ANALYTIC HOMOTOPY THEORY. RIGIDITY OF HOMOTOPY INTEGRALS

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The well-known formula of Whitehead for the Hopf invariant of a smooth mapping $f\colon S^3 \to S^2$ states that

$$H\{f,\omega\} = \int_{S^3} v \wedge f^*(\omega), \qquad \int_{S^2} \omega = 1, \quad dv = f^*(\omega). \tag{1}$$

The formula (1) has the following properties:

(I) $\delta H/\delta f = 0$ ("homotopy invariance").

(II) $\delta H/\delta\omega = 0$ and $\int_{S^2} \delta\omega = 0$ (rigidity).

(III) If $H^+\{f,\omega\} = \int_{D^3_+} v \wedge f^*(\omega)$, then H^+ is a multi-valued functional (i.e., δH^+ is a closed 1-form) on the space F of null-homotopic mappings $S^2 \to S^2$. Here the 1-form δH^+ is local, i.e., at any point it depends on the mapping $f\colon S^2 \to S^2$ and finitely many of its derivatives at this point ("weak local property").

Strongly local 1-forms on the spaces of mappings, where the primitive $H^+\{f\}$ is itself local, and not merely δH^+, are classified in [1] and [2].

The present note is a development of [3].

As was indicated in [3], the generalization of Whitehead's formula to homotopy groups of manifolds was begun in [4] and [5]. There only property (I) was considered; the algebraic formalism of these papers appears to be unnaturally complicated. The author's attention was first turned to property (III) by A. Polyakov and P. Vigman (see [3]). Property (II) was first considered in [3], together with the development of a more natural and simpler algebraic formalism, in which property (I) is self-evident (see §1 below). The present note is mainly devoted to the analysis of property (II).

1. **Homotopy groups of d-algebras. Homotopy integrals.** Let $A = \sum_{j\geqslant 0} A^j$ be a skew-commutative graded d-algebra, where $A^iA^j = (-1)^{ij}A^jA^i$, $dA^j \subset A^{j+1}$, $d(xy) = (dx)y \pm x(dy)$, $d^2 = 0$, $A^0 = k$ (a field of characteristic zero) and $H^1(A) = 0$. In what follows we shall take $k = \mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$.

The main example is the algebra of smooth differential forms of a smooth simply connected manifold, $A = \Lambda^*(M^n)$, or one of its subalgebras. Taking into account the prospect of application to field theory, we do not consider non-simply-connected generalizations.

We denote by $C^{(q)}(A)$ a minimal free skew-commutative extension of the algebra A for which

$$A \subset C^{(q)}(A), \qquad H^j(C^{(q)}(A)) = 0, \quad 0 < j \leqslant q.$$

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The algebra $C^{(q)}(A)$ is constructed as follows. Choose in the algebra $A = C_0^{(q)}$ a minimal multiplicative basis $y_{j\alpha}$ of cocycles $dy_{j\alpha} = 0$ of dimension $j \leqslant q$. By definition we put

$$C_1^{(q)} = A[\dots, v_{j-1,\alpha}, \dots], \quad dv_{j-1,\alpha} = y_{j\alpha},$$

the $v_{j-1,\alpha}$ are new free generators. The embedding $A = C_0^{(q)} \to C_1^{(q)}$ induces the zero homomorphism in cohomology in dimensions $j \leqslant q$, by construction. However, there may arise new cocyles of dimension at most q in the d-algebra $C_1^{(q)}$. We iterate the construction, replacing A by $C_1 Y$, etc.:

$$A = C_0^{(q)} \subset C_1^{(q)} \subset C_2^{(q)} \subset \cdots \subset C^{(q)}(A). \tag{2}$$

Definition 1. The *homotopy groups* of the algebra A are the k-vector spaces dual to the cohomology of the algebra $C^{(q)}(A)$

$$\pi_{q+1}(A)^* = H^{q+1}(C^{(q)}(A)); \tag{3}$$

here $k = \mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$.

Using classical results of Cartan and Serre (see [6]), it is easy to prove the following assertion.

Lemma 1. *If $\pi_1(M^n) = 0$ and $A = \Lambda^*(M^n)$, then $\pi_{q+1}(A) = \pi_{q+1}(M^n) \otimes k$, when $k = \mathbb{R}$ or $\mathbb{C}$.*

Remark. Definition 1 itself is elementary; it requires no results of algebraic topology, as is also true of Theorem 1.

Theorem 1. *Each element $z \in H^{q+1}(C^{(q)}(A))$ for $A = \Lambda^*(M^n)$, $\pi_1 = 0$, defines a homotopy invariant linear form, the homotopy integral*

$$H_z \colon \pi_{q+1}(M^n) \to k, \quad k = \mathbb{R}, \mathbb{C} \tag{4}$$

(*the generalized Whitehead–Hopf invariant*).

Proof. Suppose given a smooth mapping $f \colon S^{q+1} \times \mathbb{R} \to M^n$ (a homotopy). It induces a d-mapping

$$f^* \colon \Lambda^*(M^n) \to \Lambda^*(S^{q+1} \times \mathbb{R}).$$

Taking into account the equality $H^j(S^{q+1} \times \mathbb{R}) = 0$, $j \leqslant q$, the mapping f^* extends to a d-mapping of the extension algebra, by construction of the algebra $C^{(q)}(A)$:

$$\hat{f} \colon C^{(q)}(A) \to \Lambda^*(S^{q+1} \times \mathbb{R}), \quad H_z(f) = \int_{S^{q+1} \times t} \hat{f}(z). \tag{5}$$

To any cocycle $\tilde{z}$ in the class z there corresponds a closed $(q+1)$-form $f(z)$ on $S^{q+1} \times \mathbb{R}$; its integral over the cycle $S^{q+1} \times t$ is independent of t by virtue of Stokes' formula. The theorem is proved. □

This settles the generalization of property (I) mentioned above.

Remark. Definition 1 is applicable to different d-closed subalgebras of $\Lambda^*(M^n)$, which gives prospects of different generalizations, for example to complex manifolds and others.

2. **Minimal models as subalgebras of the algebra of forms. Deformations.** According to Sulivan, a minimal model of the $\mathbb{Q}$-homotopy type of simply connected finite complex (we shall not deal in greater generality) is a free skew-commutative d-algebra over $\mathbb{Q}$ with generators $x_{j\alpha}$ of dimension $j \neq 1$, $j > 0$, and unit $1 \in A^0$ such that $H^j(A) = H^j(M^n)$, where

$$dx_{j\alpha} = P_{j\alpha}(\dots, x_{q\beta}, \dots), \quad q < j. \tag{6}$$

Free d-algebras of this type with property (6) will be called *Sullivan algebras.*

Let a homotopy type be realized by a manifold M^n of high dimension $n \to \infty$. The realization of the minimal model as $\mathbb{Q}$-subalgebra of the algebra of forms $\Lambda^*(M^n)$ leads to difficulties. By induction, choosing in dimension $j = 2$ a basis $x_{2\alpha}$ of forms in general position for the integral lattice in the $\mathbb{R}$-cohomology, we construct a minimal model $A \subset \Lambda^*(M^n)$ as follows: if the construction has been completed for $q < j$, then we add closed generators $x_{j\alpha}$, $dx_{j\alpha} = 0$, as forms in general position with the condition that their integrals over cycles are integer-valued.[1] Suppose that in dimension $j+1$ there arose a kernel of the embedding of the already constructed part $A_{j-1} \subset \Lambda^*(M^n)$ in the cohomology mapping $H^{j+1}(A_{j-1}, \mathbb{Q}) \to H^{j+1}(M^n, \mathbb{R})$, where

$$A_{j-1} = \mathbb{Q}[\dots, x_{q\beta}, \dots], \quad q < j. \tag{7}$$

Choosing a basis $P_{j\alpha}(\dots, x_{q\beta}, \dots)$ of polynomials in the kernel in dimension $j+1$, we introduce new generators $x_{j\alpha}$ as j-forms such that

$$dx_{j\alpha} = P_{j\alpha} \neq 0.$$

Thus the algebra A_j is constructed, where the $x_{j\alpha}$ are defined up to closed forms ($dx_{j\alpha} \neq 0$)

$$x_{j\alpha} \to x_{j\alpha} + \omega_{j\alpha}, \quad d\omega_{j\alpha} = 0. \tag{8}$$

Definition 2. A homotopy type is called *m-rational* if the given procedure suffices to construct the entire minimal model through dimension $j \leqslant m$ (here n is sufficiently large).

By this procedure one can always construct a minimal model as an $\mathbb{R}$-subalgebra $A \otimes \mathbb{R} \subset \Lambda^*(M^n)$. We shall in the following consider only ∞-rational homotopy types. Obviously, starting with $m \geqslant k$, where $\sum_{j \geqslant k} H^j(M^n, \mathbb{R}) = 0$, the concepts of m-rationality and ∞-rationality coincide.

Consider a free skew-commutative d-extension

$$\bar{A} = A[\dots, v_{j-1,\alpha}, w_{j\alpha}, \dots], \quad dv_{j-1,\alpha} = w_{j\alpha},$$

of the Sullivan algebra, where we adjoin two new generators $v_{j-1,\alpha}$ and $w_{j\alpha}$ for each free generator $x_{j\alpha}$ of the algebra A satisfying a relation (6). For d-closed $x_{j\alpha}$, where $P_{j\alpha} = 0$, we define their *deformations* by the formula

$$Dx_{j\alpha} = \bar{x}_{j\alpha} = x_{j\alpha} + w_{j\alpha}.$$

Lemma 2. *If $P_{j\alpha} \neq 0$, and if the deformation is defined for all $x_{q\beta}$, then the formula*

$$P_{j\alpha}(\dots, \bar{x}_{q\beta}, \dots) = P_{j\alpha}(\dots, x_{q\beta}, \dots) + d\psi_{j\alpha}(\dots, x_{q\beta}, v_{s\beta}, w_{p\gamma}, \dots)$$

is valid in the free algebra $\bar{A}$.

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[1]The image in cohomology $H^j(A_{j-1}) \to H^j(M^n, \mathbb{Q})$ is supplemented to a full basis of $H^j(M^n, Q)$.

Next, we put

$$Dx_{j\alpha} = \bar{x}_{j\alpha} = x_{j\alpha} + \psi_{j\alpha} + w_{j\alpha}. \tag{9}$$

The following assertion is true.

Theorem 2. *The universal formula* (9) *defines a deformation of the minimal model A itself, realized homomorphically in the algebra of forms $\Lambda^*(M^n)$ of any manifold of this homotopy type, and so also a deformation of its extensions $C^{(q)}(A)$. The embeddings of the $\mathbb{Q}$-algebra A into the algebra of forms $\Lambda^*(M^n)$ as $n \to \infty$ decompose into equivalence classes up to a deformation* ("*homotopic*"); *the equivalence classes of the $\mathbb{Q}$-algebra A form a space, embedded in the* "*moduli*" *space*

$$\mathcal{H} = \sum_{j \geqslant 2} \operatorname{Ker}_j H \otimes H_j(M^n, \mathbb{R}),$$

where $\operatorname{Ker}_j H \subset \pi_j(M^n)$ denotes the kernel of the Hurewicz homomorphism $H\colon \pi_j(M^n) \to H_j(M^n, \mathbb{R})$.

Corollary 1. *If the space $\mathcal{H}$ is trivial, $\mathcal{H} = 0$, then there is only one homotopy class of minimal models over $\mathbb{Q}$ in the algebra of forms $\Lambda^*(M^n, \mathbb{R})$.*

The proof of the theorem follows easily from the construction of the algebras A and $C^{(q)}$ (see above), where the arbitrariness of choice of the operator d^{-1} leads to nonuniqueness.

The concept of deformation is important because of the following theorem (see [3] for particular cases).

Theorem 3. *If, in the formula representing the homotopy integral on $\pi_{q+1}(M^n)$ we submit all formulas to deformation, then the value of the integral does not change* (*rigidity theorem*).

Discussion of further properties will be given in the following papers.

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