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Singular Solitons and Indefinite Metric

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References: Novikov's Homepage
www.mi.ras.ru/~snovikov
click Publications, items 175,176

Physical Derivation of KdV (KP) in the Theory of Nonlinear Waves in Dispersive Media requires NONSINGULARITY of solutions $u_t(x,t) = 6uu_x - u_{xxx}$.

Spectral Theory of Rapidly Decreasing and Periodic Schrodinger Operators $L = -\partial_x^2 + u$ in the Hilbert Space $L_2(R)$ also requires NONSINGULARITY of Potential $u(x)$.

However, big Math Literature is dedicated to the singular KdV and KP Solutions, in particular, to the Theory of Rational and Elliptic Solutions. Are there any unsolved problems here? What can one do with them?

Example:

Let $j = 1, ..., n(n+1)/2$

there are Real Rational Solutions

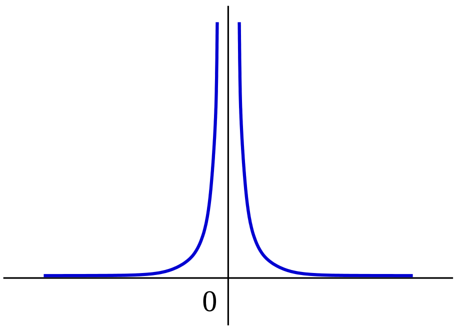
$u(x,t) = \Sigma_j 2/(x - x_j(t))^2$

and Real Elliptic Solutions

$u(x,t) = \Sigma_j 2\wp(x - x_j(t))$

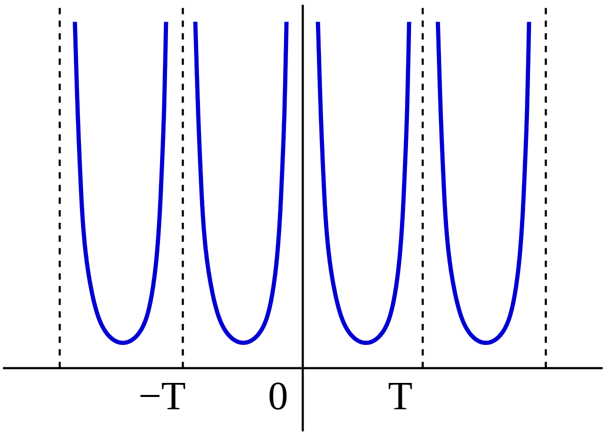
Consider following solutions:

$u(x,0) = n(n+1)/x^2$



(Rational Case)

$u(x,0) = n(n+1)\wp(x)$



(Elliptic case).

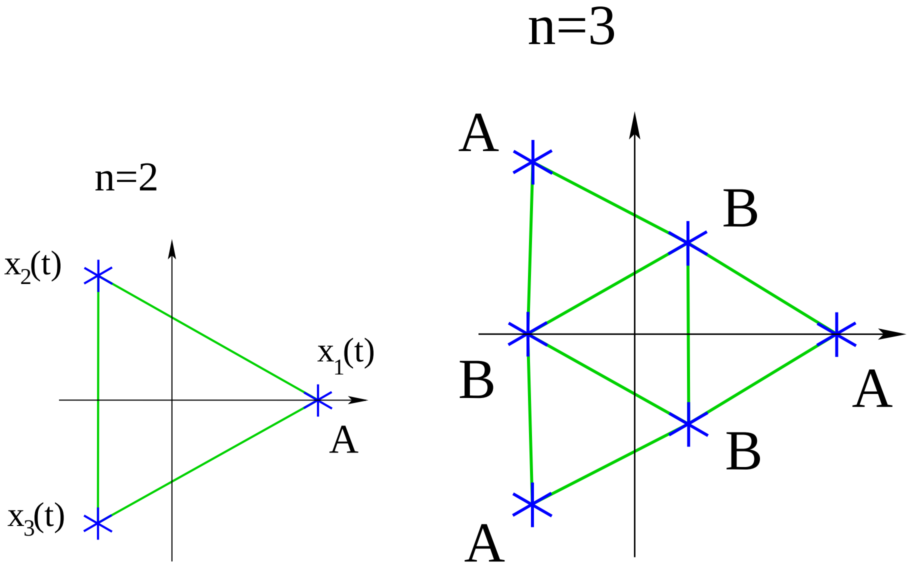
$u(x,0)$ are the famous Lame' Potentials. Hermit found Spectrum in $\mathcal{L}_2(0,T)$ with Dirichlet boundary conditions. Here T is a real period. No spectral theory was constructed on the real line. For $n = 1$ this solution is a SINGULAR TRAVELING WAVE $u =$

$2\wp(x-at)$ with 2nd order pole in the point $x=at$.

A NONSINGULAR TRAVELING KdV WAVE is

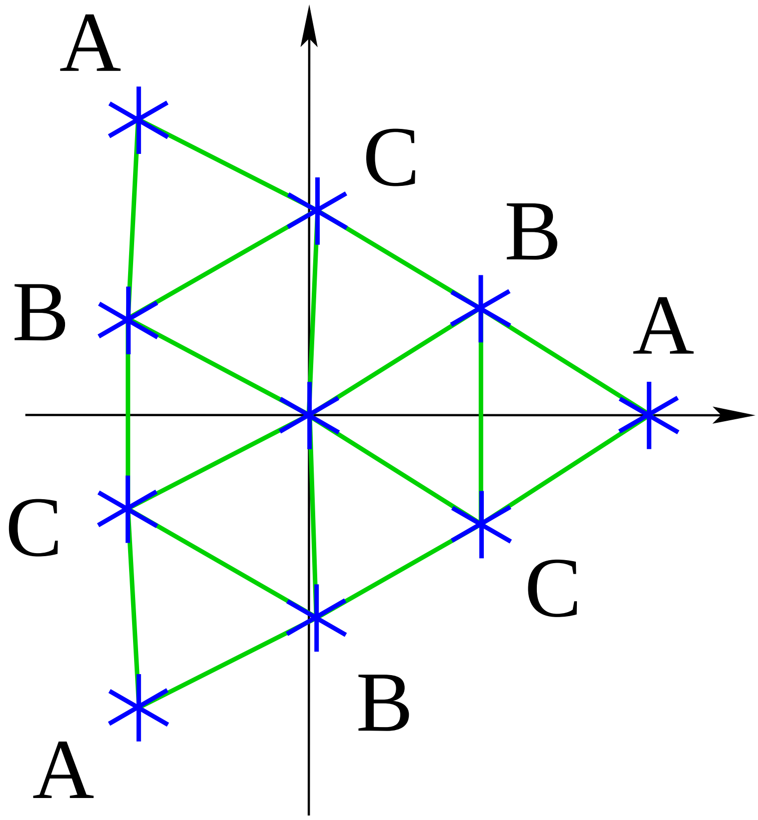
$u=2\wp(x+i\omega-at)$ where $2i\omega$ is an imaginary period .

Conclusion: For all $u(x,0)=n(n+1)\wp(x+i\omega)$ all solutions $u(x,t)$ are NONSINGULAR and FINITE-GAP. The evolution of Lame' Potentials $u(x,0) = n(n+1)\wp(x)$ or $u(x,0)=n(n+1)/x^2$ leads to the singular solutions Question: How many real poles these solutions have $u = \sum_j 2/(x - x_j(t)^2), u = \sum_j 2\wp(x - x_j(t))$?



The orbits of group $Z/3Z$ are marked here. We have $1, 1, 2, 2, 3, ...$ real poles for $n = 1, 2, 3, 4, 5....$

n=4 $\frac{n(n+1)}{2}$ =10



$x_j \sim r_j t^{1/3}$

The symmetry group $Z/3Z$ acts here

$r_j \to \zeta r_j, \zeta^3 = 1$

Our Result: The number of real

poles is equal to $[(n+1)/2]$. This number is equal to the number of negative squares for the Inner Product in the Spaces of functions on the real line where the operator $L=-\partial_x^2+u(x,t)$ is symmetric (see below).

Remark: Many Years ago Russian Scientists (Arkad'ev, Polivanov and Pogrebkov) constructed Scattering Theory for the potentials with singularities like $2/x^2$ but missed Indefinite Metric.

Consider All Real Singular "Algebraic" or "Finite-Gap" Potentials (i.e. potentials satisfying to the Stationary Higher KdV equation $[L, A] = 0$ where $L = -\partial_x^2 + u$ and A–some Ordinary Differential Operator of the odd order). Their singularities are always like $n(n+1)/(x-x_j)^2$. Every solution $L\psi = \lambda\psi$ is meromorphic in x near the real axis with negative part: $\psi(y) = a_0/y^n + a_1/y^{n-2} + ... + O(1))$ where $y =$

$x - x_j$. So we have parameters $a_0, \ldots, a_{[(n+1)/2]}$.

Take the space F of smooth periodic (quasiperiodic) or rapidly decreasing functions f defined by the set of points x_j with numbers n_j. For $n = 1$ function f should have ordinary zero at x_j, and $f(y) - (-1)^n f(-y) = O(y^{n+1})$ in every pole $y = x - x_j$. The Generic case is $n_j = 1$.

We start with ordinary positive Inner Product in the Space F extending it using analytical regularization. The proper eigenfunctions give continuous basis in the extended space. It turns out that every coefficient a_j above gives one negative square.

How Singular Solitons can be used? We used them to define right analog of Fourier Transform on Riemann Surfaces.

What is Fourier Transform on Riemann Surfaces? Which Problems need it? Why singular Solitons are important here? Todays talk is dedicated to this problem.

Example: Discrete Analog of the Fourier/Laurent decomposition on the contours which are the time-sections on Riemann Surfaces (the worldsheets) was constructed by Krichever-Novikov (1986-1990) realizing The Program of Operator Quantization of Closed Strings ("Decompose fields into Fourier Series and replace c-numbers by operators")

. We used singular solutions to the 2D Toda System.

Continuous Fourier Transform on Riemann Surfaces was invented and studied in our works using singular finite-gap solutions to KdV and KP (GN, 2003-2010).

It is associated with Indefinite Inner Product for Riemann Surfaces with genus $g > 0$.

The ordinary Fourier Transform is based on the standard orthonormal base in the Space of functions of k on the circle in Riemann Sphere:

$\Psi_n(k) = k^n = e^{in\phi}, x = n \in Z, k = e^{i\phi}, |k| = 1$ belongs to the unit circle (discrete)

$\Psi(x,k) = \exp(ikx), k \in R$. The circle is passing through infinity (continuous)

The exceptional property is

The Graded Multiplication:

$\Psi_n(k)\Psi_m(k) = \Psi_{m+n}(k)$,

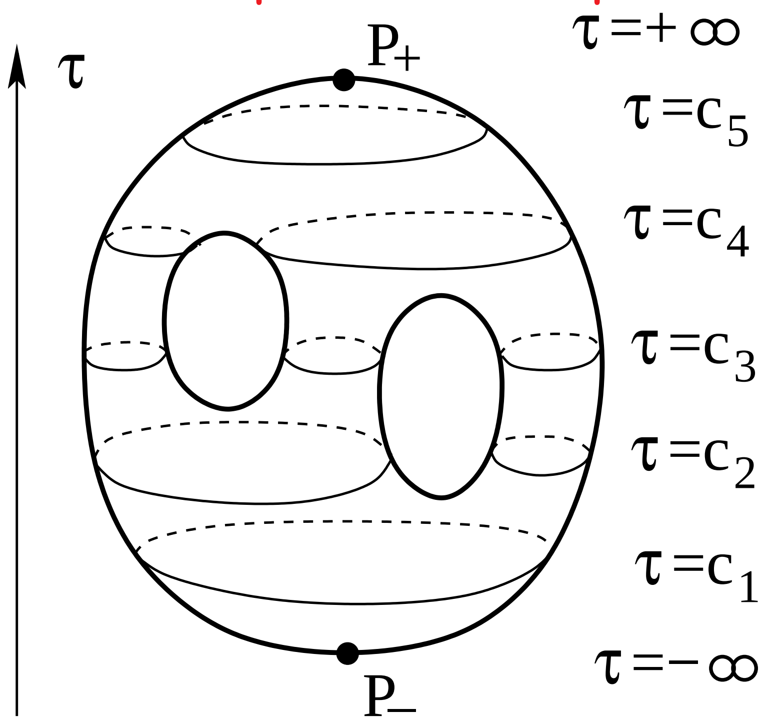
$\Psi(x,k)\Psi(y,k) = \Psi(x+y,k)$

Here genus is equal to zero $g = 0$, and k belongs to the "Canonical Contour" κ_0 on Riemann Surface $\tau = 0$ which is $|k| = 1$ in discrete case and $k \in R$ in continuous case.

There are many orthonormal bases in Mathematics and Applications ("Wavelets", for example) but only Fourier base has such multiplicative property important for Nonlinear Problems. The Notion of Resonances is bases on it.

Discrete Case and Fourier/Laurent (KN) bases on Riemann surfaces were studied in the works of Krichever and Novikov, published in Functional Analysis Applications, 1987-89, and in the work: Krichever I.M., Novikov S.P.,Riemann Surfaces, Operator Fields, Strings. Analogues of Fourier-Laurent bases.In the Memorial Volume of V.Kniznik: Physics and Mathematics of Strings, pp 356-388, Editors L.Brink, D.Friedan A.Polyakov, World Scientific, Singapore 1990

The String Diagram (World-sheet) $(\Gamma, P_+, P_-, k_+, k_-)$:



Here $1/k_+$, $1/k_-$ are local parameters near "The Infinite Points" P_- ("in")and P_+ ("out") respectively, dp is defined as meromorphic differential with 2 simple poles at P_+, P_-, $dp = dk_+/k_+ + O(1)$, $dp =$

$-dk_-/k_- + O(1)$, $\Re \oint dp = 0$ for all closed paths.

The "time" is $\tau = \Re p$

An analogue of discrete Fourier bases is defined for functions (tensor fields) at the contour $\kappa_c : -\infty < \tau = c < +\infty$

An analogue of Laurent basis for the holomorphic functions is defined for the domains between 2 contours $\kappa_{c'}$ and $\kappa_{c''}$ where $c' < c''$. All constructions are extended to the tensor fields with any tensor weights. The tensors with weights equal to 0,1,-1,2,1/2 are especially important for the string theory.

These bases were constructed for operator quantization of the closed bosonic string. For this problem it was critical to have bases with good multiplicative properties.

They are defined by the following asymptotics at the points P_+, P_- where $k_\pm(\lambda) = \infty$:

$$\Psi_j(\lambda) = \begin{cases} k_+^{j+g/2}(c_j^+ + o(1)) & (P_+) \\ k_-^{-j+g/2}(c_j^- + o(1)) & (P_-) \end{cases}$$

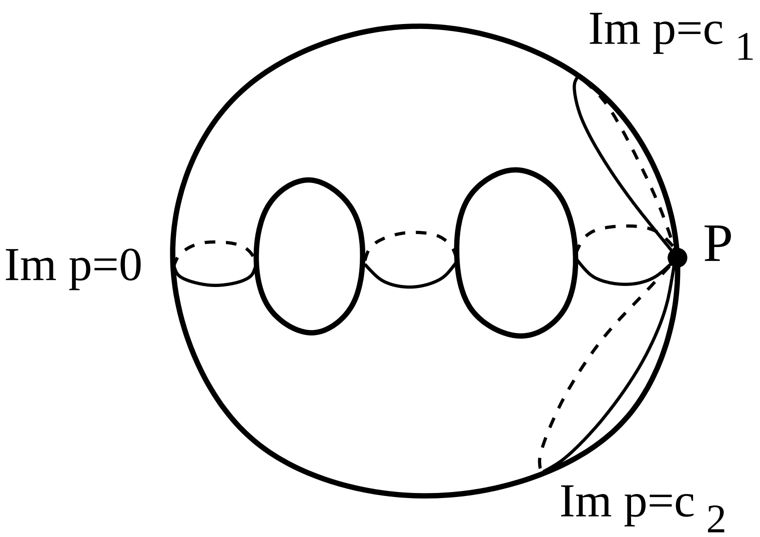
The multiplication rule is Almost Graded:

$$\Psi_l(\lambda)\Psi_m(\lambda) =$$

$$= \sum_{n=l+m-N}^{n=l+m+N} C_{lm}^n \Psi_n(\lambda)$$

Here $N = N(g)$, does not depend on l, m. Coefficients C_{lm}^n do not depend on λ.

Continuous analog of the Krichever-Novikov bases.



Step 1: Select Contour.

Let $z = 1/k$ be local parameter near P,

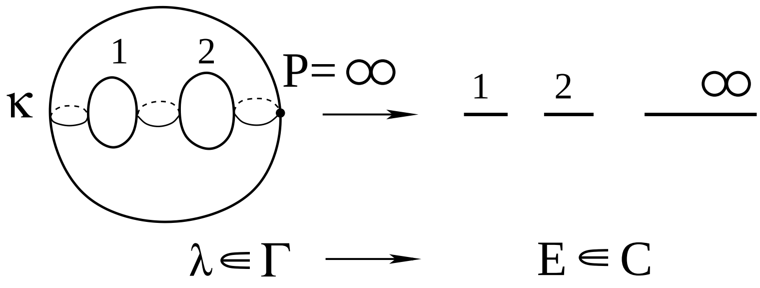
dp is a meromorphic differential with a second-order pole at $P = \infty$, $dp = dk + O(1)$, $\Im \oint dp = 0$ for all closed paths. So $\tau = \Im p$ is well-defined. The Special Canonical Contour is $\kappa_0 : \tau = \Im p = 0$.

Step 2: Take Real Inverse Spectral Data, and construct Ψ-function:

1) A compact Riemann surface Γ of genus g with an

"infinite" point $P = \infty$ and local parameter $z = 1/k$ near $P, z(P) = 0$. 2) A collection of points ("Divisor") $D = \gamma_1, \ldots, \gamma_g$ (the poles of ψ-function). 3)The "reality conditions" for Γ and poles should be satisfied.

Γ is hyperelliptic (2-sheeted over λ-plane) for KdV and may be general for KP.



1) The eigenfunction $\Psi(\lambda, x)$, $\lambda \in \Gamma$, $x \in R$, is meromorphic in $\Gamma \backslash \infty$ with simple poles $\gamma_1, \ldots, \gamma_g$, $\Psi(\lambda, x_0) = 1$.

2. $\Psi(\lambda, x) = (1 + o(1)) \exp(ik(x - x_0)), \lambda \to \infty$.

Let $g = 0$ and $\Gamma = C \cup \infty, P = \infty$. Here k is the standard coordinate $k = \lambda$. Then $p = k$, $\Psi(\lambda, x) = \exp(ikx)$

is the standard Fourier base on the real line $\Im k = 0$

A continuous analog of the Fourier bases is defined by the Special Ψ-function with following Singular Data: $\gamma_1 = \ldots = \gamma_g = \infty, x_0 = 0$

Ψ-functions form an almost-graded algebra: $\Psi(\lambda, x)\Psi(\lambda, y) = [\partial_z^g + \Sigma_{j>0}\, c_{g-j}(x, y)\partial_z^{g-j}]\Psi(\lambda, z)|_{x+y}$

Here $c_1 = \zeta(x + y)$ for $g = 1$.

We study functions of λ; x is a parameter numerating our basic functions.

The functions $\Psi(\lambda, x)$ are Singular in x. They have a pole at $x = 0$. Example: The classical periodic Lame' operators $-\partial_x^2 + g(g+1)\wp(x)$

Physical Soliton Theory (KdV) deals with regular operators $-\partial_x^2 + g(g+1)\wp(x+i\omega')$ where $2\omega'$ is an imaginary period ("traveling wave" for $g = 1$).

Do Singular Operators have any reasonable spectral theory on the whole real line x?

Classical people like Hermit considered their spectrum only at the interval $[0, T = 2\omega]$ with zero boundary conditions. We need to use the whole line in order to construct Fourier Transform with good multiplicative properties.

Ψ-functions for Regular Real Periodic Operators never form an almost-graded multiplicative system for $g>0$.

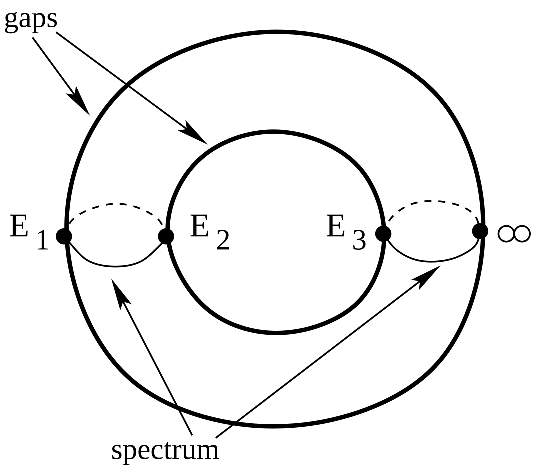
Consider real (may be singular) "finite-gap" periodic operator with spectral curve (Riemann Surface) Γ given by the equation $\mu^2=(E-E_1)\cdots(E-E_{2g+1})$ with permutation of sheets $\sigma(E,\mu)=(E,-\mu),\sigma^2=1$ and poles of Ψ-function $D=\gamma_1,...,\gamma_g$.

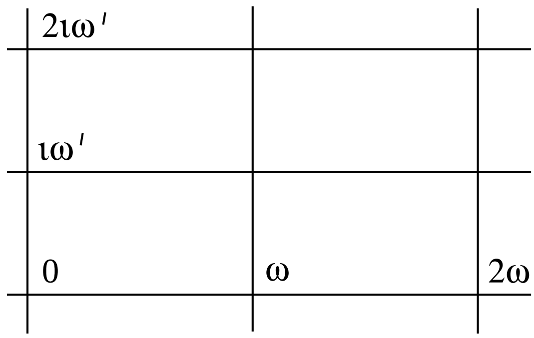
Real case corresponds to the following data:

1) Γ is real i.e. collection of branching points is invariant under complex conjugation. 2) $\tau(E,\mu) = (\bar{E},\bar{\mu})$: The whole collection of poles is invariant under τ.

Example 1: Let $g = 1$ (Γ is a torus)

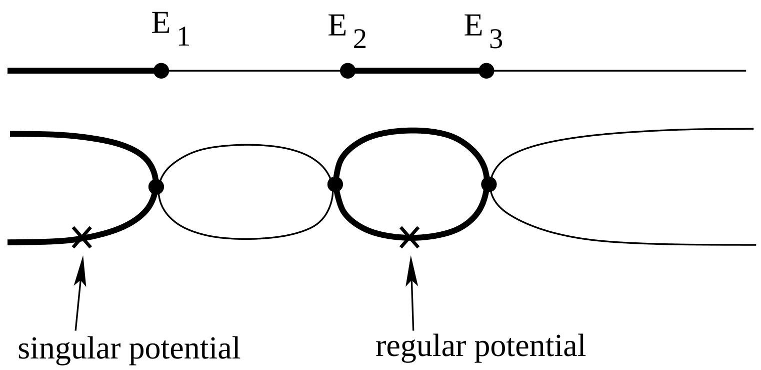
and all E_j are real, $j = 1, 2, 3$:





The lattice of periods of the Weierstrass $\wp$ -function is **rectangular with periods** $2\omega, 2i\omega'$.

The spectrum is real , and spectral gaps are $[-\infty, E_1]$ and $[E_2, E_3]$



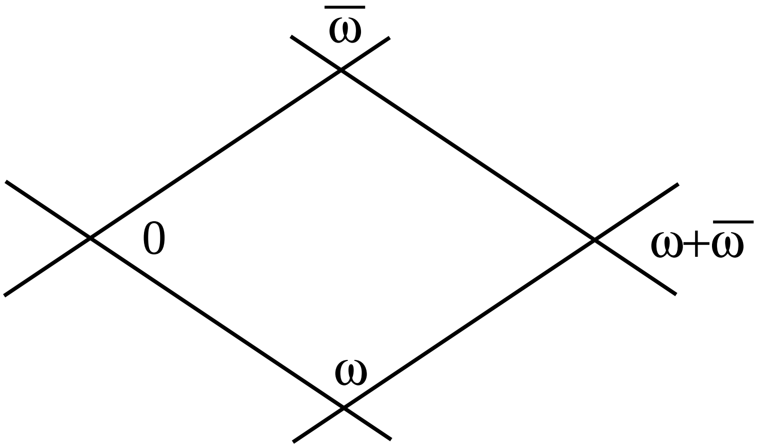
κ_0 is represented by fine lines.

The contour κ_0 has 2 components here: infinite and finite. There is only one pole γ: For Regular Case it belongs to the finite gap, for the Singular Case it belongs to the infinite gap

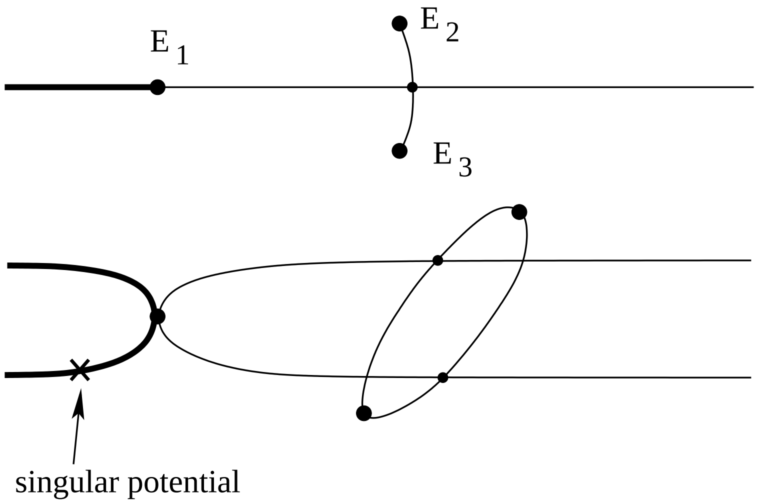
(They both are the shifted Hermit -Lame Operators but in regular case the shift is imaginary, in singular case the shift is real).

In both cases the spectrum on the whole line is the same but eigenfunctions and functional spaces on the x-line are drastically different.

Example 2. Let $g = 1, E_1 \in R, E_3 = \overline{E_2}$:



The lattice of periods is **rombic**.



κ_0 given by fine lines.

The spectrum on the whole line coincides with the projection of the contour κ_0 on the $E-line$. It contains complex arc joining $E_2, \bar{E}_2$. Spectral meaning of singular operators on the whole line was not discussed before.

Define the "spectral measure" $d\mu$. Let λ_j= projection of poles,

$\lambda = \lambda(E)$:

$$d\mu = \frac{(E-\lambda_1)...(E-\lambda_g)dE}{2\sqrt{(E-E_1)...(E-E_{2g+1})}}$$

For every smooth function on the contour κ_0 parametrized by λ, with decay fast at infinity, we define

Direct and Inverse Spectral Transform:

$$\tilde{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\kappa_0} \phi(\lambda)\Psi(\sigma\lambda, x)d\mu \quad (1)$$

$$\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_R \tilde{\phi}(x)\Psi(\lambda, x)dx \tag{2}$$

We call it Fourier Transform if all $\lambda_j = \infty$; $d\mu^F = dE/2\sqrt{(E-E_1)\dots(E-E_{2g+1})}$ and our base has good multiplicative properties

In the Regular Case this Spectral Transform is an Isometry between the Hilbert spaces with inner products

$$< \psi_1, \psi_2 >_{\kappa_0} = \int_{\kappa_0} \psi_1(\lambda)\overline{\psi}_2(\lambda) d\mu(\lambda)$$

$$< f_1, f_2 >_R = \int_R f_1(x)\overline{f}_2(x) dx$$

Consider the Singular Potentials

1) Formula for the Spectral Transform remains valid; For the Inverse Transform it remains valid

after a natural regularization. 2) Spectral Transform is an isometry between the spaces with indefinite metric described below. All singularities have a form ($n_j \in Z$): $u(x) = n_j(n_j+1)/(x-x_j)^2 + O(1)$

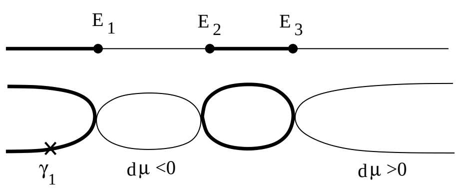
The function $\Psi(\lambda, x)$ is meromorphic in x. For $\lambda', \lambda'' \in \Gamma$ all residues of the product $\Psi(\lambda', x)\Psi(\lambda'', x)$ are equal to 0.

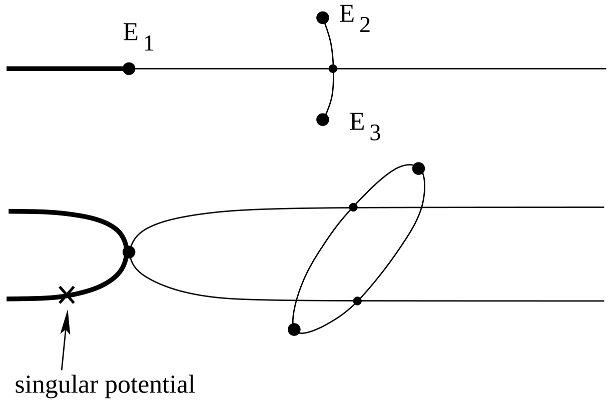
All residues in the formula for the Inverse Spectral Transform are equal to 0. Our Regularization: If we meet singularity under the integral, we go around it in the complex domain. We can go above or below, but the result is the same.

The Inner Product on the Riemann Surface (in the space of functions in κ_0) $< \psi, \phi >_{\kappa_0} =$

$$= \int_{\kappa_0} \psi(\lambda)\overline{\phi}(\tau\lambda)d\mu(\lambda)$$

Example 1. All branching points are real: τ acts identically on κ_0, the form $d\mu$ is negative somewhere. For Fourier Transform we have: $d\mu^F/dp > 0$ exactly in every second component starting from the infinite one; So we have $[(g+1)/2]$ "negative" finite components in κ_0. Example 2. Some pair of branching points is complex adjoint: τ is not identity in the nonreal components of κ_0; So the inner product is nonlocal and therefore indefinite.





Let $f(x), g(x)$ belong to the image of the Spectral Transform and meromorphic near real line. The expression $f(x)\bar{g}(\bar{x})$ is meromorphic and has zero residue in every pole.

So we can define $\int_R fg dx$ avoiding pole. This scalar product is indefinite. This procedure is well-defined in our case.

Pontryagin-Sobolev spaces. Every function $f(x) \in \mathcal{L}_2(R)$ can be written for real x as

$$f(x) = \oint \widehat{f}(\varkappa, x) d\phi(\varkappa), \varkappa = e^{i\phi}$$

Here $f(\varkappa, x+T) = \varkappa f(\varkappa, x)$. Therefore $\mathcal{L}^2(R)$ is represented as a direct integral of Bloch-Floquet spaces $f \in B_\varkappa, |\varkappa| = 1$: Our inner product has finite number r of negative squares in $B_\varkappa, r = [(g+1)/2]$ for Fourier. **Remark: Singular Bloch-Floquet eigenfunctions are known for the $k+1$-particle Moser-Calogero operator with Wei** strass elliptic pairwise potential if coupling constant is equal to $n(n+1), n \in Z$. They form a k-dimensional complex algebraic variety. No one function is known for $k > 1$ serving the discrete spectrum in the space $\mathcal{L}_2$ of the bounded domain inside of poles. Our case corresponds to $k = 1$. We believe that for

all $k > 1$ this family of eigenfunctions also serves spectral problem in some indefinite inner product in the proper space of functions defined in the whole space R^k.