vw\,dy \wedge dx = (ut - vw)\,dx \wedge dy.$

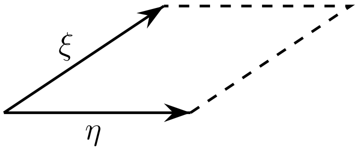
For general $n \geqslant 2$ we define:

$$X = \sum u_i\,dx^i, \quad Y = \sum v_j\,dx^j,$$
$$X \wedge Y = \sum_{i<j} (u_i v_j - v_i u_j)\,dx^i \wedge dx^j.$$

So we have can formulate axioms for multiplication: 1) $dx^i \wedge dx^j = -dx^j \wedge dx^i$; 2) numbers (functions, 0-forms) commute with everything, $u_i\,dx^j = dx^j \cdot u_i$.

Extension to products of more than 2 1-forms and to arbitrary m-forms ($m \geqslant 2$) follows from the *Associativity Axiom* $(dx \wedge dy) \wedge dz = dx \wedge (dy \wedge dz)$.

Motivation: Let 2 vectors be given in $\mathbb{R}^2$

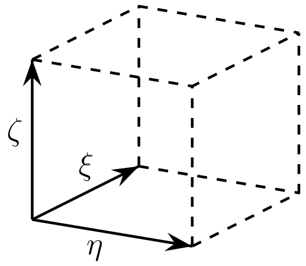


$$\text{Area} = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \xi \wedge \eta = -\eta \wedge \xi.$$

$\nearrow$

determinant

$n = 3$. Volume $\eta, \zeta, \xi \in \mathbb{R}^3$



$$\text{Volume} = \begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix}.$$

$$\text{Volume} = \eta \wedge \zeta \wedge \xi = -\zeta \wedge \eta \wedge \xi = \zeta \wedge \xi \wedge \eta.$$

Same is true in $\mathbb{R}^n$ for n vectors. Volume is a multilinear skew symmetric function of n vectors. We write it as $\eta^{(1)}, \dots, \eta^{(n)}$

$$\text{Volume} = \eta^{(1)} \wedge \eta^{(2)} \wedge \ldots \wedge \eta^{(n)}$$
$$V(\eta^{(1)}, \eta^{(2)}, \dots, \eta^{(n)})$$

linear in every variable, skew symmetric (changes sign by $(-1)^\sigma$ for any permutation σ). We write it as

$$(\eta_i^{(1)}\,dx^i) \wedge (\eta_j^{(2)}\,dx^j) \wedge \ldots \wedge (\eta_k^{(n)}\,dx^k).$$

Lecture 9

$$\underset{(f)}{(\text{0-forms})} \underset{d}{\longrightarrow} \underset{(X)}{(\text{1-forms})} \underset{d}{\longrightarrow} \underset{(\Omega)}{(\text{2-forms})}$$

$$f(x),\ x \in U \subset \mathbb{R}^n \quad \Big(X = \sum_i u_i\, dx^i\Big) \quad \Big(\Omega = \sum_{i<j} a_{ij}\, dx^i \wedge dx^j\Big)$$

$n = 2$:

$$X = u\, dx + v\, dy, \qquad \Omega = a\, dx \wedge dy.$$

Axioms.

1. $dx^i \wedge dx^j = -dx^j \wedge dx^i$;
2. functions commute with everything, $u(x)\, dx = dx \cdot u(x)$;
3. associativity (extension to k-forms for $k > 2$):
$$dx \wedge (dy \wedge dz) = (dx \wedge dy) \wedge dz;$$
4. multiplications of k-forms is bilinear ("distributive").

We have "algebra of $R(C)$-valued differential forms" $\Lambda(U, R(C))$ in the domain $U \subset \mathbb{R}^n$ with Cartesian coordinates $(x^1, \ldots, x^n)$. A 2-form is

$$\sum a_{ij}(x)\, dx^i \wedge dx^j, \quad i < j,$$

a 3-form is

$$\sum_{i<j<k} a_{ijk}(x)\, dx^i \wedge dx^j \wedge dx^k,$$

$$\ldots$$

k-form is

$$\sum_{i_1<\ldots<i_k} a_{i_1,\ldots,i_k}(x)\, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

n-form is

$$a(x)\, dx^1 \wedge \ldots \wedge dx^n.$$

Here

$$a(x) = a_{1,2,\ldots,n}(x).$$

Examples.
$k=1$: $X=\sum u_i(x)\,dx^i$, $n=2$: $X=u\,dx+v\,dy$;
$k=2$: $\Omega=\sum_{i<j} a_{ij}\,dx^i\wedge dx^j$, $n=2$: $\Omega=a\,dx\wedge dy$;
$k=2$, $n=3$:

$$\text{2-form}=a_{12}\,dx\wedge dy+a_{13}\,dx\wedge dz+a_{23}\,dy\wedge dz.$$

The "vector" $(a_{23},-a_{13},a_{12})=\overrightarrow{\eta}$ is "an axial vector" in the physical literature.

Operations. $d: f\to df=f_{x^i}\,dx^i$.
$n=2$. $d: X\to dX=(u_y-v_x)\,dx\wedge dy$, $a_{12}=u_y-v_x$.
$n\geqslant 2$.

$$dX=\sum_{i<j}\Big(\frac{\partial u_i}{\partial x^j}-\frac{\partial u_j}{\partial x^i}\Big)\,dx^i\wedge dx^j.$$

Example. 2-form $a\,dx\wedge dy+b\,dy\wedge dx=(a-b)\,dx\wedge dy$.

Change of coordinates: what do we already know?

Functions $x'\to x$ or $x(x')$, $x^1(x'^1,\dots,x'^n)$, $\dots$, $x^n(x'^1,\dots,x'^n)$. $f(x(x'))$ is the pull-back of the function f for the map $x'\to x$ (map $\varphi: U'\to U$), $f(x(x'))=\varphi^*f(x')$.

1-forms. $\varphi: U'\to U$, $x'\to x$, $x(x')$.

$$\varphi^*X=\sum_i u_i\,dx^i\xrightarrow[\text{pull-back}]{\varphi^*}\sum_{i,j}u_i(x(x'))\frac{\partial x^i}{\partial x'^j}\,dx'^j.$$

Or $u(x)\xrightarrow{\varphi^*}u(x(x'))$ as a function

$$dx^j\xrightarrow{\varphi^*}\sum_i\frac{\partial x^j}{\partial x'^i}\,dx'^i.$$

Partial cases of "pull-back":

1. Change of coordinates $x\leftrightarrow x'$, one-to-one (may be local)

$$\det\Big(\frac{\partial x}{\partial x'}\Big)=\det\widehat{J}\neq 0,$$

$$\widehat{J}=\text{Jacoby matrix}=\Big(\frac{\partial x^i}{\partial x'^j}\Big).$$

2. Restriction to some surface (curve) (number of x' is less than n).

General definition of the pull-back is given

$$f(x) \to f(x(x')) = \varphi^* f(x'),$$

$$\varphi^* : df \to \frac{\partial f}{\partial x^j}(x(x')) \frac{\partial x^j}{\partial x'^l}\, dx^l = d(\varphi^* f) \quad \text{(sum in } j, l).$$

Conclusions. The differential d of a function (0-form) commutes with the pull-back operation: $\varphi^*(df) = d(\varphi^* f)$. By definition, the exterior multiplication of forms commute also with pull-back map. So we have

Claims. 1. $\varphi^* d = d\varphi^*$ for all k-forms (proof for $k = 1$ will be given later).
2. $\varphi^*(X \wedge Y) = \varphi^*(X) \wedge \varphi^*(Y)$. Product of forms is "natural" (commutes with the pull-back for maps $\varphi : U' \to U$ including restrictions and changes of coordinates).

Lecture 10

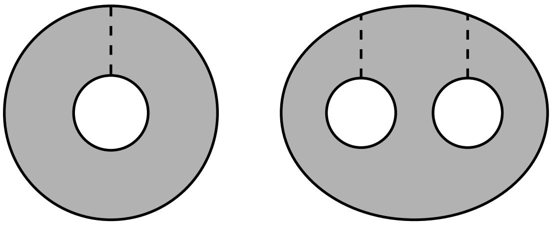
Remarks.

1. Differential of function f is 1-form $df = \sum f_{x^i}\, dx^i$ as we write it in Analysis. Vectors in Analysis we write as fist order differential operators of the form $\sum a^j(x)\partial/\partial x^j = w$.

   The scalar product of a 1-form and a vector attached to some point is by definition"the directional derivative" of a function

   $$\nabla_w f = \sum_i \frac{\partial f}{\partial x^i} a^i(x) = \langle df, w\rangle.$$

2. The homology $H^1(U)$ of a planar domains U in fact appear a lot in Complex Analysis as number of "cuts" necessary to make domain simply connected.

   Number of cuts = rank of $H_1(U)$.

   Remaining domain should be simply connected (every closed contour = boundary of a "ball").

   The homology is generated by "cycles". A cycle c is ~ 0 iff for every closed 1-form X, $dX = 0$, we have

   $$\oint_c X = 0.$$

3. Cartesian coordinates in $U \subset \mathbb{R}^n$ $(x^1, \ldots, x^n)$ "inherited" from $\mathbb{R}^n$ are $(x^1, \ldots, x^n)$ in U. Other Cartesian coordinates = set $(y^1, \ldots, y^n)$ of C^∞-functions $U \underset{y_i}{\to} \mathbb{R}$ such that $\overrightarrow{y}_0 \neq \overrightarrow{y}_1 \Leftrightarrow$ the points are distinct.

0-forms = functions,
1-forms: $\sum u_i\, dx^i$ in $U \subset \mathbb{R}^n$ or $u\, dx + v\, dy$ in $U \subset \mathbb{R}^2$,

2-forms: $\sum_{i<j} a_{ij}\, dx^i \wedge dx^j$, $U \subset \mathbb{R}^n$, $a\, dx \wedge dy$ in $\mathbb{R}^2$,
$\ldots,$
k-forms:

$$\sum_{i_1<i_2<\ldots<i_k} a_{i_1,\ldots,i_k}(x)\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad dx^i \wedge dx^j = -dx^j \wedge dx^i.$$

n-form in $\mathbb{R}^n$

$$a\, dx^1 \wedge \ldots \wedge dx^n, \quad a = a_{1,2,\ldots,n}(x),$$

an object of integration in $\mathbb{R}^n$ (ordinary)

$$\underset{D\subset\mathbb{R}^n}{\int \cdots \int} a\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n.$$

Algebraic operations: forms = linear space, multiplication $X \wedge Y$

a) functions commute with everything $f(x) \wedge X = X \wedge f(x)$;

b) $dx^i \wedge dx^j = -dx^j \wedge dx^i$, $dx \wedge dy = -dy \wedge dx$;

c) associativity $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$.

Symbols $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ for $i_1 < \ldots < i_k$ generate the space of k-forms linearly (coefficients are functions $a_{i_1,\ldots,i_k}(x)$).

The differential operation: df.
d : functions → 1-forms, $f \mapsto df$;
d : 1-forms → 2-forms, $u\, dx + v\, dy \mapsto (u_y - v_x)\, dx \wedge dy$.

Definition of the operator d for all k-forms (it is a linear operator):

1. $d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0$;
2. $d(f) = df$ for functions;
3. $d(f\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$.

$$d : k\text{-forms} \longrightarrow (k+1) - \text{forms}.$$

For the case $k = 1$ we already know

$$0\text{-forms} \xrightarrow{d} 1 - \text{forms} \xrightarrow{d} 2 - \text{forms}.$$

$$d \circ d = 0$$

$H^1(U) =$ closed $\underset{dX=0}{\text{1-forms}}$/exact $\underset{X=df}{\text{1-forms}}$.

Integration of k-forms along a parametrized k-surface

$$\varphi : \underset{(t^1,\dots,t^k)}{D^k} \longrightarrow \underset{(x^1,\dots,x^n)}{U} \subset \mathbb{R}^n, \quad x^i(t^1,\dots,t^k), \quad \overrightarrow{x}(\overrightarrow{t}).$$

Step 1. Take a k-form $X = \sum a_{i_1,\dots,i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. Pull it back to D^k (restriction to D^k), i.e.

$$\varphi^* X = \sum a_{i_1,\dots,i_k}(x(t)) \frac{\partial x^{i_1,\dots,i_k}}{\partial t^{1,\dots,k}}\, dt^1 \wedge \ldots \wedge dt^k = b(t)\, dt^1 \wedge \ldots \wedge dt^k.$$

Step 2. Integrate the restricted k-form along the body $D^k \subset \mathbb{R}^k$ $(t^1,\dots,t^k)$ (ordinary integration).

Is this program well-defined?
Is it invariant under changes of coordinates?
Is it well-defined in the category of manifolds $U \subset \mathbb{R}^n$ and C^∞-maps?

$$\varphi : \underset{\substack{\mathbb{R}^n \\ x^1,\dots,x^n}}{U} \longrightarrow \underset{\substack{\mathbb{R}^m \\ y^1,\dots,y^m}}{V}, \qquad \varphi : x \mapsto y, \qquad y^i = y^i(x^1,\dots,x^n).$$

We need to prove that

a) operations (addition and multiplication) of k-forms are "natural", i.e. they commute with change of coordinates;

b) the differential d—does it commute with φ^*? Or with change of coordinates in particular?

We define:

1. $k = 0$. $\varphi^* f(x) = f(y(x))$ for functions;
2. $k = 1$. $\varphi^* df = d(\varphi^* f)$?

$$d(\varphi^* f) = df(y(x)) = \frac{\partial f}{\partial y^i}\frac{\partial y^i}{\partial x^j}\, dx^j = d(\varphi^* f)$$

$$\varphi^*(df) = \varphi^*\Big(\frac{\partial f}{\partial y^i}\, dy^i\Big) = \frac{\partial f(y(x))}{\partial y^j}\frac{\partial y^j}{\partial x^i}\, dx^i = d(\varphi^* f) \text{ — ok.}$$

Conclusion: $\varphi^* d = d\varphi^*$.

Lecture 11.

Diffrential forms: change of coordinates and pull-back map:

$$\varphi : \underset{x'}{U'} \underset{\rightarrow}{\rightarrow} \underset{x}{U} \xrightarrow{f} \mathbb{R}, \quad f \ \text{ is a function,}$$

Theorem.

1. $\varphi^* f(x') = f(x(x'))$ (0-forms).
2. $\varphi^*(X \wedge Y) = \varphi^*(X) \wedge \varphi^*(Y)$ (commute with external product).
3. $\varphi^*(dx) = d\varphi^*(x)$. More general: φ^* commutes with operator d, mapping k-forms to $(k+1)$ forms:

$$d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0,$$
$$d(f dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Example. $n = 2$, $\Omega = dx \wedge dy$.

$$\varphi : (x') \rightarrow (x).$$
$$\varphi^*\Omega = \left(\frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial y'}dy'\right) \wedge \left(\frac{\partial y}{\partial x'}dx' + \frac{\partial y}{\partial y'}dy'\right) =$$
$$= \left(\frac{\partial x}{\partial x'}\frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'}\frac{\partial y}{\partial x'}\right) dx' \wedge dy' = \det \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{pmatrix} dx' \wedge dy'.$$

Formal properties of d in the given system of coordinates:

1. $df = \sum\limits_i \frac{\partial f}{\partial x^i} dx^i$.
2. $d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0$
3. $d(f dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$.

Corollaries:

1. $d(df) = 0$.

Proof:

$$d(df) = d\left(\sum_i \underset{\substack{\uparrow\\ \text{Key point}}}{\frac{\partial f}{\partial x^i}} dx^i\right) = \sum_{ij} \underset{\substack{\uparrow\\ \text{Symmetric}\\ \text{in (i,j)}}}{\frac{\partial^2 f}{\partial x^j \partial x^j}} \cdot \underset{\substack{\uparrow\\ \text{Skew symmetric}\\ \text{in (i,j)}}}{dx^j \wedge dx^i} = 0 \quad (!)$$

2. For 0-forms we have:

$$\varphi : U' \xrightarrow{\varphi} U \xrightarrow{f} \mathbb{R},\ x' \to x,$$
$$\varphi^* f(x') = f(x(x')),$$
$$d(\varphi^* f(x')) = \varphi^*(df) = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial x'^j} dx'^j.$$

3. Operator d has the foillowing property:

$$\underset{\substack{\uparrow\\ k\text{-form}}}{d(\Omega_1} \wedge \Omega_2) = d\Omega_1 \wedge \Omega_2 + (-1)^k \Omega_1 \wedge d\Omega_2.$$

Proof: a) Let $k = 0$, $\Omega_2 = dx^i$. $d(fdx^i) = df \wedge dx^i$. O.K. $d(dx^i) = 0$.
b) Let k and l be arbitrary: $(fdx^{i_1} \wedge \ldots \wedge dx^{i_k}) \wedge (gdx^{j_1} \wedge \ldots \wedge dx^{j_l})$.
Prove that $d \circ d = 0$?

$$d(adx^{i_1} \wedge \ldots \wedge dx^{i_k}) = da \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$
$$d(d(adx^{i_1} \wedge \ldots \wedge dx^{i_k})) = d(da \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0,$$

because

$$d(da) = 0, \quad d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0.$$

Prove that $d(f \cdot X) = df \wedge X + fdX$ (X is 1-form): $X = udx$

$$d(f \cdot udx) = d(fu) \wedge dx = \frac{\partial f}{\partial y} \wedge udx + f \cdot \frac{\partial u}{\partial y} dy \wedge dx = df \wedge X + f \cdot dX.$$

Prove that $d(X \wedge f) = dX \wedge f - X \wedge df$.

$$X = udx, \quad X \wedge f = udx \wedge f = ufdx,$$

$$
\begin{aligned}
&d(u\cdot dx\cdot f) = d(fu)\wedge dx = \frac{\partial u}{\partial y}\wedge dx\cdot f + u\cdot\frac{\partial f}{\partial y}dy\wedge dx = df\wedge X + f\cdot dX,\\
&dX\wedge f = du\wedge dx\cdot f = \frac{\partial u}{\partial y}\cdot f dy\wedge dx\\
&X\wedge df = udx\wedge\frac{\partial f}{\partial y}dy = u\frac{\partial f}{\partial y}dx\wedge dy = -u\frac{\partial f}{\partial y}dy\wedge dx.
\end{aligned}
$$

O.K.

So

$$d(\Omega_k\wedge\Omega_l) = d\Omega_k\wedge\Omega_l + (-1)^k\Omega_k\wedge d\Omega_l.$$

Examples:

$$
\begin{aligned}
n=2:\quad & d(udx) = u_y dy\wedge dx = -u_y dx\wedge dy\\
& d(vdy) = v_x dx\wedge dy.
\end{aligned}
$$

$n=3:\quad udx,\quad vdy,\quad wdz,\quad (u,v,w) -$ "vector" X

$$
\begin{aligned}
\underline{\text{curl}}\ X:\quad dX &= \frac{\partial u}{\partial y}dy\wedge dx + \frac{\partial u}{\partial z}dz\wedge dx + \frac{\partial v}{\partial x}dx\wedge dy + \frac{\partial v}{\partial z}dz\wedge dy + \frac{\partial w}{\partial x}dx\wedge dz =\\
&= \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\wedge dy + \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)dy\wedge dz + \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)dz\wedge dx
\end{aligned}
$$

Curl X – "vector" with components:

$$\underset{1\leftrightarrow(23)}{(w_y - v_z)}\ dy\wedge dz + \underset{2\leftrightarrow(31)}{(u_z - w_x)}\ dz\wedge dx + \underset{3\leftrightarrow(12)}{(v_x - u_y)}\ dx\wedge dy.$$

This formula is true in any coordinate system (as a 2-form).

Association of 2-form with vector is O.K. for $n = 3$ in orthogonal positive coordinates and is invariant under rotations only (det $= +1$).

Lecture 12.

Claim:

$$d(\overset{k}{\Omega_1} \wedge (\overset{l}{\Omega_2}) = d\Omega_1 \wedge \Omega_2 + (-1)^k \Omega_1 \wedge d\Omega_2.$$

Let $k = 1$, $l = 0$.
Proof: Let $\Omega_1 = X = udx$, $\Omega_2 = f$.

1) $d(fX) = df \wedge X + f \cdot dX = d(fudx) = df \wedge udx + f(du \wedge dx)$,

2) $d(Xf) = dX \cdot f \overset{?}{-} X \wedge df = d(\underset{\curvearrowleft}{udx \cdot f}) = \overbracket{du \wedge \underbracket{dx \cdot f}} + u\, df \wedge dx = dX \cdot f - X \wedge df.$

O.K.
Claim: $d \circ d \equiv 0$.
Proof:

$$1) \qquad d(df) = 0 = \sum_{ij} \frac{\partial^2 f}{\partial x^j \partial x^j} dx^j \wedge dx^i.$$

$$1) \qquad d(df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = d(df) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} = 0.$$

O.K.
Examples: n-forms in $\mathbb{R}^n$:

$$\Omega = a\, dx^1 \wedge \ldots \wedge dx^n = (-1)^\sigma a\, dx^{i_1} \wedge \ldots \wedge dx^{i_n},$$

where

$$\sigma = \begin{pmatrix} 1 & \ldots & n \\ i_1 & \ldots & i_n \end{pmatrix}.$$

Integration:

$$\underset{D \subset \mathbb{R}^n}{\int \cdots \int} a dx^1 \wedge \ldots \wedge dx^n$$

denotes **ordinary** integral.

Change of variables for n-form in $\mathbb{R}^n$: $x(x')$

$$\Omega = a\, dx^1 \wedge \ldots \wedge dx^n \Rightarrow a\, J\, dx'^1 \wedge \ldots \wedge dx'^{n,}$$

where denotes the Jacobian:

$$\mathcal{J} = \det \left(\frac{\partial x}{\partial x'} \right).$$

Proof for $\mathbf{n = 2}$: $dx^1 \wedge \ldots \wedge dx^n = dx \wedge dy$.

$$dx = \alpha dx' + \beta dy', \quad dy = \gamma dx' + \delta dy', \quad \text{where} \quad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = J = \det \left(\frac{\partial x}{\partial x'} \right),$$

$$dx \wedge dy = \alpha\delta \, dx' \wedge dy' + \beta\gamma \, dy' \wedge dx' = (\alpha\delta - \beta\gamma) \, dx' \wedge dy' = \mathcal{J} \, dx' \wedge dy'.$$

Proof for any n: Let

$$\eta^j = \frac{\partial x^j}{\partial x'^1} dx'^1 + \ldots + \frac{\partial x^j}{\partial x'^n} dx'^n.$$

Consider the product

$$\eta^1 \wedge \ldots \wedge \eta^n = \text{Sum of terms: } \frac{\partial x^1}{\partial x'^{j_1}} \cdots \frac{\partial x^n}{\partial x'^{j_n}} \, dx'^{j_1} \wedge \ldots \wedge dx'^{j_n} =$$
$$= \det \left(\frac{\partial x^i}{\partial x'^j} \right) dx'^1 \wedge \ldots \wedge dx'^n.$$

O.K.

What is the image (pull-back) of the form $\mathbf{dx^{i_1}} \wedge \ldots \wedge \mathbf{dx^{i_k}}$? As above,

$$x = x(x'), \quad dx^i \to \eta^i,$$

$$\underset{i_1 < \ldots < i_k}{\eta^{i_1} \wedge \ldots \wedge \eta^{i_k}} = \sum_{j_1 < \ldots < j_k} \mathcal{J}^{i_1 \ldots i_k}_{j_1 \ldots j_k} \, dx'^{j_1} \wedge \ldots \wedge dx'^{j_k}$$

For any $k \leq n$ we have

$$\Omega = \sum_{i_1 < \ldots < i_k} a_{i_1, \ldots, i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Let $\varphi : x' \to x$, i.e. $x = x(x')$.

$$\varphi^* \Omega = \sum_{i_1 < \ldots < i_k} \frac{\partial x^I}{\partial x'^J} \, dx'^{j_1} \wedge \ldots \wedge dx'^{j_k},$$

where

$$I = (i_1 < \ldots < i_k), \quad J = (j_1 < \ldots < k_k),$$

$$\frac{\partial x^I}{\partial x'^J} = \det \left(\frac{\partial x^{i_1} \ldots \partial x^{i_k}}{\partial x'^{j_1} \ldots \partial x'^{j_k}} \right) \quad \text{are } k \times k \text{ minors in the Jacoby matrix.}$$

Examples.

$$\begin{aligned}
&k=0: && \varphi^* a(x') = a(x(x')),\\
&k=1: && \varphi^* X = \varphi^*(u_i dx^i) = u_i(x(x'))\frac{\partial x^i}{\partial x'^j}\, dx'^j, \quad \text{(sum in } i,j),\\
&k=2,\ i<j: && \varphi^*(u_{ij} dx^i \wedge dx^j) = u_{ij}(x(x'))\left(\frac{\partial x^i}{\partial x'^l}\, dx'^l\right) \wedge \left(\frac{\partial x^j}{\partial x'^s}\, dx'^s\right) =\\
& && = u_{ij}(x(x'))\mathcal{J}\begin{pmatrix} i & j\\ l & s\end{pmatrix} dx'^l \wedge dx'^s,\\
& && \text{where } \mathcal{J}\begin{pmatrix} i & j\\ l & s\end{pmatrix} \text{ is 2-minor} \begin{pmatrix} i & j\\ l & s\end{pmatrix} \text{ in Jacoby matrix,}\\
& && \dots\\
&k=n && \varphi^*(a dx^1 \wedge \ldots \wedge dx^n) = a(x(x')) \cdot \mathcal{J} \cdot dx'^1 \wedge \ldots \wedge dx'^n.
\end{aligned}$$

a) Restriction to l-surface in $\mathbb{R}^n$: $(x^1, \ldots, x^l) \to \mathbb{R}^n$.
b) Change of variables $\mathbf{l = n}$: $U' \to U$, **one-to-one** in U.
1-space: 0, 1 forms:

$$f,\ udx.$$

2-space: 0, 1, 2 – forms:

$$f(x), \quad udx + vdy, \quad a\, dx \wedge dy.$$

2-forms are like scalar functions, but under change of variables we have:

$$a \to a \cdot \det\left(\frac{\partial x}{\partial x'}\right).$$

3-space: 0, 1, 2, 3 – forms:

$$f(x),\ \ udx+vdy+wdz,\ \ a_{12}\, dx\wedge dy + a_{13}\, dx\wedge dz + a_{23}\, dy \wedge dz,\ \ b\, dx\wedge dy\wedge dz.$$

3-forms are like scalar functions:

$$\varphi:\ b(x) \to b(x(x'))\mathcal{J}, \quad x' \to x.$$

2-forms are like 1-forms ("vectors"):

$$\underset{\sigma=(123),\ +}{a_{12} \to a_3}, \quad \underset{\sigma=(132),\ -}{a_{13} \to -a_2}, \quad \underset{\sigma=(231),\ +}{a_{23} \to a_1}.$$

What is “Curl”? $d:$ 1-forms $\to$ 2-forms.

$$d : u_i dx^i \to \sum_{i<j} \left(\frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j} \right) dx^i \wedge dx^j,$$

$$a_{12} = (v_x - u_y), \quad a_{13} = (w_x - u_z), \quad a_{23} = (w_y - v_z),$$

Here the “star operator” $*$ from k-forms into $n-k$-forms: $* : \bigwedge^k \to \bigwedge^{n-k}$ is used.

It depends on Riemannian metric. This formula is written in the orthonormal coordinates in euclidean space

Lecture 13.

Forms in 3-space: $(x, y, z) = (x^1, x^2, x^3)$.

1. Let $X = udx + vdy + wdz$.

We have

$$d(udx+vdy+wdz) = (v_x - u_y)\, dx \wedge dy + (w_x - u_z)\, dx \wedge dz + (w_y - v_z)\, dy \wedge dz.$$

Denote:

$$a_{12} = (v_x - u_y), \quad a_{13} = (w_x - u_z), \quad a_{23} = (w_y - v_z),$$

To define "Curl" denote:

$$\underset{\sigma=(123),\ +}{a_{12} \to a^3}, \quad \underset{\sigma=(132),\ -}{a_{13} \to -a^2}, \quad \underset{\sigma=(231),\ +}{a_{23} \to a^1}.$$

The "vector" (a^1, a^2, a^3) is called the "Curl" of X: $dX = \operatorname{curl} X$.

2. Let $\vec{u} = (a^1, a^2, a^3)$.

$$d(a_{12}\, dx \wedge dy + a_{13}\, dx \wedge dz + a_{23}\, dy \wedge dz) = \left(\frac{\partial a^1}{\partial x^1} + \frac{\partial a^2}{\partial x^2} + \frac{\partial a^3}{\partial x^3} \right) dx \wedge dy \wedge dz.$$

We write:

$$\div \vec{u} = \sum_{i=1}^{3} \frac{\partial a^i}{\partial x^i}.$$

$d \circ d = 0$, therefore

$$\operatorname{curl}(df) = 0, \quad \operatorname{div} \operatorname{curl}(X) = 0.$$

Examples: 2-forms in $\mathbb{R}^3$ are:

a) "Vorticity" of vector field: $X = \sum u_i dx^i$ is by definition a 2-form $\Omega = dX$ written as a vector field in euclidean space and orthonormal coordinates.

b) Magnetic field H is a closed 2-form. $A = \sum a_i dx^i$ is called "Vector potential".

$$dA = H \quad \leftrightarrow \quad dH = 0 \ \ (H \text{ is closed}).$$

Constant 2-forms in $\mathbb{R}^n$ = skew symmetric matrices: $(a_{ij} = -a_{ji}) = A$. e^{At} is a rotation.

Let $n = 3$,

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

Claim: $e^{At} \in SO_n$ for all $t \to A^t = -A$.

Proof:

$$e^{At} = 1 + At + O(t^2),$$
$$< e^{At}\eta, e^{At}\xi > = < \eta, \xi > + t\,(< A\eta, \xi > + < \eta, A\xi >) + O(t^2) = < \eta, \xi >$$

For $t = 0$ we have: $< A\eta, \xi > = - < \eta, A\xi >$. The basis is orthonormal, therefore $A^t = -A$.

"Lie Algebra" of SO_n consists of constant 2-forms. $e^{At} \in SO_n$.

For $n = 3$ it looks like 3-vectors (Euler).

Consider 4-space: $x^0 = ct$, x^1, x^2, x^3. Let $F = F_{ij}\,dx^i \wedge dx^j$ be a 2-form ("Electromagnetic Field"). $dF = 0$ – **Faraday laws**.

$$F = \underbrace{E_1\,dx^1 \wedge dx^0 + E_2\,dx^2 \wedge dx^0 + E_3\,dx^3 \wedge dx^0}_{\text{electric field}} + \underbrace{\sum_{\substack{\alpha,\beta=1,2,3 \\ \alpha<\beta}} H_{\alpha\beta}\,dx^\alpha \wedge dx^\beta}_{\text{magnetic field}}.$$

We assume that $(x^0, \vec{x}) = (ct, \vec{x}) = (\text{time}, \text{space})$.

In 3-space $\mathbb{R}^3(x^1, x^2, x^3)$ we have 1-form E and 2-form H, depending on t as of parameter.

$$d^{(4)}F = 0 \to \quad d^{(3)}H = 0$$
$$\text{and} \quad \frac{\partial E_\alpha}{\partial x^\beta}\,dx^\beta \wedge dx^\alpha \wedge dx^0 + \frac{\partial H_{\alpha\beta}}{\partial x^0} dx^0 \wedge dx^\alpha \wedge dx^\beta = 0,$$

therefore

$$\frac{\partial E_\alpha}{\partial x^\beta} - \frac{\partial H_{\alpha\beta}}{\partial x^0} = 0.$$

Finally in the 3-space $\mathbb{R}^3$ we have:

$$d^{(3)}E - \frac{\dot{H}}{c} = 0, \quad \text{where} \quad x^0 = ct, \quad \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}.$$

Lecture 14.

Differential forms and physics.

1. **Non-relativistic physics.** Space $\mathbb{R}^3$, $(x,y,z)=\vec{x}$, time t is a parameter.

   Electric charge: e.

   Electric field: $E=(E_1,E_2,E_3)$ is a 1-form $\sum\limits_{\alpha} E_\alpha dx^\alpha, \quad \alpha=1,2,3.$

   In $\mathbb{R}^3$: $x^1=x$, $x^2=y$, $x^3=z$, $E=E(\vec{x},t)$.

   Newtonian equations: $m\ddot{x}^i=eE_i+$ other forces.

   Magnetic field is a 2-form $B_{\alpha\beta}$, $\alpha,\beta=1,2,3$.

   $$B(\vec{x},t)=\sum_{\alpha<\beta} B_{\alpha\beta}\,dx^\alpha\wedge dx^\beta,$$

   Magnetic (Lorentz) force acting on the charged particle moving with speed (v^1,v^2,v^3) is $f_\alpha=e/cB_{\alpha\beta}v^\alpha$ where $\alpha,\beta=1,2,3$.

   Faraday law: $d^{(3)}B=0$ (is closed), or

   $$\frac{\partial B_{12}}{\partial x^3}-\frac{\partial B_{13}}{\partial x^2}+\frac{\partial B_{23}}{\partial x^1}=0,$$

   $$B^1=B_{23},\quad B^2=-B_{13},\quad B^3=B_{12},\quad \vec{B}=(B^1,B^2,B^3).$$

   $$\operatorname{div}\vec{B}=0 \quad\leftrightarrow\quad \frac{\partial B^1}{\partial x^1}+\frac{\partial B^2}{\partial x^2}+\frac{\partial B^3}{\partial x^3}=0.$$

2. **Relativistic physics.**
   $x^0=ct$, $c\cong 300000\frac{\text{km}}{\text{sec}}$ (speed of light in vacuum).

   4-space: $(x^0,x^1,x^2,x^3)=(ct,x^1,x^2,x^3)$.

   Electromagnetic field:

   $$F=\sum_{j<j} F_{ij}\,dx^i\wedge dx^j,\quad i,j=0,1,2,3,$$

   $$\begin{aligned}&F=E\wedge dx^0+B=E\wedge cdt+B=\\&=E_1\,dx\wedge dx^0+E_2\,dy\wedge dx^0+E_3\,dz\wedge dx^0+\\&+B_{12}\,dx\wedge dy+B_{13}\,dx\wedge dz+B_{23}\,dy\wedge dz.\end{aligned}$$

The 1^{st} pair of Maxwell Laws: $dF = 0$ in the 4-space.

$$dF = d^{(3)}E \wedge dx^0 + d^{(3)}B + \frac{\partial B}{\partial x^0} \wedge dx^0.$$

Here $d^{(3)}$ denotes d in the 3-space $\mathbb{R}^3(x, y, z)$, and $d^{(4)}$ denotes d in $\mathbb{R}^4$.

$$x^0 = ct,$$

therefore

$$\frac{\partial B}{\partial x^0} = \frac{1}{c}\frac{\partial B}{\partial t} = \frac{1}{c}\sum_{\alpha<\beta}\frac{\partial B_{\alpha\beta}}{\partial t}\, dx^\alpha \wedge dx^\beta =$$
$$= \frac{1}{c}\frac{\partial B_{12}}{\partial t}\, dx \wedge dy + \frac{1}{c}\frac{\partial B_{13}}{\partial t}\, dx \wedge dz + \frac{1}{c}\frac{\partial B_{23}}{\partial t}\, dy \wedge dz.$$

$$d^{(3)}E = \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial E_3}{\partial x} - \frac{\partial E_1}{\partial z}\right) dx \wedge dz +$$
$$+ \left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right) dy \wedge dz, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z.$$

Conclusion:

(a) $d^{(3)}B(\vec{x}, t) = 0$

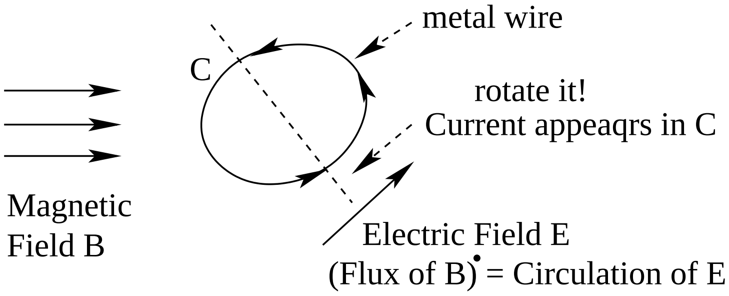
(b) $d^{(3)}E + \frac{1}{c}\frac{\partial B}{\partial t} = 0.$

Corollary:

(a) $B = d^{(3)}A$ (A is vector-potential).

$$B_{12} = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}, \quad B_{13} = \frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}, \quad B_{23} = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}.$$

(b) $\frac{1}{c}\dot{B} = -d^{(3)}E$ in $\mathbb{R}^3$.



Lecture 15.

Integration of differential forms. Consider the n-forms in the n-space $U \subset \mathbb{R}^n$, $x^1, \ldots, x^n$.

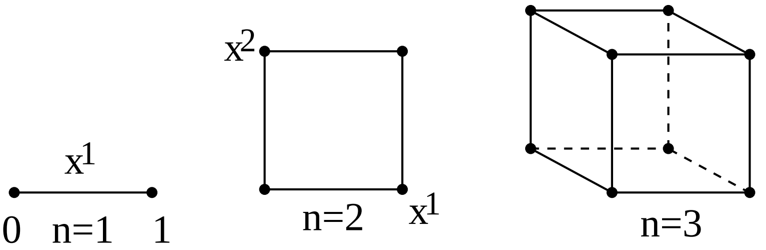
$$\Omega = f(x)\, dx^1 \wedge \ldots \wedge dx^n.$$

By definition,

$$\underset{D^n \subset U}{\int \cdots \int} f(x)\, dx^1 \wedge \ldots \wedge dx^n = \int \left(\ldots \int \left(\int f(x) dx^1 \right) dx^2 \ldots \right) dx^n,$$

is the ordinary integral.

Let D^n be n-cube $(0 \leq x^i \leq 1)$.



$D^n = \underset{x^1}{I^1} \times \ldots \times \underset{x^n}{I^1}$.

What do we know: Let us have a 1 to 1 change of variables: $x = x(x')$. Change of variables for n-form in $\mathbb{R}^n$: $x(x')$

$$\Omega = f(x)\, dx^1 \wedge \ldots \wedge dx^n = \overbrace{f(x(x'))\, \det \left(\frac{\partial x^i}{\partial x'^j} \right) dx'^1 \wedge \ldots \wedge dx'^n}^{\Omega'}.$$

Let $D \cong D' \subset (x')$ (i.e. we have a change of coordinates). Then

$$\int_D \Omega = \int_{D'} \Omega'.$$

Integration of k-forms in n-space along k-surface.

$$\varphi : \underset{x'}{U} \to \underset{x}{\mathbb{R}^n}, \quad U \subset \mathbb{R}^k, \quad x = x(x').$$

Let

$$\Omega = \sum_I a_I(x) dx^I, \quad a_I = a_{i_1 \ldots i_k}, \quad dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad i_1 < \ldots < i_k.$$

Then

$$\varphi^* \Omega = \sum_I a_I(x(x')) \left(\frac{\partial x^I}{\partial x'} \right) dx'^1 \wedge \ldots \wedge dx'^k,$$

where

$$\left(\frac{\partial x^I}{\partial x'} \right) = \det \left(\frac{\partial x^{i_1} \ldots x^{i_k}}{\partial x'^1 \ldots x'^k} \right).$$

Examples.

$$k = 1: \quad \varphi^* \Omega = \sum_i \frac{\partial x^i}{\partial t} a_i(x(t))\, dt, \quad x' = t,$$

$$k = 2,: \quad \varphi^* \Omega = \sum_{i_1 < i_2} a_i(x(t)) \left(\frac{\partial x^{i_1}, x^{i_2}}{\partial u, v} \right) du \wedge dv, \quad x' = (u, v),$$

$$\text{where} \quad \left(\frac{\partial x^{i_1}, x^{i_2}}{\partial u, v} \right) = \left(\frac{\partial x^{i_1}}{\partial u} \frac{\partial x^{i_2}}{\partial v} - \frac{\partial x^{i_1}}{\partial v} \frac{\partial x^{i_2}}{\partial u} - \right),$$

$$\ldots$$

$$k = n \quad \varphi^* \Omega = a_{1 \ldots n}(x(x')) \cdot \underset{\underset{\text{Jacobian}}{\uparrow}}{\mathcal{J} \left(\frac{x}{x'} \right)} \cdot dx'^1 \wedge \ldots \wedge dx'^n.$$

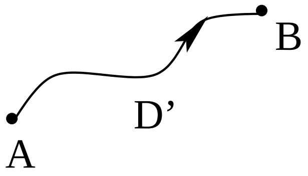
Claim: Consider domain D^k with the boundary ∂D^k. Then:

$$\underset{\underset{\text{Domain}}{\uparrow}}{\int_{D^k}} d\Omega \quad = \quad \underset{\underset{\text{Boundary (oriented?)}}{\uparrow}}{\int_{\partial D^k}} \Omega.$$

Example (XVII century) – Newton-Leibnitz.

$$k = 1: \quad \int_{D'} df = \int_{\partial D'} f = f(b) - f(a)$$

Consider a curve in n-space $\mathbb{R}^n$.



$$D' : \quad \begin{array}{l} x^i(t), \quad i = 1, \dots, n \\ t = a, \quad \vec{x}(a) = A \\ t = b, \quad \vec{x}(b) = B \end{array} \qquad .$$

$$\int\limits_{D'} \frac{\partial f}{\partial x^i} dx^i \underset{\text{def}}{=} \int\limits_a^b \frac{\partial f}{\partial x^i}(x(t)) \frac{\partial x^i}{\partial t} dt = \int\limits_a^b \underbrace{\left(\frac{\partial f}{\partial x^i}(x(t)) \frac{\partial x^i}{\partial t} dt \right)}_{\varphi^*(df)} = \int\limits_a^b \varphi^*(df) =$$

$$= \int\limits_a^b \Phi(t)\, dt = f(b) - f(a), \qquad \text{where} \qquad \Phi(t) = \left(\frac{\partial f}{\partial x^i}(x(t)) \frac{\partial x^i}{\partial t} \right)_{x=x(t)} .$$

What is orientation for k $>$ 1?

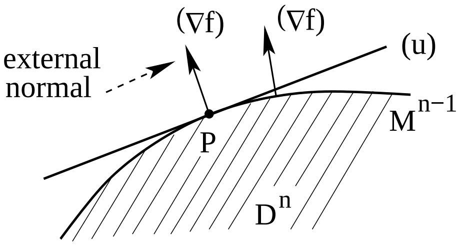
For linear space over $\mathbb{R}$ orientation is a choice of basis up to linear change with det > 0.

Orientation in $\mathbb{R}^n$ or in $U \subset \mathbb{R}^n$ is given by Cartesian coordinates $(x^1, \dots, x^n)$ up to change of coordinates $x = x(x')$ such that $\mathcal{J} > 0$.

Another point of view: orientation provided at the point $P \in U \subset \mathbb{R}^n$ is given by the basis $(\partial/\partial x^1, \dots, \partial/\partial x^n)$ in the “tangent” space $T_P(\mathbb{R}^n)$ generated by $\partial/\partial x^i$.

Consider a submanifold in $\mathbb{R}^n$ defined by an equation (globally!):

$$M^{n-1} = \{ f(x^1, \dots, x^n) = 0 \}, \quad \text{such that} \quad (df)_{M^{n-1}} \neq 0.$$



“Tangent vectors” to the submanifold M are $(\partial/\partial u^1, \dots, \partial/\partial u^{n-1})$ if $(u^1, \dots, u^{n-1})$ are local coordinates in M^{n-1} near the point P.

Take vectors

$$\left(\nabla f, \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^{n-1}}\right), \qquad \text{where} \qquad \nabla f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i}\right) \frac{\partial}{\partial x^i}.$$

Compare this basis the **chosen** oriented basis $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$.

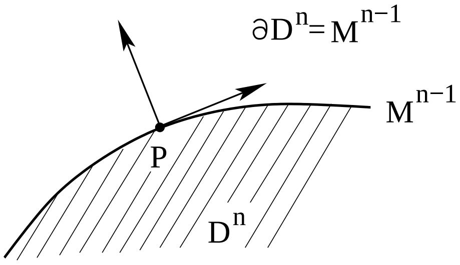
$$(\nabla f, \tau_u) = \left(\nabla f, \frac{\partial}{\partial u}\right) \xleftrightarrow{+} \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = \tau_x.$$

Induced orientation at M^{n-1} is such that determinant is $+$ (> 0), $M^{n-1} = \partial D^n$.

Lecture 16.

Orientation:

M^{n-1} − hypersurface in $\mathbb{R}^n$ given by equation $f\,(x^1, \dots, x^n) = 0$, such that $df \neq 0$ in all points x where $f(x) = 0$.



$$\nabla f = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right),$$
$$\nabla f \neq 0 \quad \text{on} \quad M.$$

Orientation in $\mathbb{R}^n$ with given coordinates $(x^1, \dots, x^n)$.

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \qquad - \qquad \text{basis in } T_P\,(\mathbb{R}^n)$$

Let

$$\frac{\partial f}{\partial x^1} \neq 0 \quad \text{at} \quad P \in M^n$$

$$(u^1, \dots, u^{n-1}) = (x^2, \dots, x^n)$$

- local coordinates in M near P. “Implicit function theorem”:

$$x^1 = \Phi\,(x^2, \dots, x^n) \qquad \text{if} \qquad \frac{\partial f}{\partial x^1} \neq 0\Big|_P$$

near P on the surface given by the equation $f\,(x^1, \dots, x^n) = 0$.

Basis

$$\left(\nabla f\Big|_P, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\right) = A\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$$

$$\det A > 0 \qquad \leftrightarrow \qquad x^2, \dots, x^n \quad \text{given} \quad (+)$$

$$A\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \left(\nabla f\Big|_P, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\right)$$

Definition:

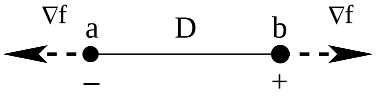
$$\det A \;>\; 0 \qquad \leftrightarrow \qquad (x^2,\, \ldots,\, x^n) \quad \text{are oriented coordinates in } M^{n-1}$$

$$\det A \;<\; 0 \qquad \leftrightarrow \qquad (-x^2,\, \ldots,\, x^n) \quad \text{are oriented coordinates in } M^{n-1}$$

"Orientation of ∂D induced by orientation of $\mathbb{R}^n$"

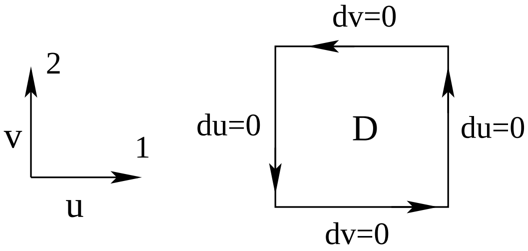
Example:

1. $n \,=\, 1$



$\partial D \;=\; b \,-\, a\,.$

2.



Orientation of D induces orientation on ∂D.

Theorem.

$$\iint_D d\,\Omega \;\;=\; \int_{\partial D} \Omega$$

Proof for 1 - form Ω. Let $x^1 \,=\, u\,,\;\; x^2 \,=\, v$ and

$$\Omega \;\;=\;\; f(u,\, v)\; dv \quad , \qquad\qquad d\,\Omega \;\;=\;\; \frac{\partial f}{\partial u}\; du \wedge dv$$

$$\iint_D (f_u\; du \wedge dv) \quad = \quad \int_{\partial D} f\; dv \quad = \quad \int_0^1 f(1,\, v)\; dv \;\; - \;\; \int_0^1 f(0,\, v)\; dv$$

$$\iint_D f_u\; du \,\wedge\, dv \;\;=\;\; \int_0^1 dv \left(\int_0^1 f_u(u,\, v)\, du \right) \;\;=\;\; \int_0^1 dv\; [f(1,\, v) \,-\, f(0,\, v)]$$

So we have

$$\iint_D f_u\; du \,\wedge\, dv \quad = \quad \int_{\partial D} f\, dv \quad = \quad \iint_D d\,\Omega$$

Theorem is proved for $n = 2$.

For any $n \geq 2$ we have

$$D = \text{cube } I^n \quad , \qquad x = x^1, \ldots, x^n \quad , \qquad 0 \leq x^i \leq 1$$

$$\int_{\partial D} f \, dx^2 \wedge \cdots \wedge dx^n \quad = \int f_{x^1} \, dx^1 \wedge \cdots \wedge dx^n$$

$$u = x^1 \, , \, v = (x^2, \ldots, x^n)$$

$$\int_{\partial D} f \, dv \quad = \quad \iint_D f_u \, du \wedge dv$$

$$\int_{\partial D} f \, dv \quad = \quad \int_{I^{n-1}} dv \left(\int_0^1 f_u \, du \right) \quad =$$

$$= \quad \int_{I^{n-1}} dv \, (f(1, v) - f(0, v)) \quad = \quad \iint_D f_u \, du \wedge dv.$$

So

$$\int_{\partial D} \Omega \quad = \quad \iint_D d\,\Omega$$

Theorem is proved.

Lecture 17. Linear Algebra (Minicourse).

Linear space over field k $(= \mathbb{R}, \mathbb{C})$

$$L \; : \qquad \text{basis} \quad (e_1, \ldots, e_n) \quad \text{in} \quad L$$

Every vector

$$e \;=\; \sum_i \eta^i \, e_i \qquad \left[e \;=\; (\eta^1, \, \ldots, \, \eta^n)\right]$$

Linear Map

$$L \quad \overset{A}{\to} \quad M \qquad (e'_1, \, \ldots, \, e'_n)$$

$$A(e_i) \;=\; a_i^j \, e'_j \quad , \qquad a_i^j \ \text{is} \ m \times n \ \text{matrix}$$

$$A(e) \;=\; \sum_i \eta^i \; A(e_i) \;=\; \left(\eta^i \, a_i^j\right) \, e'_j \qquad (\text{sum in } i, \, j)$$

For "components" (η^i)

$$A(e) \;=\; \left(\eta^i \, a_i^j\right) \, e'_j$$

$$(\eta^1, \, \ldots, \, \eta^n) \;\to\; (\eta'^1, \, \ldots, \, \eta'^n) \quad , \qquad \eta'^j \;=\; a_i^j \, \eta^i$$

Adjoint operator A^t: spaces of "covectors" L^*, M^*.

$$L^* \;=\; \left\{\tilde{e}^1, \, \ldots, \, \tilde{e}^n\right\} \quad , \qquad M^* \;=\; \left\{\tilde{e}'^1, \, \ldots, \, \tilde{e}'^n\right\}$$

$$\langle \tilde{e}^i, e_k \rangle \;=\; \delta_k^i \quad , \qquad \langle \tilde{e}'^j, e'_p \rangle \;=\; \delta_k^j$$

$$\langle A^t \zeta, \eta \rangle \;=\; \langle \zeta, A\eta \rangle$$

$$\zeta \;=\; u_j \, e'^j \quad , \qquad \eta \;=\; \eta^i \, e_i \quad , \qquad A^t \zeta \;=\; \left(a_i^j \, u_j\right)$$

(vectors η, covectors ζ).

"Kernel" of operator $A : L \to M$

$$S \subset L \quad , \qquad A(S) = 0$$

$$\text{rank}\, A \;=\; \dim L \;-\; \dim S$$

"Cokernel" of A :

$$T \;=\; M/A(L)$$

Change of basis

$$(e_1, \dots, e_n) \ , \qquad e_i = r_i^j \, e''_j \quad (\text{sum over } j) \ , \qquad e''_j \in L \ , \qquad e = R(e'')$$

$$e'' \ \overset{R}{\to} \ e \ \overset{A}{\to} \ e' \in M \qquad\qquad (\text{change in } L)$$

$$L \ \overset{R}{\to} \ L \ \overset{A}{\to} \ M \ \xrightarrow{Q^{-1}} \ M \qquad (\text{change in } M)$$

$$A \ \Rightarrow \ Q^{-1} A R$$

Case $L = M$ (eigenvalue problem)

$$M = L \quad , \qquad Q = R$$

$$A \ \Rightarrow \ R^{-1} A R \qquad (!)$$

Theory of inner products

Inner Product $\langle \eta_1, \eta_2 \rangle$ = number, $\eta_1, \eta_2 \in L$.

1. Bilinear

$$\langle \eta + \zeta, \eta_1 \rangle \;=\; \langle \eta, \eta_1 \rangle \;+\; \langle \zeta, \eta_1 \rangle$$

(same in the variable η_1).

2. $\langle \lambda \eta_1, \eta_2 \rangle \;=\; \lambda \, \langle \eta_1, \eta_2 \rangle$

3. $\langle \eta_1, \lambda \eta_2 \rangle \;=\; \langle \eta_1, \eta_2 \rangle \cdot \lambda$
or $\langle \eta_1, \lambda \eta_2 \rangle \;=\; \langle \eta_1, \eta_2 \rangle \cdot \bar{\lambda}$ ("hermitian"), $k = \mathbb{C}$, $\lambda \in \mathbb{C}$.

Gramm matrix $G = (g_{ij})$

$$\langle e_i, e_j \rangle = g_{ij}$$

Change of basis: $e_i = a_i^j \, e'_j$

$$g_{ij} = \langle a_i^p e'_p \, , \, a_i^k e'_k \rangle = a_i^p \, a_j^k \, g'_{pk}$$

Conclusion

$$G = A^t \, G \, A$$

$$\eta_1 = \eta_1^i \, e_i \quad , \qquad \eta_2 = \eta_2^j \, e_j \quad , \qquad \langle \eta_1, \eta_2 \rangle = \eta_1^i \, \eta_2^j \, g_{ij}$$

Nondegenerate Inner Product

$$\det G \neq 0$$

$k = \mathbb{R}$: sign of $\det G$ is invariant under change of basis $e = A \, e'$.

Proof: $\det G' = (\det A)^2 \, \det G$.

Symmetric Inner Product

$$\langle \eta_1, \eta_2 \rangle = \langle \eta_2, \eta_1 \rangle$$

Theorem. Every symmetric Inner Product can be written in some basis as

$$G = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & -I \end{array} \right) \quad , \qquad \eta^2 = \sum_{j=k+1}^{k+p} \eta_j^2 - \sum_{j=k+p+1}^{n} \eta_j^2$$

"rank G" $= n -$ dimension of **"Kernel"**

$$G : \; L \to L^*$$

$$G^t : \; (L^*)^* \to L^* \quad , \qquad L^{**} = L$$

$$G^t \;=\; G \quad - \quad (\text{“symmetric”})$$

Nondegenerate case:

$$G \quad = \quad \left(\begin{array}{ccc|ccc} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 \end{array}\right)$$

Type (p, q) – “signature” , $p + q = n$

Euclidean case $q = 0$ (or $p = 0$) : $\mathbb{R}^n$.

Lorentzian case $p = 1\,,\ q = n - 1$: $\mathbb{R}^{1,n-1}$

Hermitian Case is needed in Quantum Theory.

Lecture 18. Linear Algebra - II.

Theory of inner products:

L – linear space, $w, v \in L$.

$$\langle w, v\rangle = \text{number}$$

bilinear:

$$\langle \lambda w, v\rangle = \langle w, \lambda v\rangle = \lambda \langle w, v\rangle$$

nondegenerate:

$$\forall v \quad \exists w \quad : \qquad \langle v, w\rangle \neq 0$$

Inner Product

$$G : \quad L \to L^*$$

$$e_1, \ldots, e_n \quad - \quad \text{basis } L \quad , \qquad e^1, \ldots, e^n \quad - \quad \text{basis } L^*$$

$$\langle e_i, e_j\rangle = g_{ij} \quad , \qquad G(e_i) = g_{ij}\, e^j \quad \text{(sum)}$$

Gramm Matrix

$$\det G \neq 0 \quad \leftrightarrow \quad \text{nondegenerate Inner Product}$$

Symmetric:

$$\langle v, w\rangle = \langle w, v\rangle \quad , \qquad g_{ij} = g_{ji} \quad , \qquad G^t = G$$

Lemma. Let $M \subset L$ is subspace such that restriction $G|_M$ is nondegenerate. The orthogonal complement space $M^\perp$ exists and unique;

$$\forall v \in M \ , \ w \in M^\perp \ : \qquad \langle v, w\rangle = 0$$

Proof: Choose basis $e_1, \ldots, e_k \in M$, $e_{k+1}, \ldots, e_n$ $(\dim M = k)$ in L. We have

$$G_M = (g_{ij}) \ , \quad i, j \leq k \quad , \qquad \det G_M \neq 0$$

Find all vectors $w \in M^{\perp}$

$$w = \sum_{i \leq k} \alpha_i e_i + \sum_{j > k} \beta_j e_j = e + \sum_{j > k} \beta_j e_j \quad , \qquad e \in M$$

Linear Equations

$$\langle w, M \rangle = 0 \qquad \leftrightarrow \qquad \langle w, e_1 \rangle = 0 \, , \; \ldots \, , \; \langle w, e_k \rangle = 0$$

Or for the variables $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_{n-k})$

$$\sum \alpha_i \langle e_i, e_1 \rangle + \sum \beta_j \langle e_j, e_1 \rangle = 0$$

$$\ldots$$

$$\sum \alpha_i \langle e_i, e_k \rangle + \sum \beta_j \langle e_j, e_k \rangle = 0$$

- k linear homogeneous equations.

Rank of system $= k$ (minor g_{ij} , $i, j = 1, \ldots, k$ is $\neq 0$).

So we have linear space $\mathbb{R}^{n-k}$ of solutions

$$M^{\perp} \simeq \mathbb{R}^{n-k} \quad , \qquad \dim M^{\perp} = n - k$$

(any field k instead of $\mathbb{R}$).

Lemma is proved.

Proof of Theorem:

$$\langle v, w \rangle \quad - \quad \text{symmetric inner product}$$

$$\exists \, v_1 = e_1 \quad \text{such that} \quad \langle v_1, v_1 \rangle \neq 0$$

Normalize v_1 such that $\langle v_1, v_1 \rangle = \pm 1$.

Take orthogonal space, dim $= n - 1$.

Iterate procedure. We are coming to

$$e_1 = v_1 \, , \; \ldots \quad , \qquad \langle e_i, e_j \rangle = \begin{cases} +1 & \text{or} \\ -1 & \text{or} \\ 0 & \end{cases}$$

So we can reduce our matrix G to the form

$$G = \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & -I & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \begin{array}{c} p \\ q \\ r \end{array} , \qquad p + q + r = n$$

Theorem is proved.

Nondegeneracy: $r = 0$. "Signature" (p, q), p pluses, $q = n - p$ minuses.

Cases:

$$q = 0 : \quad \text{Euclidean Inner Product}$$
$$ds^2 = \sum \left(dx^i\right)^2 , \qquad G = I$$

$$q = n - 1 : \quad \text{Minkowki Inner Product}$$
$$(+ \ - \ - \ \dots \ -) \qquad \text{(Relativity)}$$
$$ds^2 = \left(dx^0\right)^2 - \sum_{\alpha=1}^{n-1} \left(dx^\alpha\right)^2 , \qquad G = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -I \end{array}\right)$$

Isometries

$$A : \ \mathbb{R}^n \to \mathbb{R}^n$$

Let A be Linear Map $A^t = A^{-1}$ (orthogonal)

$$\langle Av, Aw\rangle = \langle v, w\rangle = \textbf{general case}$$

Euclidean space $\mathbb{R}^2$ $(G = 1)$.

$$A = P^\alpha \cdot \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} , \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \alpha = 0, 1$$

Minkowski case: $\mathbb{R}^{1,1}(x^0, x^1)$.

$$A = P^\alpha \cdot T^\beta \cdot \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}$$

$$P \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;, \qquad T \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \;, \qquad \alpha,\, \beta \;=\; 0,\, 1$$

"Lorentz Transformations":

$$A \;=\; \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \;, \qquad \left\{ \begin{array}{l} x^0 \;=\; ct \\ x^1 \;=\; x \end{array} \right.$$

$$\begin{array}{rcl} x^0 & = & \cosh\psi\; x'^0 \;+\; \sinh\psi\; x'^1 \\ x^1 & = & \sinh\psi\; x'^0 \;+\; \cosh\psi\; x'^1 \end{array} \;, \qquad \left\{ \begin{array}{l} x'^0 \;=\; ct' \\ x'^1 \;=\; x' \end{array} \right.$$

$$\left. \begin{array}{rcl} ct & = & a\,ct' \;+\; b\,x' \\ x & = & b\,ct' \;+\; a\,x' \end{array} \right\} \;, \qquad a^2 \;-\; b^2 \;=\; 1$$

"Physical Parametrization":

$$a \;=\; \frac{1}{\sqrt{1-v^2}} \;, \qquad b \;=\; \frac{v}{\sqrt{1-v^2}} \;, \qquad v \;=\; w/c \quad (\text{“speed”})$$

$$t \;=\; a\,t' \;+\; \frac{b}{c}\,x' \;, \qquad x \;=\; c\,b\,t' \;+\; a\,x'$$

Assumption: $v \ll 1$ $(w \ll c)$. So we have

$$t \;\simeq\; t' \;, \qquad x \;\simeq\; x' + w\,t$$

$$a \;\simeq\; 1 \;,\quad b \;\simeq\; 0 \;, \qquad c\,b \;\simeq\; w \quad \text{(speed)}$$

\- Galilean Transformation.

Lecture 19.

Theorem. Every skew-symmetric inner product can be reduced to the form

$$G \quad = \quad \left(\begin{array}{cc|c|cc|ccc} 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & -1 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{array}\right)$$

("canonical basis"). $\det G \neq 0 \quad \Rightarrow \quad n = 2k\,.$

Proof. Find vectors v, w such that $\langle v, w\rangle \neq 0$. Normalize them such that $\langle v, w\rangle = 1$. Take subspace $M = \operatorname{Span}\{v,\, w\}$. Restriction of inner product to M is

$$G_M \quad = \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \qquad \det G_M \neq 0$$

Take orthogonal complement $M^{\perp}$ and so on.

Theorem is proved.

Remark. Skew-symmetric inner product is a 2 - form

$$\Omega \quad = \quad \sum g_{ij}\, dx^i \wedge dx^j \quad , \qquad g_{ij} \; = \; -\, g_{ji}$$

Let $\det G \neq 0$. We choose a basis $(e_1, \dots, e_k, e'_1, \dots, e'_k)$ and coordinates $(q^1, \dots, q^k, p_1, \dots, p_k)$ such that

$$\Omega \quad = \quad \sum_{i=1}^{k} dq^i \wedge dp_i$$

$$G \quad = \quad \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$$

Riemannian Geometry:

At every point $x = (x^1, \dots, x^n) \in U$ positive quadratic form is given

$$ds^2 = g_{ij}\, dx^i\, dx^j \quad , \qquad g_{ij} = g_{ji}$$

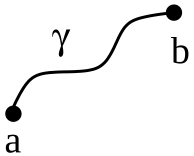
a) $ds^2 > 0$ – **Riemannian Geometry:** $\mathbb{R}^n$.

b) $\det g_{ij} \neq 0$ – **Pseudoriemannian** Geometry of the type (p, q) (**signature**) : $\mathbb{R}^{p,q}$.

Case $p = 1$: – Lorentzian Geometry (Relativity).

Euclidean (Pseudoeuclidean) Space

$$g_{ij}(x) = \text{const}$$

Arc length: 

$$\gamma = \left\{x^i(t)\right\} \quad , \qquad i = 1, \dots, n \quad , \qquad a \le t \le b$$

- curve (smooth).

Length

$$l(\gamma) = \int_a^b \sqrt{g_{ij}(x(t))\ \dot{x}^i\, \dot{x}^j}\ dt = \int_a^b |\dot{x}|\ dt$$

$$|\dot{x}|^2 = g_{ij}\ \dot{x}^i\, \dot{x}^j$$

“Pseudoriemannian Case”: (Lorentzian, $\mathbb{R}^{1,n-1}$)

timelike vector – $|\dot{x}|^2 > 0$, $g_{ij}\ \dot{x}^i\, \dot{x}^j > 0$

lightlike vector – $|\dot{x}|^2 = 0$, $g_{ij}\ \dot{x}^i\, \dot{x}^j = 0$

spacelike vector – $|\dot{x}|^2 < 0$, $g_{ij}\ \dot{x}^i\, \dot{x}^j < 0$

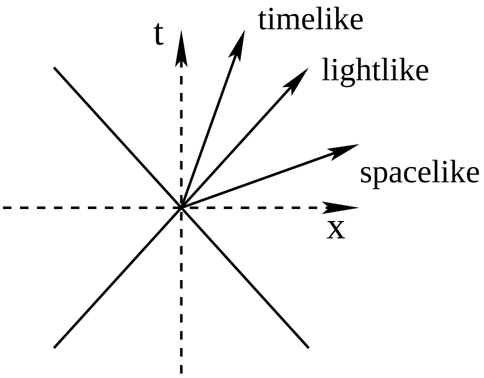
Symplectic Geometry:

$$x^1, \dots, x^n \ , \qquad \Omega = g_{ij}\ dx^i \wedge dx^j \ , \qquad g_{ij} = -g_{ji} \quad (G)$$

$$\det g_{ij} \neq 0 \quad , \qquad n = 2k \quad , \qquad \underline{d\Omega = 0}$$

Lorentzian Geometry $\mathbb{R}^{1,1}$:

$$(x^0, x^1) \quad , \qquad x^0 = ct \ , \quad x^1 = x$$



timelike vector $-$ $\langle\zeta,\zeta\rangle > 0$

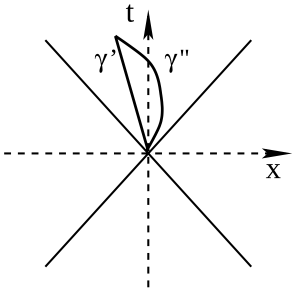
lightlike vector $-$ $\langle\zeta,\zeta\rangle = 0$

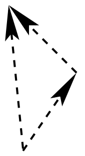
spacelike vector $-$ $\langle\zeta,\zeta\rangle < 0$

World - line of any object $=$ curve $(x^0(\tau), x^1(\tau))$.

"Real object" (nonzero mass): $(\dot{x}^0)^2 - (\dot{x}^1)^2 > 0$.

Zero mass (light-like): $(\dot{x}^0)^2 - (\dot{x}^1)^2 = 0$.



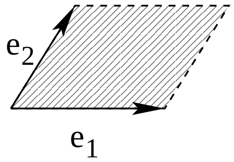


Statement: length $(\gamma') >$ length (γ'').

Axiom:

$$\text{"Living time of object"} \quad = \quad \frac{1}{c} \text{ length (World line)}$$

Volume in euclidean geometry



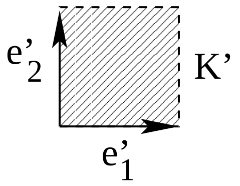
$$g_{ij} \;=\; \langle e_i, e_j \rangle \qquad (G)$$

Claim:

$$\mathrm{vol}\,(K) \;=\; \sqrt{\det G}$$

Why:

$$e_i \;=\; \lambda_i^j \; \tilde{e}_j \quad , \qquad \langle \tilde{e}_j, \tilde{e}_k \rangle \;=\; \delta_{jk} \qquad \text{(orthonormal)}$$

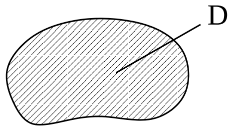


$$\mathrm{vol}\,(K') \;=\; 1 \quad , \qquad \mathrm{vol}\,(K) \;=\; \det\, \left(\lambda_i^j\right) \;=\; \det \Lambda$$

$$G \;=\; \Lambda\, \Lambda^t \quad , \qquad \det G \;=\; (\det \Lambda)^2$$

Conclusion: $\mathrm{vol}\,(K) \;=\; \sqrt{\det G}$.

Definition: in any pseudoriemannian geometry we define



$$\mathrm{vol}\ D \;=\; \int_D \sqrt{|\det G|}\ dx^1 \wedge \ldots \wedge dx^n$$

$$\det G \;=\; g^2 \quad , \qquad \Omega \;=\; g\ dx^1 \wedge \ldots \wedge dx^n$$

In orientable manifolds volume is integral of n - form (differential) .

Change of coordinates:

$$x\ (x') \quad , \qquad \hat{J} \;=\; \left(\frac{\partial x^i}{\partial x'^j} \right) \ ,$$

$$G' \;=\; \hat{J}\, G\, \hat{J}^t \quad , \qquad \det G' \;=\; \det G \,\cdot\, |J|^2$$

Lecture 20. Volume and Differential Forms. Symplectic Manifolds.

Inner Product in $\mathbb{R}^n \ni v,\, w \;:\; \langle w, v\rangle$

Bilinear

$$\langle \lambda w, v\rangle \;=\; \langle w, \lambda v\rangle \;=\; \lambda\, \langle w, v\rangle$$

Nondegenerate

$$\forall\, v \;\exists w \;:\; \langle v, w\rangle \;\neq\; 0$$

Gramm matrix (basis $e_1, \ldots, e_n$)

$$G \;=\; (g_{ij}) \;=\; \langle e_i, e_j\rangle$$

Nondegeneracy: $\det G \;\neq\; 0\,.$

Inner Product $=$ Map $G : \; L \to L^*$

$$\langle e_i, e^j\rangle \;=\; \delta^i_j \quad , \qquad G(e_i) \;=\; g_{ij}\, e^j$$

Symmetric Inner Product: $g_{ij} \;=\; g_{ji}\,, \quad \langle v, w\rangle \;=\; \langle w, v\rangle\,.$

Theorem 1. There exists basis $e_1, \ldots, e_n$ such that g_{ij} is diagonal and

$$g_{ij} \;=\; \begin{cases} +1 & \text{or} \\ -1 & \text{or} \\ 0 & \end{cases} \quad , \qquad G \;=\; \begin{array}{cc} \begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} p \\ q \\ r \end{array} \end{array}$$

$p + q + r \;=\; n\,.$

Nondegenerate: $r \;=\; 0\,.$

$(p,\, q) \;=\;$ "signature" .

Euclidean: $q \;=\; 0$ (or $p \;=\; 0$) : $ds^2 \;=\; \sum (dx^i)^2$

Lorentzian: $p \;=\; 1$ (or $q \;=\; 1$) : $ds^2 \;=\; (dx^0)^2 \,-\, \sum (dx^\alpha)^2$
$(x^0 \;=\; ct)\,.$

"Orthogonal Maps":

$$\langle Av, Aw \rangle \;=\; \langle v, w \rangle$$

Skew - Symmetric Inner Product: $g_{ij} \;=\; -\,g_{ji}\;, \quad \langle v, w \rangle \;=\; -\,\langle w, v \rangle\;.$

Theorem 2. There exists basis

$$(e_1,\,\ldots,\,e_k,\,e'_1 = e_{k+1},\,\ldots,\,e'_k = e_{2k},\,e''_1,\,\ldots,\,e''_s)$$

such that

$$g_{ij} \;=\; \langle e_i, e_j \rangle \;=\; \begin{array}{cc} \begin{pmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} k \\ k \\ s \end{array} \end{array}$$

For $s \,=\, 0$ we have $n \,=\, 2k$. “Canonical Coordinates”:

$$\begin{array}{ccccccc} (q^1, & \ldots, & q^k, & p_1, & \ldots, & p_k) \\ & (e_i) & & & (e'_i) & \end{array}$$

Define “symplectic 2 - form” in that basis

$$\Omega \;=\; \sum_{i=1}^{k} dq^i \wedge dp_i \quad , \qquad (s = 0)$$

$$\det\, g_{ij} \;=\; 1$$

Property:

$$\Omega^k \;=\; \underbrace{\Omega \,\wedge\,\ldots\,\wedge \Omega}_{k \text{ times}} \;=\; k!\;\, dq^1 \wedge dp_1 \wedge\,\ldots\,\wedge dq^k \wedge dp_k \;=$$

$$=\;\; k!\;\,(-1)^k\;\, dq^1 \wedge\,\ldots\,\wedge dq^k \wedge dp_1 \wedge\,\ldots\,\wedge dp_k$$

So for the volume element

$$d^n\sigma \;=\; \sqrt{g_{ij}}\;\, dq^1 \wedge\,\ldots\,\wedge dq^k \wedge dp_1 \wedge\,\ldots\,\wedge dp_k$$

we have

$$\frac{1}{k!}\,\Omega^k \quad = \quad d^n\sigma$$

For any coordinates Symplectic 2 - form is

$$\Omega \quad = \quad \sum_{i<j} \tilde{g}_{ij}\; dx^i \wedge dx^j \qquad\qquad \text{(basis } \tilde{e}\text{)}$$

$$\det\,\tilde{G} \;\neq\; 0 \quad , \qquad n \;=\; 2k$$

$$\Omega \quad = \quad \sum_{i=1}^{k} dq^i \wedge dp_i$$

in canonical basis (e) .

Linear Change: matrix $\hat{J}$:

$$\tilde{e} \;=\; \hat{J}\,e \quad , \qquad\qquad \det\,G \;=\; \left(\det\,\hat{J}\right)^2 \det\,\tilde{G}$$

So we have:

Volume element

$$d^n\tilde{\sigma} \quad = \quad \sqrt{\det\,\tilde{G}}\;\; dx^1 \wedge \,\ldots\, \wedge dx^n \quad = \quad \left(\det\,\hat{J}\right)\; dx^1 \wedge \,\ldots\, \wedge dx^n$$

$$d^n\sigma \quad = \quad dq^1 \wedge \ldots \wedge dq^k \wedge dp_1 \wedge \ldots \wedge dp_k \quad = \quad \frac{1}{k!}\,\Omega^k \quad = \quad \frac{1}{k!}\,\Omega \wedge \ldots \wedge \Omega$$

$$\det\;\tilde{G} \;=\; \left(\det\,\hat{J}\right)^2$$

Volume element

$$d^n\tilde{\sigma} \quad = \quad \left(\det\,\hat{J}\right)\; dx^1 \wedge \,\ldots\, \wedge dx^n \quad = \quad \frac{1}{k!}\,\Omega^k$$

Result:

$$\det\,\tilde{G} \;=\; \left(\det\,\hat{J}\right)$$

$$\det \hat{J} \quad - \quad \text{Pfaffian}$$

Theorem. A Pfaffian

$$\det \hat{J} \quad = \quad \sqrt{\det\,(g_{ij})}$$

is a polynomial of the skew-symmetric matrix G.

Proof. We have for the volume element

$$d^n\sigma \quad = \quad \sqrt{\det G}\ dx^1 \wedge \ldots \wedge dx^n \quad = \quad \frac{1}{k!}\ \underbrace{\Omega \wedge \ldots \wedge \Omega}_{k \text{ times}}$$

because multiplication of forms is invariant under changes of coordinates. Elements of $\Omega \wedge \ldots \wedge \Omega$ are polynomials.

Theorem is proved.

Inner Product of vectors

$$\langle v, w\rangle \ = \ \sum g_{ij}\, v^i\, w^j \ , \qquad v \ = \ v^i\, e_i \ , \quad w \ = \ w^j\, e_j \ , \qquad v,\, w \ \in \ L$$

Inner product of covectors

$$v^*,\, w^* \ \in \ L^* \ , \qquad v^* \ = \ v_i\, e^i \ , \quad w^* \ = \ w_j\, e^j \ , \qquad \langle v^*, w^*\rangle \ = \ \sum v_i\, w_j\, g^{*ij}$$

and

$$\left(g^{*ij}\right) \quad = \quad G^* \quad = \quad G^{-1}$$

Why?

$$L \quad \overset{G}{\underset{G^*}{\rightleftarrows}} \quad L^*$$

$G^* \ = G^{-1} \quad - \quad$ definition .

Example.

Symplectic Inner Product (2 - form) : $g_{ij} \ = \ -\, g_{ji}$.

$$\Omega \quad = \quad g_{ij}\; dx^i \wedge dx^j \quad , \qquad i,\, j \ = \ 1,\, \ldots,\, n$$

$$\Omega \quad = \quad \sum dq^j \wedge dp_j \quad , \qquad j \ = \ 1,\, \ldots,\, k$$

Inner Product of gradients is given by matrix $(g^{*ij}) = G^{-1}$.

Example. Hamiltonian System (H).

$$\frac{df(x)}{dt} \quad = \quad \{\nabla f \,,\, \nabla H\} \quad = \quad g^{*ij}\,\frac{\partial f}{\partial x^i}\,\frac{\partial H}{\partial x^j}$$

Lecture 21. Duality Operator and Maxwell's Equations.

In presence of Riemannian (or Pseudoriemannian) metric ($g_{ij} = g_{ji}$) duality operator is defined for k - forms $\Omega \in \Lambda^k(U^n)$:

$$* \, \Omega \quad \text{is a} \;\; k - \text{form}$$

$$* * \Omega \;=\; \pm \Omega$$

Define it first for Euclidean Metric: $g_{ij} = \delta_{ij}$ using orthonormal basis $(e_1, \ldots, e_n)$, $\langle e_i, e_j \rangle = \delta_{ij}$. Dual basis of 1 - forms is

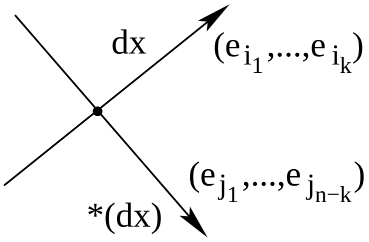
$$dx^1, \ldots, dx^n \quad : \qquad dx^i\,(e_j) \;=\; \delta^i_j$$

Definition:

$$* \left(dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) \;=\; dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}} \; (-1)^\sigma$$

where $i_1 < \ldots i_k$, $j_1 < \cdots < j_{n-k}$,

$$\sigma \;=\; \begin{pmatrix} 1 & \ldots & \ldots & \ldots & \ldots & n \\ i_1 & \ldots & i_k & j_1 & \ldots & j_{n-k} \end{pmatrix}$$



(Dual means "orthogonal"?)

We have:

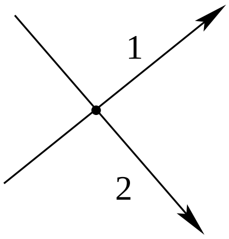
$$* * (\Omega) \;=\; (-1)^{k(n-k)} \, \Omega$$

Examples:

$$k = 0 : \quad *(1) \;=\; dx^1 \wedge \cdots \wedge dx^n \;=\; d^n\sigma \qquad \text{(volume)}$$

$$* \, (d^n\sigma) \;=\; 1$$

$$k = 1, \;\; n = 2$$



$$\begin{aligned} *\,(1) &= (2) \\ *\,(2) &= -(1) \\ *^2 \quad &= -1 \end{aligned}$$

$$k = 1 \quad , \quad n = 3 \qquad\qquad k = 2 \quad , \quad n = 3$$

$$\begin{aligned} *\,(1) &= (23) \\ *\,(2) &= -(13) \\ *\,(3) &= (12) \end{aligned} \qquad\qquad \begin{aligned} *\,(12) &= (3) \\ *\,(13) &= -(2) \\ *\,(23) &= (1) \end{aligned}$$

$$*^2 \; = \; 1$$

$$k = 1 \quad , \quad n = 4 \qquad\qquad k = 3 \quad , \quad n = 4$$

$$\begin{aligned} *\,(0) &= (123) \\ *\,(1) &= -(023) \\ *\,(2) &= (013) \\ *\,(3) &= -(012) \end{aligned} \qquad\qquad \begin{aligned} *\,(012) &= (3) \\ *\,(013) &= -(2) \\ *\,(023) &= (1) \\ *\,(123) &= -0 \end{aligned}$$

$$*^2 \; = \; -1$$

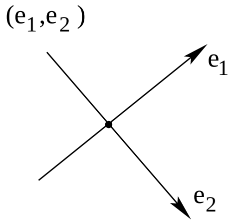
$$k = 2 \quad , \quad n = 4$$

$$\begin{aligned} *\,(01) &= (23) \\ *\,(02) &= -(13) \\ *\,(03) &= (12) \end{aligned} \qquad\qquad \begin{aligned} *\,(12) &= (03) \\ *\,(13) &= -(02) \\ *\,(23) &= (01) \end{aligned}$$

$$*^2 \; = \; 1$$

Claim: This map is well - defined on the linear space of k - forms in Euclidean space and do NOT depend on orthogonal basis.

Proof for $k = 1\,, \quad n = 2\,$:



$$\begin{aligned} *\,(e_1) &= \quad e_2 \\ *\,(e_2) &= -e_1 \end{aligned}$$

Take any unit vector $v = a\,e_1 + b\,e_2$. We have

$$w = *(v) = a\,e_2 - b\,e_1$$

So : $|v| = |w|$ and $\langle v, w\rangle = 0$.

OK.

Now define $*$ operator for Pseudoeuclidean (Lorentzian) metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix} \quad , \qquad ds^2 = (dx^0)^2 - \sum_{\alpha=1}^{n} (dx^\alpha)^2$$

$$\begin{aligned} e_0, \dots, e_n \quad &- \quad \text{basis} \qquad (x^0 = ct,\, x^1, \dots, x^n) \\ dx^0, \dots, dx^n \quad &- \quad \text{dual basis} \qquad \langle dx^i, e_j\rangle = \delta^i_j \end{aligned}$$

Definition.

$$*\,(i_1, \dots, i_k) = (-1)^s\, (j_1, \dots, j_{n+1-k})\, (-1)^\sigma$$

$$\sigma = \begin{pmatrix} 0 & \dots & \dots & \dots & \dots & n \\ i_1 & \dots & i_k & j_1 & \dots & j_{n+1-k} \end{pmatrix}$$

s = number of times we have "negative" squares among the indices $(i_1, \dots, i_k)$.
We have: either $s = k$ (all $i_k > 0$) or $s = k - 1$ ($i_1 = 0$).

Examples.

$$*\,(0) = (12\dots n) \;, \qquad *\,(1) = (02\dots n) \;, \qquad *\,(2) = -\,(013\dots n) \;, \qquad \dots$$

$$*\,(0\, i_1 \dots i_{k-1}) = (-1)^{k-1}\, (j_1, \dots, j_{n+1-k})\, (-1)^\sigma$$

$$\sigma \;=\; \begin{pmatrix} 1 & \dots & \dots & \dots & \dots & n \\ i_1 & \dots & i_{k-1} & j_1 & \dots & j_{n+1-k} \end{pmatrix}$$

So we have for $n = 3$: **(Minkowski Space)**

$$(e_0\; e_1\; e_2\; e_3) \quad , \qquad ds^2 \;=\; (dx^0)^2 - \sum_{\alpha=1}^{3} (dx^\alpha)^2$$

$$*\,(1) \;=\; dx^0 \wedge \cdots \wedge dx^3 \quad , \qquad *\,(dx^0 \wedge \cdots \wedge dx^3) \;=\; -\,1$$

Minkowski space $(x^0,\; x^1,\; x^2,\; x^3)$:

$$*\,(0) \;=\; (123)\;, \qquad *\,(1) \;=\; (023)\;, \qquad *\,(2) \;=\; -\,(013)\;, \qquad *\,(3) \;=\; (012)$$

$$*\,(01) \;=\; -\,(23) \quad , \qquad *\,(02) \;=\; (13) \quad , \qquad *\,(03) \;=\; -\,(12)$$

$$*\,(12) \;=\; (03) \quad , \qquad *\,(13) \;=\; -\,(02) \quad , \qquad *\,(23) \;=\; (01)$$

$$*\,(012) \;=\; (3)\;, \qquad *\,(013) \;=\; -\,(2)\;, \qquad *\,(023) \;=\; (1)\;, \qquad *\,(123) \;=\; (0)$$

Electric Field

$$E \;=\; E_x\, dx \;+\; E_y\, dy \;+\; E_z\, dz$$

Magnetic Field

$$B \;=\; B_{12}\, dx \wedge dy \;+\; B_{13}\, dx \wedge dz \;+\; B_{23}\, dy \wedge dz$$

Let us put

$$B_x \;=\; B_{23} \quad , \qquad B_y \;=\; B_{31} \quad , \qquad B_z \;=\; B_{12}$$

Electromagnetic Field

$$F \;=\; E \wedge dx^0 \;+\; B$$

$$*\,F \;=\; E_x\, dy{\wedge}dz + E_y\, dz{\wedge}dx + E_z\, dx{\wedge}dy + (B_x\, dx \;+\; B_y\, dy \;+\; B_z\, dz) \wedge dx^0$$

Maxwell's Equations:

1st pair (Faraday Law):

$$d\,F \;=\; 0 \qquad \Rightarrow \qquad dB \;=\; 0 \quad , \qquad dE \;=\; \frac{1}{c}\,\frac{\partial B}{\partial t} \quad , \qquad\qquad (x^0 \;=\; c\,t)$$

2nd pair:

$$d \ast F \;\;=\;\; \ast\,(4 - \text{Current})$$

"4 - Current" $=$ $J^{(4)}$ (1 - form)

4 - Current:

$$(\rho,\ -J_x,\ -J_y,\ -J_z) \;\;=\;\; J^{(4)} \;\;=\;\; \big(\rho,\ -J^{(3)}\big)$$

$$\rho\,dx^0 \;-\; J_x\,dx \;-\; J_y\,dy \;-\; J_z\,dz \;\;=\;\; J^{(4)}$$

Claim:

$$d \ast J^{(4)} \;\;=\;\; 0$$

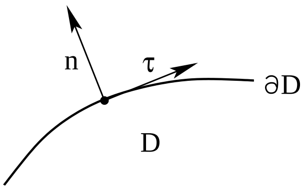
$$\ast\,J^{(4)} \;\;=\;\; \rho\,dx \wedge dy \wedge dz \;-\; dx^0 \,\wedge\, (J_x\,dy \wedge dz \;+\; J_y\,dz \wedge dx \;+\; J_z\,dx \wedge dy)$$

$$\frac{1}{c}\,\frac{\partial \rho}{\partial t} \;\;=\;\; -\;\text{div}\;J^{(3)}$$

Lecture 22. Stokes Formula.

$$\int_D d\,\Omega_k \;=\; \int_{\partial D} \Omega_k$$

D - any body, ∂D - its boundary (orientation is induced).

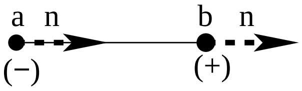


τ - tangent to ∂D,
n - external normal,
$(n,\, \tau)$ - orientation of ∂D,
τ - orientation of ∂D.

Examples.

$k \,=\, 0$ (Newton - Leibnitz)

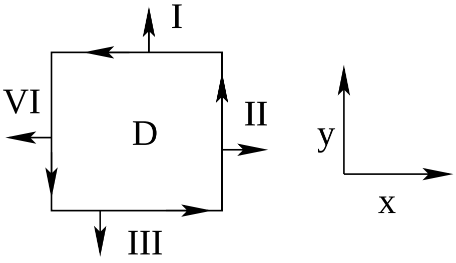
$\Omega_0 \,=\, f(x),$



$$\int_a^b d\,f \;=\; f(b) \,-\, f(a),$$

$D \,=\, [a,\, b], \quad \partial\, D \,=\, (a \,\cup\, b)$

$k \,=\, 1$



$(x,\, y)$ - orientation in $\mathbb{R}^2$.
Cube D - $n = 2$: square I^2.

(I) :	$n \,=\, y$,	$\tau \,=\, -x$	$(y,\, -x) \;\sim\; +$
(II) :	$n \,=\, x$,	$\tau \,=\, y$	$(x,\, y) \;\sim\; +$
(III) :	$n \,=\, -y$,	$\tau \,=\, x$	$(-y,\, x) \;\sim\; +$
(IV) :	$n \,=\, -x$,	$\tau \,=\, -y$	$(-x,\, -y) \;\sim\; +$

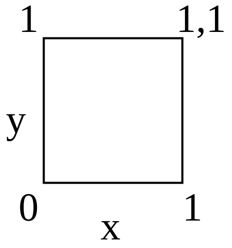
Proof of theorem for I^2:
Let $D \,=\, I^2$ and $\Omega_1 \,=\, f(x,\, y)\, dx$.

We have:

$$d\,\Omega_1 \;=\; \frac{\partial f}{\partial y}\;dy\wedge dx \;=\; -\,f_y\;dx\wedge dy$$

$$\iint_D d\,\Omega_1 \;=\; -\int_0^1\int_0^1 dy\,dx\;f_y \;=\; -\int_0^1 dx\;(f(x,\,1)\,-\,f(x,\,0))$$

$$\int_{\partial D}\Omega_1 \;=\; -\int_0^1 f(x,\,1)\,dx \;+\;\int_0^1 f(x,\,0)\,dx \;=\;\int_0^1 dx\;[f(x,\,0)\,-\,f(x,\,1)]$$



OK. This is proved.

Proof for $D\,=\,I^n$, any $n\geq 2$:

same as for $n=2$:

$$\int_{\partial D}\Omega_{n-1} \;=\; \int_D d\,\Omega_{n-1}$$

Let

$$\Omega_{n-1} \;=\; f(x^1,\ldots,x^n)\;dx^1\wedge\cdots\wedge dx^{n-1}$$

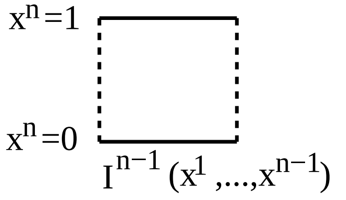
$$d\Omega_{n-1} \;=\; \frac{\partial f}{\partial x^n}\;dx^n\wedge dx^1\wedge\cdots\wedge dx^{n-1} \;=\; (-1)^{n-1}\;dx^1\wedge\cdots\wedge dx^n\;\frac{\partial f}{\partial x^n}$$

We have

$$\int_D d\,\Omega_{n-1} \;=\; (\pm)\int_0^1\ldots\int_0^1 dx^1\ldots dx^{n-1}\int_0^1\frac{\partial f}{\partial x^n}\,dx^n \;=$$

$$=\;(\pm)\int_0^1\ldots\int_0^1 dx^1\ldots dx^{n-1}\;\big(f(x^1,\ldots,x^{n-1},1)\;-\;f(x^1,\ldots,x^{n-1},0)\big)$$

$$\int_{\partial D}\Omega_{n-1} \;=\; \pm\left(\int_{x^n=1}\Omega_{n-1}\;-\;\int_{x^n=0}\Omega_{n-1}\right)$$



$$\Omega_{n-1} \;=\; f(x^1,\dots,x^{n-1},x^n)\,dx^1\wedge\cdots\wedge dx^{n-1}$$

We see that results are the same (check sign!).

Theorem is proved for $D = I^n$, $k = n-1$.

Now we prove this theorem for every manifold U of any dimension M, $U \subset \mathbb{R}^M$, Ω_{n-1} - (n-1)-form in U.

$$\varphi : \quad I^n \;\to\; U$$

- "singular cube" (smooth map).

We need to prove

$$\varphi^*\,\Omega \;=\; \tilde{\Omega}_{n-1}$$

(pull-back) in cube I^n.

By definition

$$\int\limits_{\varphi\,:\,I^{n-1}\to U} \Omega \qquad = \qquad \int\limits_{I^{n-1}} \varphi^*\Omega$$

So our theorem follows from the result above.

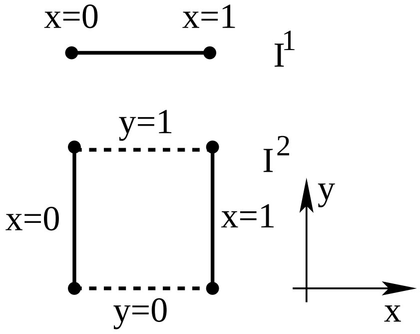
OK.

Lecture 23. Algebraic Boundary.

Stokes formula:

$$\int_D d\Omega \;=\; \int_{\partial D} \Omega$$

We proved it for cube I^n, $\{0 \le x_j \le 1\}$, $j = 1, \ldots, n$. But every convex body is isomorphic to cube up to change of coordinates. We calculate now boundary of cubes and simplices with orientation.



$$\partial I^n \;=\; \bigcup_{i=1}^{n} \left(I^{n-1}_{x_i=1} \cup I^{n-1}_{x_i=0}\right)$$

Orientation of cube is given by coordinates $(x_1, \ldots, x_n)$, basis $e_1, \ldots, e_n$, where $e_j \;=\; \partial/\partial x_j$.

Calculate orientation of $I^{n-1}_{x_i=0}$ and $I^{n-1}_{x_i=1}$ induced by $(x_1, \ldots, x_n)$ orientation of I^n :

We have $(n \;=\; x_i)$ for $I^{n-1}_{x_i=1}$ and $(n \;=\; -x_i)$ for $I^{n-1}_{x_i=0}$. We have $\tau \;=\; (x_1, \ldots, \hat{x}_i, \ldots x_n)\,(-1)^{?}$. For $(n,\, \tau)$ we have

$$(x_i, x_1, \ldots, \hat{x}_i, \ldots x_n)\,(-1)^{?} \;=\; (-1)^{?+i-1}\,(x_1, \ldots, x_n)$$

So $? + i - 1 = 0$, $? = i - 1 \pmod 2$.

Final answer is

$$\partial I^n \;=\; \sum_{i=1}^{n} (-1)^{i-1} \left(I^{n-1}_{x_i=1} \;-\; I^{n-1}_{x_i=0}\right)$$

("Algebraic Boundary").

Stokes Formula is

$$\int_{I^n} d\,\Omega_{n-1} \;=\; \sum_{i=1}^{n} (-1)^{i} \left(\int_{I^{n-1}_{x_i=1}} \Omega \;-\; \int_{I^{n-1}_{x_i=0}} \Omega\right)$$

$$\Omega \;=\; \sum_{i=1}^{n} A_i(x_1,\dots,x_n)\; dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^n$$

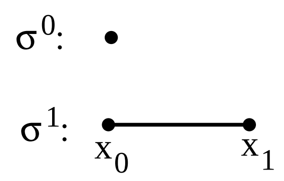
Now we calculate Algebraic Boundary for simplex $\sigma^n = (x_0,\dots,x_n)$, $x_j -$ points in $\mathbb{R}^n$ (independent).

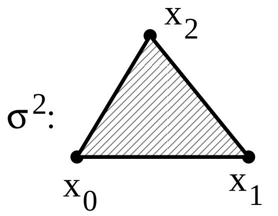
It means that

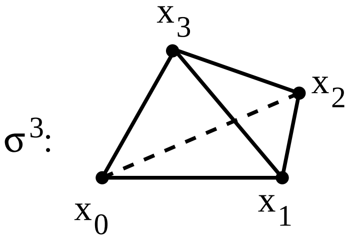
$$(x_0x_1,\dots,x_0x_n) \;=\; (e_1,\dots,e_n)$$

form a basis in $\mathbb{R}^n$.

$$x \in \sigma^n \quad \leftrightarrow \quad x = \sum_{j=0}^{n} \alpha^j\, x_j \quad , \qquad 0 \le \alpha^j \le 1 \quad , \qquad \sum \alpha^j \;=\; 1$$







Lemma 1. Orientation of simplex $(x_0,\dots,x_n)$ changes by sign $(-1)^\kappa$ for the permutation of vertices

$$(x_0,\dots,x_n) \;\xrightarrow{\kappa}\; (x_{j_0},\dots,x_{j_n})$$

Proof. Consider

$$\kappa \;=\; \begin{pmatrix} x_0 & x_1 & \dots & x_n \\ x_1 & x_0 & \dots & x_n \end{pmatrix} \quad , \qquad x_0 \leftrightarrow x_1$$

Basis $e_1 = x_0x_1$, $\dots$, $e_n = x_0x_n$ should be replaced by basis $e'_1 = x_1x_0$, $e'_2 = x_1x_2$, $\dots$, $e'_n = x_1x_n$. We have $e'_1 = -e_1$ and

$$e_2 \;=\; x_0x_2 \;=\; x_0x_1 + x_1x_2 \;=\; e_1 + e'_2$$

$$\dots$$

$$e_n \;=\; x_0x_n \;=\; x_0x_1 \,+\, x_1x_n \;=\; e_1 \,+\, e'_n$$

So, this basis has **opposite** orientation because $e'_1 \;=\; -e_1$. Iterating this argument, we obtain all permutations.

Lemma is proved.

Lemma 2. Simplex $\sigma_i^{n-1} \;=\; (x_0, \ldots, \hat{x}_i, \ldots, x_n)$ enters into boundary $\partial\sigma^n \;=\; \partial(x_0, \ldots, x_n)$ with sign $(-1)^{i-1}$.

Proof. Let $i = 0$, $\sigma_0^{n-1} \;=\; (x_1, \ldots, x_n)$ with basis $\tau' \;=\; (x_1x_2, \ldots, x_1x_n) \;=\; (e'_2, \ldots, e'_n)$.

We have $n \;\equiv\; x_0x_1 \;=\; e_1$.

Basis $(n,\, \tau')$ is

$$(x_0x_1 \;=\; e_1,\, x_1x_2 \;=\; x_1x_0{+}x_0x_2 \;=\; -e_1{+}e_2 \;=\; e'_2, \ldots,\, e'_{n-1} \;=\; x_1x_0{+}x_0x_n \;=\; -e_1{+}e_n)$$

So, we have orientation:

$$(n,\, \tau') \;=\; (-1)^{?}\,(n,\, \tau) \;\simeq\; (n,\, \tau')$$

Final result:

$$\partial\,\sigma^n \;=\; \sum_{i=0}^{n} (-1)^i\,\sigma_i^{n-1}$$

(Algebraic Boundary).

Definition.

$$\text{“Chain”} \quad = \quad \text{linear combination of} \; \begin{cases} \text{cubes (cubic chain)} \\ \text{simplexes (simplicial chain)} \end{cases}$$

$$C_n \;=\; \sum_s \lambda_s\, I^n_{(s)} \qquad \text{(cubic)}$$

$$C_n \;=\; \sum_s \lambda_s\, \sigma^n_{(s)} \qquad \text{(simplicial)}$$

Boundary of chains

$$\partial C_n \;=\; \sum_s \lambda_s \, \partial I^n_{(s)} \;=\; \sum_s \lambda_s \sum_{i=1}^{n} (-1)^{i-1} \left(I^{n-1}_{x_i=1} - I^{n-1}_{x_i=0} \right)$$

$$\partial C_n \;=\; \sum_s \lambda_s \, \partial \sigma^n_{(s)} \;=\; \sum_s \lambda_s \sum_{i=0}^{n} (-1)^{i-1} \, \sigma^{n-1}_i$$

Lemma. $\partial \circ \partial \;=\; 0$.

Topology uses the so-called

(Singularcubes $I^n_{(s)} \; : \; I^n \xrightarrow{\psi_s} U$, andSingularsimplexes $\sigma^n_{(s)} \; : \; \sigma^n \xrightarrow{\psi_s} U$)

Their algebraic boundary can be defined naturally as well as singular chains as finite linear combination of singular simplices (cubes). The cycles are chains with zero boundary. Singular homology group is defined as factor of space of cycles by the "exact " cycles which are algebraic boundaries of singular chains. The space of cycles is very big but homology group is not, it is topologically (homotopy) invariant.

Lecture 24. Differential Forms and Homotopy Poincare Lemma.

Definition. Homotopy process is a map F (C^∞ - map)

$$\begin{array}{ccc} U \times R & \xrightarrow{F} & V \\ x, t & \to & F(x,t) = y \end{array}$$

U, V - manifolds (open domains in euclidean spaces $U \subset \mathbb{R}^M$, $V \subset \mathbb{R}^N$ or other)

$$F(x,\, t = \text{const}) \;=\; f_t \;:\; U \to V$$

(deformation or homotopy of map f_t).

Maps $f_1 = f$ and $f_0 = g$ are called “homotopic maps” (C^∞ - homotopy).

Theorem. For every closed differential k - form Ω in $\Lambda^k(V)$ we have

$$f^*\Omega \;-\; g^*\Omega \;=\; du \quad , \qquad u \in \Lambda^{k-1}(U) \quad ,$$

Λ^k - space of all C^∞ differential forms on any manyfold.

Proof. We have by definition $d\Omega = 0$.
Remind.

$$d\,f^* \;=\; f^*\,d \quad , \qquad d\,g^* \;=\; g^*\,d$$

So

$$d\,(f^*\,\Omega) \;=\; d\,(g^*\,\Omega) \;=\; 0$$

Take homotopy process

$$F \;:\; U \times R \to V$$

$$F(x,t) \;: \qquad F|_{t=0} \;=\; g \;\; , \qquad F|_{t=1} \;=\; f$$

Consider k - form $F^*\Omega$ in $\Lambda^k(U \times R)$.

Every form A in $U \times R$ has the following form

$$A \;=\; a \;+\; b \wedge dt \qquad (k - \text{form})$$

where

$$a \;=\; \sum a_{i_1 \dots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \;=\; a(x,t)$$

$$b \;=\; \sum b_{j_1 \ldots j_{k-1}}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}} \;=\; b(x,t)$$

$$a(x,t) \;=\; A|_{t=\text{const}}$$

We have:

$$\underset{U \times R}{dA} \;=\; \underset{U}{da} \;+\; (-1)^k\, \dot{a} \wedge dt \;+\; \underset{U}{db} \;\wedge\; dt$$

$$\dot{a} \;=\; \frac{\partial a}{\partial t}\,(x,t) \quad , \qquad a \;=\; A|_{t=\text{const}} \qquad (\text{put } dt = 0)$$

Define operator

$$\mathcal{D} \;:\; \Lambda^k(U \times R) \;\to\; \Lambda^{k-1}(U)$$

$$\mathcal{D}\, A \;=\; \int_0^1 b \wedge dt \quad , \qquad b \;=\; b(x,t)$$

Lemma 1:

$$\pm\, (d\,\mathcal{D} \,\pm\, \mathcal{D}\, d)\; A \;=\; A|_{t=1} \;-\; A|_{t=0}$$

(sign depends on dimension and plays no role here).

Proof of Lemma.

$$\mathcal{D}\, A \;=\; \int_0^1 b \wedge dt$$

$$\mathcal{D} \;\; \underset{U \times R}{(dA)} \;=\; \mathcal{D} \;\Big(\; \underset{U}{da} \;+\; (-1)^k\, \dot{a} \wedge dt \;+\; \underset{U}{db} \;\wedge\; dt \;\Big) \;=$$

$$=\; (-1)^k\, (a|_{t=1} \;-\; a|_{t=0}) \;+\; d_U \left(\int_0^1 b(x,t) \wedge dt \right) \;=$$

$$=\; (-1)^k\, (A|_{t=1} \;-\; A|_{t=0}) \;+\; d_U \mathcal{D} A \;=\; \mathcal{D}\, dA$$

Lemma is proved.

Apply lemma to the form $A \;=\; F^*(\Omega)$ in $U \times R$. We have

$$A|_{t=1} \;=\; f^*\, \Omega \quad , \qquad A|_{t=0} \;=\; g^*\, \Omega$$

$$dA \;=\; d\, F^*(\Omega) \;=\; F^*(d\Omega) \;=\; 0$$

because $d\Omega \;=\; 0$ (closed form).

So we have

$$f^*(\Omega) \; - \; g^*(\Omega) \; = \; d\,u \quad , \qquad u = \mathcal{D}\,A$$

Theorem is proved.

Poincare Lemma: Let $U \, = \,$ Ball D^n . Every closed k - form in the Ball is **exact**.

Proof. Map

$$f \;:\; \; [0,1] \times D^n \, \to \, D^n$$

such that $f(x) \, = \, x$ is homotopic to the map g , $g(x) \, = \, 0$.

Obviously we have

$$g^*(\Omega) \; = \; 0 \quad , \qquad 1 \, \leq \, k \, \leq \, n - 1$$

So

$$f^*(\Omega) \; = \; \Omega_k \quad , \qquad \Omega_k \, - \, 0 \; = \; du$$

Poincare Lemma is proved.

Cohomology:

$$H^k(U) \quad = \quad \text{Closed forms} \,/\, \text{Exact forms}$$

$$\Omega \; \sim \; \Omega' \qquad \text{iff} \qquad d\Omega \, = \, d\Omega' \, = \, 0 \; , \qquad \Omega \, - \, \Omega' \; = \; du$$

Cohomology Ring is given by Product of forms.

Examples:

1. $U \, = \,$ point
$H^0(U) \, = \, \mathbb{R}$, $H^k(U) \, = \, 0, \; k \, \neq \, 0.$

1′. $U \, = \, l$ points
$H^0(U) \, = \, \mathbb{R}^l$, $H^k(U) \, = \, 0, \; k \, \neq \, 0.$

2. $U \, = \,$ Ball
$H^0(U) \, = \, \mathbb{R}$, $H^k(U) \, = \, 0, \; k \, \neq \, 0.$

3. $U \, = \, S^1$: $H^0 \, = \mathbb{R}$, $H^1 \, = \mathbb{R}$, $H^k \, = \, 0$, $k \, \neq \, 0, 1.$

4. $U \, = \, S^n$: $H^0 \, = \, H^n \, = \, \mathbb{R}$, $H^k \, = \, 0$, $k \, \neq \, 0, n$ (not proved yet).

5. $U \, = \,$ connected domain in $\mathbb{R}^2$, $H^0 \, = \, \mathbb{R}$, $H^1 \, = \, \mathbb{R}^p$ ($p \, = \,$ number of holes), $H^2 \, = \, 0$ (!) .

Lecture 25. Examples of important closed differential forms.

0 - forms: constant, $dc = 0$

1 - forms:

$$1) \quad d\varphi = \frac{x\,dy - y\,dx}{\rho^2} \quad , \qquad \rho^2 = x^2 + y^2$$

$$2) \quad f(z)\,dz \quad , \qquad \frac{\partial f}{\partial \bar{z}} = 0 \qquad \text{(Cauchy)}$$

$$f(z) = z^n \quad , \qquad f = \frac{1}{z} \quad , \qquad f = \frac{P(z)}{Q(z)}$$

n - **torus** $\mathbb{T}^n$: $(x^1, \dots, x^n)$, $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$
(periodic functions)

$$f(x^1, \dots, x^i + 1, \dots, x^n) = f(x^1, \dots, x^n)$$

Forms: 1, dx^i, $dx^i \wedge dx^j$, $\dots$, $dx^1 \wedge \dots \wedge dx^n$
(basis of cohomology)
$H^*(\mathbb{T}^n) = \Lambda^k$ (exterior algebra)

2 - forms Ω , $d\Omega = 0$ in $\mathbb{R}^3 \setminus 0$:

$$\frac{x\,dy \wedge dz - y\,dx \wedge dz + z\,dy \wedge dx}{r^3} \quad , \qquad r^2 = x^2 + y^2 + z^2$$

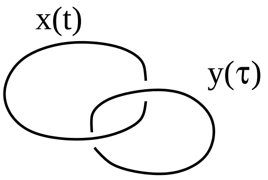
$\Omega = \sin\theta\,d\theta \wedge d\varphi$ on S^2 $(r = 1)$ - area form in S^2 .

Gauss 2 - form (for calculation of Linking Number)

$$\vec{x}(t) = \vec{x}(t+1) \quad , \qquad \vec{y}(\tau) = \vec{y}(\tau + 1) \qquad \text{in } \mathbb{R}^3$$

$$\Omega = \frac{(d\vec{x} \times d\vec{y}\,,\ \vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3} \quad - \quad 2 - \text{form in } \mathbb{R}^6 \ : \ (x^1,\, x^2,\, x^3,\, y^1,\, y^2,\, y^3)$$

minus diagonal Δ: $x^i = y^i, i = 1, 2, 3.$
$d\,\Omega = 0$ (calculation)



Property (Gauss): if $\vec{x}(t)$ does not cross $\vec{y}(\tau)$ then

$$\frac{1}{4\pi}\int_0^1\int_0^1 dt\,d\tau\;\frac{(\dot{\vec{x}}\times\vec{y}'\,,\,\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3} \quad = \quad \text{integer}$$

represents a topological invariant (linking number).

Degree of map: Take any n-form in n-sphere Ω such the $\int_{S^n}\Omega \neq 0$, for example, volume form in S^n : ($\Omega = d\varphi$ for $n=1$, $\Omega = \sin\theta\, d\theta \wedge d\varphi$ for $n=2$)

$$\int_{S^n}\Omega \;=\; 2\pi$$

Definition. For $f_t \,:\, S^n \to S^n$

$$\frac{1}{2\pi}\int_{S^n} f_t^*(\Omega)$$

divided by $\int_{S^n}\Omega$, is homotopy invariant $dI/dt \;=\; 0$
It is called Degree of Map.
It is integer.

Proof for $n=1$:

$$\begin{array}{ccccc} f & : & S^1 & \to & S^1 \quad , \quad \Omega = d\varphi \\ & & \psi & & \varphi \end{array}$$

$$f\,(\psi+2\pi) \;=\; f(\psi) \,+\, 2\pi m$$

Partial cases: $m \;=\;$ degree of map $\varphi = f(\psi)$

$$1) \qquad m \;=\; \frac{1}{2\pi}\int_0^{2\pi} f^*\,(d\varphi) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{d\psi}\,d\psi \;=\; m$$

2) Let n be any and f is orientation preserving diffeomorphism (one-to-one), so

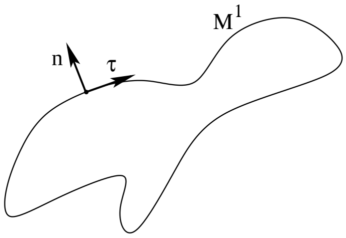
$$\int_{S^n} f^*\,(\Omega) \;=\; \int_{S^n}\Omega$$

, so **degree =1**.

Gauss Map

First case: Curve $M^1 \subset \mathbb{R}^2$, $(x(s), y(s))$

Gauss map:



$$M^1 \xrightarrow{G} S^1 \quad , \quad s \to \tau(s)$$

$$\tau = (dx/ds, dy/ds) \quad , \qquad |\tau| = 1$$

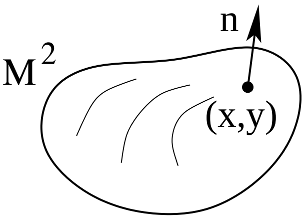
Definition: $G^*(d\varphi) = k\, ds$, s is length, k - curvature.

Corollary:

$$\frac{1}{2\pi} \oint_{M^1} G^*\, d\varphi = m \in \mathbb{Z}$$

(number of rotations).

Second case: $M^2 \subset \mathbb{R}^3$ is a boundary of some convex body



$$G : M^2 \to S^2 \quad , \quad (x, y) \to \vec{n}(x, y)$$

$$\Omega = \text{area form} \quad , \quad \iint_{S^2} \Omega = 2\pi$$

$$G^*\,\Omega = K\, d^2\sigma$$

$d^2\sigma$ - area element in M^2.

Corollary:

$$\iint_{M^2} K\, d^2\sigma = 4\pi$$

G is one-to one: degree of map $= 1$.

Kelvin Integral

$X -$ 1 - form in S^3 ("vector field" $X = \sum a_i\, dx^i$)
$X \wedge dX -$ 3 - form in S^3 ("vorticity" $dX = \Omega =$ "curl X" $-$ closed 2 - form)

"Hopf invariant"

$$S^3 \xrightarrow{f} S^2$$

(Whitehead, 1950's). Let

$$dX \;=\; f^*\,\Omega \quad , \qquad \int_{S^2} \Omega \;=\; 0$$

Claim: Kelvin - Whitehead Integral is homotopy invariant.

$$I(t) \;\equiv\; \int_{S^3} X \wedge dX \quad , \qquad dX \;=\; f_t^*(\Omega) \quad , \qquad X(t)\,,\; dX(t)$$

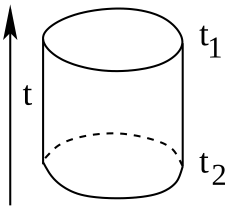
Proof:

$$\frac{dI}{dt} \;=\; 0\,?$$

Consider Homotopy Process

$$F \;=\; \{f_t\} \,:\; S^3 \times \mathbb{R} \;\to\; S^2 \quad , \qquad F^*\,\Omega \;=\; dX$$

"Kelvin Integral" in $S^3 \times \mathbb{R}$:



3 - form: $X \wedge dX \;,\; dX \;=\; F^*(\Omega)\,.$

It is closed 3 - form in $S^3 \times \mathbb{R}$. Why?

$$0 \;=\; F^*\,(\Omega \wedge \Omega) \;=\; dX \wedge dX \;=\; d\,(X \wedge dX)$$

So

$$\int_{t=\text{const}} X \wedge dX \;=\; I(t)$$

is t - independent.

OK.

Lecture 26.

Riemannian Metric in $U \; : \; x^1, \ldots, x^n :$

$$ds^2 \; = \; \sum_{i,j} g_{ij}(x) \; dx^i \, dx^j \quad , \qquad g_{ij} \; = \; g_{ji} \; , \qquad dx^i \, dx^j \; = \; dx^j \, dx^i$$

Riemannian: $ds^2 \; > \; 0\,.$

Pseudo-Riemannian: $\det g_{ij} \; \neq \; 0$ (indefinite).

Lorentzian Case: Signature $(p, \, q) \; = \; (1, \, n-1)\,.$

Inner Product in every point x:

$$\langle e_i \, , \, e_j \rangle \; = \; g_{ij} \; , \qquad e_i \; \leftrightarrow \; \frac{\partial}{\partial x^i} \qquad - \qquad \textbf{basis}$$

$$n \; = \; 2 \quad : \qquad ds^2 \; = \; F \, du^2 \; + \; 2G \, du \, dv \; + \; H \, dv^2$$

$$g_{11} \; = \; F(u,v) \; , \quad g_{12} \; = \; G(u,v) \; , \quad g_{22} \; = \; H(u,v)$$

(1^{st} quadratic form).

Euclidean Metric

$$ds^2 \; = \; \sum_{i=1}^{n} \; \left(dx^i\right)^2$$

Pseudoeuclidean Metric

$$ds^2 \; = \; \left(dx^0\right)^2 \; - \; \sum_{i=1}^{n} \; \left(dx^i\right)^2$$

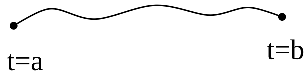
(Lorentzian).

Curve in $\mathbb{R}^n$: $(x^1(t), \, \ldots, \, x^n(t))\,.$ What is restriction of metric to curve?

$$ds^2|_\gamma \; = \; g_{ij}\left(x(t)\right) \, \frac{dx^i}{dt} \, \frac{dx^j}{dt} \; = \; \langle \dot{x}(t) \, , \, \dot{x}(t) \rangle \; = \; |\dot{x}|^2$$

"Length" (arc length) is:

$$l(\gamma) \;=\; \int_a^b |\dot{x}|\, dt$$



It is NOT a differential 1-form.

Surfaces: $x^i(u,v)\ ,\quad i = 1,\ldots,n \qquad (M^2)\,.$

Restriction of metric:

$$ds^2|_{M^2} \;=\; g_{ij}\,(x(u,v))\;\,(dx^i)|_{M^2}\,(dx^j)|_{M^2}$$

$$dx^i|_{M^2} \;=\; \frac{\partial x^i}{\partial u}\,du \;+\; \frac{\partial x^i}{\partial v}\,dv$$

Euclidean Case: $M^2 \subset \mathbb{R}^n$:

$$ds^2|_{M^2} \;=\; \sum_{i=1}^n (dx^i)^2|_{M^2} \;=\; \sum_{i=1}^n \left(\frac{\partial x^i}{\partial u}\,du \;+\; \frac{\partial x^i}{\partial v}\,dv\right)^2 \;=$$

$$=\; F\,du^2 \;+\; 2G\,du\,dv \;+\; H\,dv^2$$

$$F \;=\; \sum_{i=1}^n \left(\frac{\partial x^i}{\partial u}\right)^2 , \qquad G \;=\; \sum_{i=1}^n \left(\frac{\partial x^i}{\partial u}\right)\left(\frac{\partial x^i}{\partial v}\right) , \qquad H \;=\; \sum_{i=1}^n \left(\frac{\partial x^i}{\partial v}\right)^2$$

$n \,=\, 3 \;:\; (x,\,y,\,z)$:

Let surface is given as

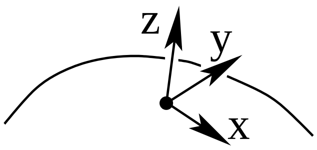
$$z \;=\; f(x,y)\ , \qquad x \,=\, u\ ,\ \ y \,=\, v$$

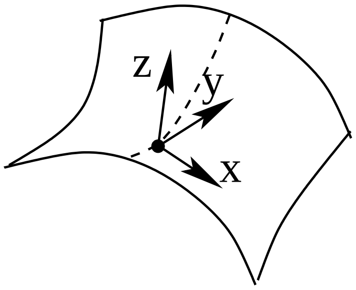
$$ds^2 \;=\; dx^2 + dy^2 + dz^2 \;=\; dx^2\left(1 + f_x^2\right) \,+\, dy^2\left(1 + f_y^2\right) \,+\, 2dxdy\;(f_x f_y)$$

$$f_x \;=\; \frac{\partial f}{\partial x} \quad , \qquad f_y \;=\; \frac{\partial f}{\partial y}$$

Let

$$f_x|_{(x,y)=(0,0)} \;=\; f_y|_{(x,y)=(0,0)} \;=\; 0$$





$$z \;=\; a\,x^2 \;+\; 2b\,xy \;+\; c\,y^2 \;+\; o\,(x^2+y^2)$$

Definition: "Curvature form" at the point $(0,0)$ is equal to 2^{nd} differential d^2z if

$$z \;=\; f(x,y) \qquad \text{and} \qquad f_x|_{(x,y)=(0,0)} \;=\; f_y|_{(x,y)=(0,0)} \;=\; 0$$

Riemannian Metric g_{ij} at the surface is equal to

$$g_{11} \;=\; 1 \,+\, z_z^2 \quad , \qquad g_{12} \;=\; z_x\, z_y \quad , \qquad g_{22} \;=\; 1 \,+\, z_y^2$$

if $z \,=\, f(x,y)\,.$

It is unit matrix at the point $(0,0)$ if $z_x \,=\, z_y \,=\, 0$ at this point.

Curves: $\gamma \subset \mathbb{R}^2$, $x(t)$, $y(t)\,.$

$$ds|_\gamma \;=\; \sqrt{(dx)^2+(dy)^2} \;=\; \sqrt{\dot{x}^2+\dot{y}^2}\; dt$$

"Natural Parameter" $=$ arc length.

$$x(s)\ ,\ y(s)\ , \qquad \tau(s) \;=\; \left(\frac{dx}{ds}\,,\,\frac{dy}{ds}\right)$$

Lemma 1. $|\tau(s)| \,=\, 1\,.$

Proof:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \qquad \text{by definition.}$$

Lemma 2. $d\tau/ds \perp \tau$.

Proof.

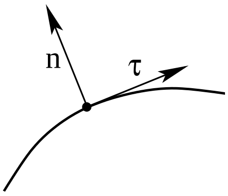
$$\frac{d}{ds}\langle\tau\,,\,\tau\rangle = 0 = 2\langle\frac{d\tau}{ds}\,,\,\tau\rangle$$

OK.

Conclusion.

$$\frac{d\tau}{ds} = k\,n$$

where n is normal vector.



Lemma 3. $dn/ds = -k\,\tau(s)$.

Proof.

$$A(s) = \{\tau(s)\,,\,n(s)\} \in SO_2$$

$$\langle A(s)\eta\,,\,A(s)\zeta\rangle = \langle\eta\,,\,\zeta\rangle$$

$$A(s=0) = I$$

$$\frac{d}{ds}\langle A(s)\eta\,,\,A(s)\zeta\rangle = 0 = \langle\dot{A}\eta\,,\,\zeta\rangle + \langle\eta\,,\,\dot{A}\zeta\rangle$$

Conclusion: $\dot{A}^t = -A$

$$\frac{d}{ds}(\tau\,,\,n) = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}(\tau\,,\,n)$$

Statement: (Frenét Formulas are partial cases of the following general property:)

$$A(t) \in O_n \qquad \Rightarrow \qquad (dA/dt)A^{-1} \quad - \quad \text{skew symmetric}$$

as well as $A^{-1}(dA/dt)$.

Lecture 27. Curvature.

$$\gamma \; : \quad x(t) \; , \; y(t) \quad \text{in} \quad \mathbb{R}^2 \quad , \qquad \vec{x} \; = \; (x, \, y)$$

Let t be a "Natural parameter"

$$t \; = \; s \; : \quad |(\dot{x}, \dot{y})|^2 \; = \; \dot{x}^2 \, + \, \dot{y}^2 \; = \; 1$$

Arc Length

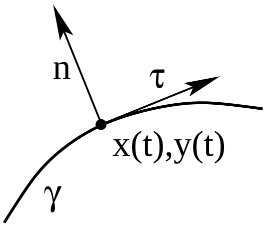
$$l \; = \; \int_a^b |\dot{\vec{x}}| \; dt \; = \; b \, - \, a \quad , \qquad t \; = \; s$$

$$\dot{\vec{x}} \; = \; \tau(s) \quad , \qquad |\tau| \; = \; 1$$

Lemmas.
Curvature.

$$1. \qquad \frac{d\tau}{ds} \; = \; k \, n \quad , \qquad n \; \perp \; \tau$$

Proof.



$$\langle \tau, \tau \rangle \; = \; 1 \quad , \quad \frac{d\langle \tau, \tau \rangle}{ds} \; = \; 0 \; = \; \langle \tau, \frac{d\tau}{ds} \rangle$$

$$\frac{d\tau}{ds} \; = \; k \, n$$

OK.

$$2. \qquad A \, \in \, O_n \quad , \qquad A \; = \; A(t) \quad , \qquad A(0) \; = \; 1$$

We have

$$B^t \; = \; - \, B \quad , \qquad B \; = \; \left. \frac{dA}{dt} \right|_{t=0}$$

Proof.

$$\langle A\eta, A\zeta \rangle \; = \; \langle \eta, \zeta \rangle \quad , \qquad A \; = \; A(t) \quad , \qquad A(t) \; = \; 1 \, + \, B \, t \, + \, O(t^2)$$

$$\frac{d}{dt}\langle A\eta, A\zeta\rangle \;=\; 0 \;=\; \frac{d}{dt}\left[\langle \eta,\zeta\rangle \;+\; t\,\langle B\eta,\zeta\rangle \;+\; t\,\langle \eta, B\zeta\rangle \;+\; O(t^2)\right]\Big|_{t=0}$$

Conclusion:

$$\langle B\eta,\zeta\rangle \;=\; -\,\langle \eta, B\zeta\rangle$$

OK.

Conclusion. Let

$$A(s) \;=\; [\tau(s)\,,\, n(s)] \;, \qquad \frac{dA}{ds}\bigg|_{s=s_0} \;=\; A(s_0)\;B(s_0)\;, \qquad B^t \;=\; -\,B$$

$$\frac{d\tau}{ds} \;=\; k\,n \quad , \qquad \frac{dn}{ds} \;=\; -\,k\,\tau$$

It is true because

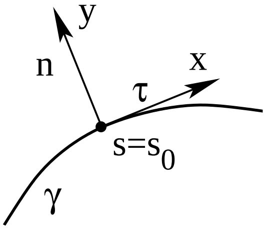
$$A(s) \;=\; A(s_0)\,A(s_0)^{-1}\,A(s) \quad ,$$

$$A(s_0)^{-1}\,A(s) \;=\; 1 \;+\; B(s_0)\,(s\,-\,s_0) \;+\; O\left((s\,-\,s_0)^2\right) \;\in\; O_2 \quad ,$$

$$B^t(s_0) \;=\; -\,B(s_0)$$

Another definition of curvature:

$$x(s)\;,\; y(s) \quad , \qquad x(s_0) \;=\; 0\;,\; y(s_0) \;=\; 0$$



$$y \;=\; f(x) \quad (\text{curve } \gamma) \quad , \qquad \frac{dy}{dx}\bigg|_{x=0} \;=\; 0$$

$$y \;=\; y_0 \;+\; a\,x^2 \;+\; O(x^2)/2$$

Definition: $k \;=\; a$

$$\tau \;=\; e_x\,|_{x=0} \quad , \qquad n \;=\; e_y\,|_{x=0}$$

We have

$$s \;=\; s(x) \quad , \qquad \frac{ds}{dx}\bigg|_{x=0} \;=\; 1$$

Calculate $dn/dx\,|_{x=0} \;=\; ?$

For the curve $F(x,\, y) \;=\; 0$ we have

$$n \;=\; \frac{(F_x,\, F_y)}{\sqrt{F_x^2 + F_y^2}} \quad , \qquad F(x,\, y) \;=\; y \,-\, f(x) \quad , \qquad n \;=\; \frac{(-f_x,\, 1)}{\sqrt{1 + f_x^2}}$$

$$\left.\frac{dn}{dx}\right|_{x=0} \;=\; -\,(f_{xx},\, 0) \;=\; \left.\frac{dn}{ds}\,\frac{ds}{dx}\right|_{x=0} \;=\; \left.\frac{dn}{ds}\right|_{s=0}$$

So

$$\frac{dn}{ds} \;=\; -\,k\,\tau \;=\; -\,f_{xx}$$

(Frenét) .

Conclusion: $k \;=\; f_{xx}$.

OK.

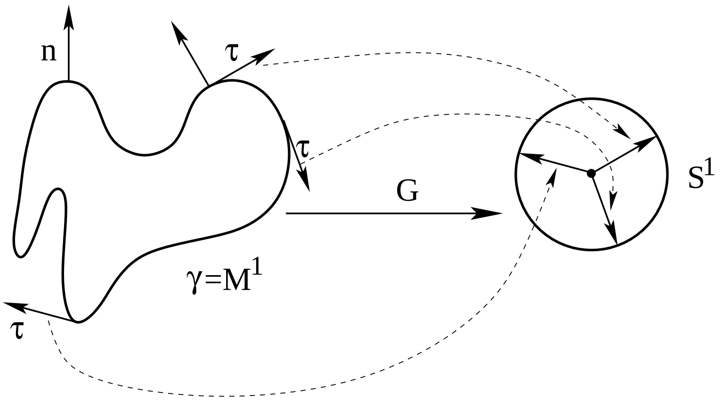
One more definition of Curvature (differential 1 - form)

$$\gamma \;=\; M^1 \;\subset\; \mathbb{R}^2 \;, \qquad x(s)\;,\;\; y(s)\;, \qquad |(\dot{x}(s),\, \dot{y}(s))| \;=\; 1$$

$(\tau(s) \,\neq\, 0$ for all $s)$.

Gauss map

$$\gamma \;=\; M^1 \;\xrightarrow{G}\; S^1$$



Another form

$$(x,\, y) \;\to\; \vec{n}(x,\, y) \quad , \qquad (x,\, y) \;\in\; \gamma \;=\; M^1$$

Definition. $k\,ds \;=\; G^*(d\varphi)$ (pull back) .

We have $(x(s),\, y(s)) -$ periodic functions

$$x\,(s+T) \;=\; x(s) \quad , \qquad y\,(s+T) \;=\; y(s)$$

$$s \;-\; \text{local parameter on } \gamma \subset \mathbb{R}^2$$

$$\varphi \;-\; \text{local parameter on } S^1 \subset \mathbb{R}^2$$

$$G(s+T) \;=\; G(s) \;+\; m \cdot (2\pi) \;. \qquad \text{Why?}$$

Because

$$G(s) \;=\; \varphi \quad , \qquad \varphi + 2\pi m \;\simeq\; \varphi$$

Claim. $m \in \mathbb{Z}$.

$$m \;=\; \text{“degree of map”} \;, \qquad m \;=\; \frac{1}{2\pi} \oint k\,ds \quad (!)$$

Lecture 28. Curvature.

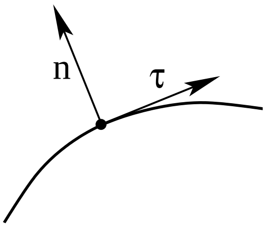
Curves:

$$\gamma \;=\; M^1 \;\subset\; \mathbb{R}^2 \quad , \qquad x(t)\ ,\ y(t)$$

Arc length

$$ds \;=\; \sqrt{\dot{x}^1 \,+\, \dot{y}^2}\; dt \quad , \qquad l(\gamma) \;=\; \int_a^b ds$$

Curvature: $\pm\, k \;=\; |d\tau(s)/ds|\,.$



$$|\tau(s)| \;=\; 1 \quad , \qquad d\tau/ds \;=\; k\,n$$

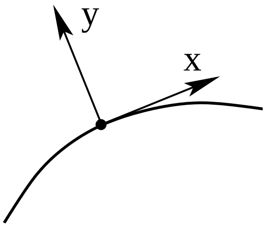
Oriented frame: $(\tau,\, n)\,.$

Frené:

$$\frac{d\tau}{ds} \;=\; k\,n \quad , \qquad \frac{dn}{ds} \;=\; -\,k\,\tau$$

$$A(s) \;=\; (\tau(s)\; n(s)) \quad , \qquad B(s) \;=\; A^{-1}(s)\,\frac{dA}{ds} \;=\; \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$$

Curvature: $y \;=\; f(x)\,.$



$$\left.\frac{dy}{dx}\right|_{x=0} \;=\; 0 \quad , \qquad k\,|_{x=0} \;=\; \frac{d^2y}{dx^2}$$

(last lecture).

$$\left.\frac{ds}{dx}\right|_{x=0} \;=\; 1 \quad , \qquad ds^2\,\big|_{x=0} \;=\; dx^2 \quad , \qquad ds^2 \;=\; dx^2\left(1 \,+\, \left(\frac{dy}{dx}\right)^2\right)$$

Gauss Map

$$G \quad : \qquad M^1 \;\to\; S^1 \quad , \qquad G(x) \;=\; n(x) \qquad (\text{or } \; G(x) \;=\; \tau(x))$$

$F(x,\, y) \;=\; 0$:

$$n \;=\; \frac{(F_x,\, F_y)}{\sqrt{F_x^2 + F_y^2}} \quad , \qquad F \;=\; y \,-\, f(x) \quad : \qquad n(x) \;=\; \frac{(-f_x,\, 1)}{\sqrt{1 + f_x^2}}$$

$$\tau(x) \quad : \qquad \text{rotate } \; n \; \text{ by } \; 90^o$$

Definition:

$$G^*(d\varphi)/2\pi \;=\; k\, ds \quad , \qquad \oint_{S^1} d\varphi/2\pi \;=\; 1 \qquad (?)$$

Claim: For closed curve $M^1 \;=\; c$:

$$\oint_c k\, ds \;=\; m \;\in\; \mathbb{Z}$$

What is “degree of map” ?

$$G \qquad : \qquad \underset{(s)}{M^1} \quad \to \quad \underset{(\varphi)}{S^1}$$

closed curve $x\,(s + T) \;=\; x(s)$, $y\,(s + T) \;=\; y(s)$:

$$G\,(s + T) \quad = \quad G(s) \;+\; 2\pi m \;, \qquad m \;\in\; \mathbb{Z}$$

Proof: Points $G(s)$ and $G(s + T)$ are **the same** on S^1 , $\varphi \,+\, 2\pi m \;\simeq\; \varphi$.
OK.

Proof that $G^*(d\varphi)/2\pi \;=\; k\, ds$ (later).

Curvature: $n \;=\; 2$:

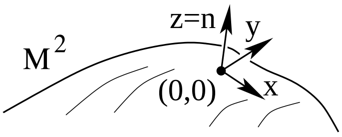
$$M^2 \;\subset\; \mathbb{R}^3 \quad : \qquad x\,(u,\, v) \;,\;\; y\,(u,\, v) \;,\;\; z\,(u,\, v)$$

Riemannian Metric: $dx^2 + dy^2 + dz^2 = ds^2$

$$g_{11} = F = x_u^2 + y_u^2 + z_u^2 \quad , \qquad g_{12} = G = x_u\, x_v + y_u\, y_v + z_u\, z_v \quad ,$$

$$g_{22} = H = x_v^2 + y_v^2 + z_v^2$$

Let



$$u = x \ , \quad v = y \ , \quad z = f(x, y)$$

and

$$z_x|_{(0,0)} = z_y|_{(0,0)} = 0$$

$$z = a + \frac{1}{2}\left(f_{xx}\, x^2 + 2\, f_{xy}\, x\, y + f_{yy}\, y^2\right) + o\,(x^2 + y^2) \qquad \text{(near(0, 0))}$$

Definition.

$$k_{ij} = \left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_{(0,0)} \quad , \qquad x^1 = x \ , \quad x^2 = y$$

$$k_{11} + k_{22} = k_{\text{mean}}$$

$$k_{11}\, k_{22} - k_{12}^2 = K \qquad \text{(Gaussian Curvature)}$$

$$\text{Area Form} = \sqrt{\det g_{ij}}\ dx^1 \wedge dx^2 = d^2\sigma \qquad (= dx \wedge dy \ \text{ at } (0, 0))$$

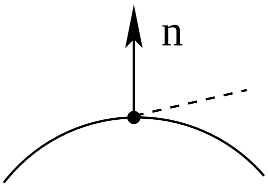
Theorem.

$$K\ d^2\sigma = G^*\,(\Omega)$$

$-$ pull-back Ω is an area 2 - form on $S^2 \subset \mathbb{R}^3$

$$G \quad : \qquad (x,\, y,\, z) \;\to\; n\,(x,\, y,\, z)$$

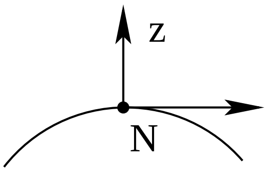
$$M^2 \quad \to \quad S^2$$



$$\Omega \quad = \quad \sin\theta \; d\theta \wedge d\varphi \qquad \text{(spherical coordinates)}$$

$$\Omega \;=\; \Big(x\, dy \wedge dz \;-\; y\, dx \wedge dz \;+\; z\, dx \wedge dy \Big)\Big|_{r^2=1} \quad , \qquad r^2 \;=\; x^2 + y^2 + z^2$$

Describe form Ω in special coordinates:



$$z' \;=\; \sqrt{1 - x'^2 - y'^2} \qquad \text{near pole } N \quad (\text{in } S^2)$$

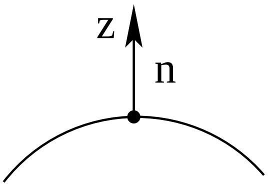
$$z'_{x'}\,|_{(0,0)} \;=\; z'_{y'}\,|_{(0,0)} \;=\; 0$$

Sphere S^2

$$(g_{ij} \;=\; \delta_{ij})\,|_{(0,0)} \;, \qquad g'_{11} \;=\; 1 \;, \quad g'_{12} \;=\; 0 \;, \quad g'_{22} \;=\; 1 \quad \text{for} \quad x' \;=\; 0 \,, \;\; y' \;=\; 0$$

$$d^2\sigma \;=\; \Phi(x',\, y')\; dx' \wedge dy' \quad , \qquad \Phi\,(0,\, 0) \;=\; 1 \quad \text{in} \quad S^2 \qquad (x' \;=\; 0 \,, \;\; y' \;=\; 0)$$

Define Guass Map in such coordinates:



$$M^2 \;\subset\; \mathbb{R}^3 \quad , \qquad z \;=\; f\,(x,\, y) \quad ,$$

$$z \;\perp\; M^2 \quad \text{for} \quad x' \;=\; 0 \,, \;\; y' \;=\; 0$$

$$G\,(x,\, y) \quad : \qquad (x,\, y) \;\to\; n \quad , \qquad n \;=\; \frac{(-f_x,\, -f_y,\, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

Gauss

$$G \;\; : \qquad n \;=\; n\,(x,\, y) \quad , \qquad n\,(0,\, 0) \;=\; (0,\, 0,\, 1) \qquad (x',\, y',\, z')$$

$(x,\, y)\ -\ $ coordinates on S^2:

$$x' \;=\; -\,\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\ , \qquad y' \;=\; -\,\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\ , \qquad z' \;=\; \frac{1}{\sqrt{1+f_x^2+f_y^2}}$$

$$dx' \;\to\; \frac{\partial x'}{\partial x}\, dx \;+\; \frac{\partial x'}{\partial y}\, dy\ , \qquad dy' \;\to\; \frac{\partial y'}{\partial x}\, dx \;+\; \frac{\partial y'}{\partial y}\, dy$$

At the point $x \,=\, 0\,,\ y \,=\, 0$ we have

$$dx' \;\xrightarrow{G^*}\; \left(\frac{\partial x'}{\partial x}\, dx \;+\; \frac{\partial x'}{\partial y}\, dy\right)\Bigg|_{(0,0)} \;=\; -\,f_{xx}\, dx \;-\; f_{xy}\, dy$$

$$dy' \;\xrightarrow{G^*}\; \left(\frac{\partial y'}{\partial x}\, dx \;+\; \frac{\partial y'}{\partial y}\, dy\right)\Bigg|_{(0,0)} \;=\; -\,f_{xy}\, dx \;-\; f_{yy}\, dy$$

$$\Omega \;=\; dx' \wedge dy' \qquad (x \,=\, 0,\ y \,=\, 0)$$

$$\Omega \;\xrightarrow{G^*}\; G^*\, dx' \,\wedge\, G^*\, dy' \;=\; (-f_{xx}\, dx \,-\, f_{xy}\, dy) \wedge (-f_{xy}\, dx \,-\, f_{yy}\, dy) \;=$$

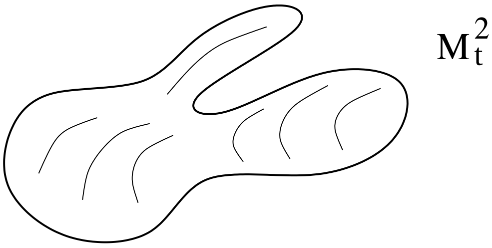
$$=\; \left(f_{xx}\, f_{yy} \;-\; f_{xy}^2\right)\, dx \wedge dy \;=\; K\ dx \wedge dy \qquad (x \,=\, 0,\ y \,=\, 0)$$

Theorem is proved.

Corollary: Let $M^2 \,\subset\, \mathbb{R}^3$ is a **closed** oriented surface.

Then

$$\iint_{M^2} K\ d^2\sigma$$



remains unchanged under smooth deformation M_t^2 of closed surface $M^2 \,\subset\, \mathbb{R}^3$.

Proof:

$$\iint_{M_t^2} K\ d^2\sigma \;=\; \iint_{M_t^2} G^*\,\Omega\ , \qquad d\,\Omega \;=\; 0 \quad \text{in} \quad S^2$$

So $K\,d^2\sigma$ changes by exact form. By Stokes Formula we have

$$\frac{d}{dt}\iint_{M^2} K\ d^2\sigma \quad = \quad 0$$

OK.

Lecture 29.

Gauss Map:

$$G \ : \quad M^2 \quad \to \quad S^2 \subset \mathbb{R}^3$$

Curvature

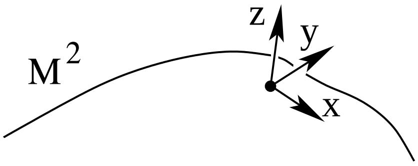
$$G^*(\Omega) \quad = \quad K \ d^2\sigma$$

$$\Omega \quad = \quad \sin\theta \ d\theta \wedge d\varphi \qquad\qquad (\text{area in } S^2)$$

$$\Omega \quad = \quad dx' \wedge dy' \,|_{(0,0)} \quad , \qquad\qquad z' \quad = \quad \sqrt{1 - x'^2 - y'^2}$$

Surface

$$M^2 \ : \qquad z \ = \ f(x,y) \qquad\qquad f_x|_{(0,0)} \ = \ f_y|_{(0,0)} \ = \ 0$$



$$d^2\sigma \,|_{(0,0)} \quad = \quad dx \wedge dy$$
$$ds^2 \,|_{(0,0)} \quad = \quad dx^2 \, + \, dy^2$$

$$\text{Gauss Map} \ : \qquad n \ = \ \frac{(-f_x, \ -f_y, \ 1)}{\sqrt{1 + f_x^2 + f_y^2}} \quad , \qquad\qquad (x, \, y) \ \xrightarrow{G} \ n\,(x, \, y)$$

$$x' \ = \ -\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}} \ , \qquad y' \ = \ -\frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}} \ , \qquad z' \ = \ \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}$$

$$dx'|_{(0,0)} \ = \ -f_{xx}\,dx \ - \ f_{xy}\,dy \quad , \qquad dy'|_{(0,0)} \ = \ -f_{yx}\,dx \ - \ f_{yy}\,dy$$

$$dx' \wedge dy' \ = \ \Omega\,|_{(0,0)} \quad \to \quad K\,dx \wedge dy \ = \ G^*(\Omega)$$

$$K\,|_{(0,0)} \;=\; f_{xx}\,f_{yy} \;-\; f_{yx}^2$$

Remark.

$$\iint_{S^2} \Omega \;=\; 4\pi$$

Normalization

$$\iint_{S^2} \frac{\Omega}{2\pi} \;=\; 2 \qquad \text{(Euler characteristics).}$$

Theorem.

a) For every closed oriented surface $M^2 \subset \mathbb{R}^3$ integral

$$\iint_{M^2} G * \Omega \;=\; \iint_{M^2} K\, d^2\sigma$$

does not change under deformation of the surface M_t^2.

b) For a convex body $\partial D = M^2$ we have

$$\iint_{M^2} K\, d^2\sigma \;=\; 4\pi$$

Proof.

a) We have homotopy process $G_t \,:\, M_t^2 \to S^2$. By theorem,

$$G_{t_1}^*\,\Omega \;-\; G_{t_2}^*\,\Omega \;=\; du$$

So

$$\frac{d}{dt} \iint_{M^2} G_t^*\,\Omega \;=\; 0$$

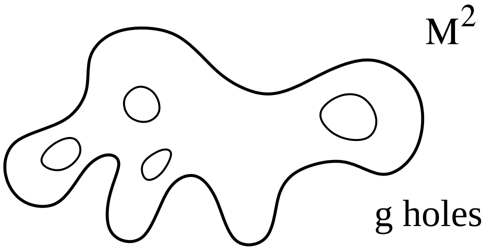
b) For $M^2 = S^2$ we have

$$\iint_{S^2} \Omega \;=\; 4\pi \quad , \qquad G \equiv 1 \;:\; S^2 \to S^2$$

Boundary of the convex body can be obtained from S^2 by deformation M_t^2.

Theorem is proved.

Remark:



$$\frac{1}{2\pi}\iint_{M^2} K\,d^2\sigma \;=\; 2 - 2g$$

Euler Characteristics, **does NOT depend on embedding**.

Reminder: Theorem (homotopy)

$$F \;:\; \underset{(x,\,t)}{M\times\mathbb{R}} \;\to\; U$$

- **smooth homotopy**.

$$f_t \;=\; F\,|_t \;:\; M \;\to\; U$$

Claim For the closed form Ω we have

$$f_1^*(\Omega) \;-\; f_0^*(\Omega) \;=\; du$$

Proof:

$$D\,\omega \;=\; \int_0^1 \omega_1\,dt \quad , \qquad \omega \;=\; \omega_0 \;+\; \omega_1\,dt$$

$$D\,\omega \;\in\; \Lambda^{k-1}(M) \quad , \qquad \omega \;\in\; \Lambda^k\,(M\times\mathbb{R})$$

$$d_{(M\times\mathbb{R})}\,\omega \;=\; d_M\omega \;\pm\; \dot\omega_0\,dt \;+\; d_M\omega_1\wedge dt$$

$$D\,d_{(M\times\mathbb{R})}\,\omega \;=\; \int_0^1 \dot\omega_0\,dt \;+\; d_M\int_0^1 \omega_1\,dt \;=\; \omega_0\,|_{t=1} \;-\; \omega_0\,|_{t=0} \;+\; d_M\,D\,\omega$$

Now let $d\,\omega \;=\; 0$. We have

$$D\,d_{(M\times\mathbb{R})}\,\omega \;=\; 0 \qquad \Rightarrow \qquad \omega_0\,|_{t=1} \;-\; \omega_0\,|_{t=0} \;=\; -\,d_M\,D\,\omega$$

Apply to $\omega = F^*(\Omega) \subset \Lambda^k(M \times \mathbb{R})$

Theorem follows.

Lecture 30.

Differential forms and homotopy:

Degree of map $S^n \xrightarrow{f} S^n$:

Take n - form Ω:

$$\int_{S^n} \Omega = 1$$

Degree of Map

$$\int_{S^n} f^*(\Omega)$$

Claim: For the closed form Ω and homotopy

$$\begin{array}{ccccc} F & : & S^n \times \mathbb{R} & \to & S^n \\ & & (x, t) & & \end{array}$$

we have

$$f_1^*(\Omega) - f_0^*(\Omega) = du$$
$$d\Omega = d(f^*\Omega) = 0$$

So

$$\int_{S^n} f_1^*(\Omega) = \int_{S^n} f_0^*(\Omega)$$

Remark: $\deg f \in \mathbb{Z}$ (proof for $n = 1$ was given):

$$f(x+T) = f(x) + m \cdot 2\pi$$

$$S^1 \xrightarrow{f} S^1 \quad \mathrm{mod}\ 2\pi \quad , \qquad x + T \sim x \quad \mathrm{mod}\ 2\pi$$

$$f(x) + 2\pi m \sim f(x) \quad \text{in} \quad S^1 \quad \mathrm{mod} \quad 2\pi$$

Hopf invariant

$$f : \quad S^3 \to S^2$$

Take Ω in S^2,

$$\int_{S^2} \Omega \;=\; 1$$

Find ω in S^3 such that

$$d\,\omega \;=\; f^*\,\Omega$$

Calculate **“Kelvin-Whitehead Integral”**

$$\int_{S^3} \omega \wedge d\omega \;=\; I(f)$$

Theorem. For

$$F \;:\; \begin{matrix} S^3 \times \mathbb{R} & \to & S^2 \\ (x,\, t) & & \end{matrix} \quad , \qquad f_t \;=\; F\,|_{t=\text{const}}$$

we have

$$\frac{dI}{dt} \;=\; 0$$

Proof.

Take $F^*(\Omega)$ in $\Lambda^2(S^3 \times \mathbb{R})$.

Find $\omega \in \Lambda^1(S^3 \times \mathbb{R})$ such that $d\,\omega \;=\; F^*(\Omega)$ in $S^3 \times \mathbb{R}$.

Calculate

$$\int_{S^3:\, t=\text{const}} \omega \wedge d\omega \;=\; I(t)$$

We have $\Omega \wedge \Omega \;=\; 0$ in $\Lambda^4\,(S^2) \;\equiv\; 0$.

$$\text{So} \quad F^*\,(\Omega) \wedge F^*\,(\Omega) \;=\; 0 \quad \text{in} \quad \Lambda^4\,\big(S^3 \times \mathbb{R}\big)$$

$$d\,(\omega \wedge d\omega) \;=\; d\omega \wedge d\omega \;=\; F^*\,(\Omega) \wedge F^*\,(\Omega) \;=\; 0$$

By Stokes we have $dI/dt \;=\; 0$.

OK.