ALGEBRAIC TOPOLOGY AT THE STEKLOV MATHEMATICAL INSTITUTE OF THE ACADEMY OF SCIENCES OF THE USSR

S. P. NOVIKOV

ABSTRACT. This article is a survey of the author's results in algebraic topology and its applications. It considers questions in the classification of simply-connected smooth manifolds, the topological invariance of the rational Pontryagin classes, the foundations of "Hermitian algebraic K-theory", the calculation of various types of bordism rings and the construction of a general theory of bordism as an "extraordinary" homology theory, questions in the topology of foliations on smooth manifolds, the construction of an analogue of classical Morse theory for mappings of manifolds into a circle, and some questions in the calculus of variations leading to "many-valued" functionals.

Bibliography: 67 titles.

§1. HISTORICAL NOTES

During of 1950's the development of algebraic topology was particularly intense, and there was a complete change in the nature of the problems being solved and the techniques being used in this field. In algebraic homotopy theory we saw the introduction and development of far-reaching apparatus, based upon homological algebra discovered and developed here. It had effects in the theory of fibrations and certain remarkable so-called "general categorical" properties of homology. This apparatus made it possible to set up a system of standard methods for calculating the homology and homotopy groups of various spaces, and to make considerable progress with the classical problem of calculating (for example) the homotopy groups of spheres. The new methods were successfully applied to the investigation of differentiable, complex and algebraic manifolds; they created new branches of algebra, and exerted widespread influence on analysis, especially complex analysis and the theory of dynamical systems. Several of the important topological ideas of this period arose in the publications of Soviet mathematicians of the 1940's and early 1950's working in the Steklov Mathematical Institute of the Academy of Sciences of the USSR.

We should mention here in the first place the work on the topology of manifolds and homotopy theory by L. S. Pontryagin and his students V. A. Rokhlin, M. M. Postnikov, V. G. Boltyanskiĭ and R. V. Gamkrelidze (the theory of characteristic classes, the first deep results in the calculation of the homotopy groups of spheres, the beginning of the theory of bordism, homotopy invariants, the study of obstructions to the extension of maps and of sections of fibrations, and the algebraicity of characteristic cycles of algebraic manifolds). The topological activity of the Soviet

1985 *Mathematics Subject Classification* (1985 *Revision*). Primary 55-02, 57-02, 58-02; Secondary 55-03, 57-03, 01A65.

UDC. 515.14/148.

Translated by C. A. ROBINSON.

school began to fall sharply in the first half of the 1950's, as the leading specialists moved on to other fields. The development of topology in that period was carried on by a large group of research workers, in the front rank of whom we whould name J. Leray, H. Cartan, J.-P. Serre, R. Thom. A. Grothendieck and A. Borel (France), N. E. Steenrod, S. Eilenberg, R. Bott, J. W. Milnor, S. Smale, M. A. Kervaire and S. S. Chern (USA), J. H. C. Whitehead, M. F. Atiyah and J. F. Adams (England), F. Hirzebruch (West Germany) and a number of others.

In the period from 1956 to 1958 a great deal of work was done by topologists of the Steklov Institute and the Faculty of Mechanics and Mathematics ("Mekhmat") of Moscow University, notably M. M. Postnikov, V. G. Boltyanskil and A. S. Shvarts, in disseminating the new methods mentioned above. We remark that the corresponding work in creating a new school of topology in Leningrad was carried out in the 1960's by V. A. Rokhlin in Mekhmat at Leningrad University; the dissemination of the contemporaneous topological methods in algebraic geometry took place at the end of the 1950's and during the first half of the I960' in a seminar at the Steklov Institute and Moscow University under the direction of I. R. Shafarevich.

Our object is solely to give a description of the key results of algebraic topology which were obtained by members of the Steklov Institute and by their immediate students or participants in their seminars, during the new period which began in the late 1950's. The Steklov Institute's group of specialists in algebraic methods of homotopy theory was founded in the 1958–1960 Mekhmat seminar of M. M. Postnikov at Moscow University, which grew out of the seminars of Postnikov, Boltyanskiĭ and Shvarts mentioned above. Its participants (in addition to the present author) were L. N. Ivanovskiĭ, B. G. Averbukh, and D. V. Anosov, who was also actively working in topology. The first well-known results of topology in the new period were obtained by participants in this seminar: the author and B. G. Averbukh.

§2. Bordism

The elements of bordism groups are equivalence classes of oriented (respectively, arbitrary) closed smooth manifolds with respect to oriented (arbitrary) "cobordism". A manifold is said to be equivalent to zero if it bounds an oriented (arbitrary) film, i.e. a manifold with boundary; the sum is induced by the union of manifolds. The absence of p-torsion in these groups for $p > 2$ was proved by Averbukh [1], who began applying the algebraic-topological methods of Cartan, Serre and others to the theory of Thom. It is worth remarking here that the 2-torsion of the oriented bordism groups was determined by Rokhlin and Wall [37], [60] by geometrical methods. As we know, the theory of bordism was founded by Pontryagin and Rokhlin, and subsequently greatly advanced by Thom, who completely solved the problem in the unoriented case (see the survey [21]). The first investigation of the p-components of the stable homotopy groups of spheres for $p > 2$, which was carried out in the winter of 1958/59 using the new "Adams spectral sequence" method which had appeared in 1958 [39], was the first scientific activity of the author. This proved a theorem about the existence of arbitrarily long nonzero compositions $S^n \to S^{n-k_1} \to \cdots \to S^{n-\sum k_j}$, $k_j > 0$, in the homotopy groups of spheres; the main part, however, was a purely algebraic construction of a new analogue of the Steenrod operations in the cohomology of the so-called graded "Hopf algebras with symmetric diagonal" which are used in the process of calculation

[18]. The superiority of Adams' new method, which was specifically designed for the calculation of stable homotopy, over the more classical Cartan–Serre method of calculating homotopy groups was not immediately apparent. The Adams spectral sequence is based upon the very same "building blocks", namely Steenrod operations, Eilenberg–Mac Lane complexes, and the Leray spectral sequence induced by a filtration of the space. However, for the determination of "stable" homotopy classes of maps it is possible to use "Adams resolutions" in a natural and beautiful way in place of the very simple decomposition of a space into Eilenberg–Mac Lane complexes using a so-called Postnikov tower, and to organize everything into a more beautiful and invariant algebraic scheme based upon successive application of the ideas of homological algebra. The view which first arose was that the Adams method could, in principle, only give the same results in the calculation of stable homotopy groups as the Cartan–Serre method. The error of this opinion was exposed quickly enough when the method was applied to bordism theory: in addition to solving the questions posed earlier, it showed that it was natural and necessary to widen the class of problems being posed here. The author and Milnor introduced the new bordism rings of stably almost complex (unitary), symplectic, special unitary and other classes of manifolds with group structures. (These rings have the union of manifolds as sum, and the direct product of manifolds as multiplication.) The group structure is imposed upon the stable tangential fibration (more precisely, the normal fibration to the manifold in R^n), the group being $G = U(n)$, $G = \mathrm{Sp}(n)$, $G = SU(n)$, etc., whereas the classical bordism theories relate to the groups $G = O(n)$ and $G = SO(n)$ (corresponding to the bordism theories of arbitrary and oriented smooth manifolds). The corresponding cobordism rings are denoted by $\Omega_G = \sum \Omega_i^G$, $G = O, SO, U, SU, \mathrm{Sp}, 1$. If the group G is the identity, the corresponding bordism ring represents a reinterpretation of earlier results of Pontryagin, and coincides with the direct sum of the stable homotopy groups of spheres: $\Omega_i^{(1)} = \pi_{n+i}(S^n)$, $i < n-1$, the multiplication operation in the ring being induced by composition of maps of spheres into spheres.

In addition to the proof of the absence of p-torsion ($p > 2$) in most of these rings ($G \neq 1$), a method of investigating the 2-torsion was obtained, and the multiplicative structure of the bordism rings Ω_G was completely determined case by case. The unitary bordism ring Ω_U turned out to be particularly simple and (as became apparent later) important for a variety of purposes: it proved to be isomorphic to a polynomial ring with one polynomial generator in every even positive dimension (see [19]; for more detailed proofs and generalizations, see [20]).

It subsequently became clear that a significant portion of these results of the author's and the above-mentioned result of Averbukh had also been obtained by Milnor, who had these ideas somewhat earlier but first published them in the paper [50], contemporary with [19], and even then only in part. Although Milnor's paper, like [19], is based upon Adam's method, they are substantially different in technique. Milnor's program for computing the multiplicative structure of the fundamental bordism rings Ω_{SO} and Ω_U was based upon a geometrical idea and was not completed by him in [50]. On the other hand, the author's paper [19], based upon a reduction to algebra, completed this calculation very simply within the framework of Adams' method. Milnor's program was completed in 1965 by Stong and Hattori, using K-theory [57].

By using these results and a method of Thom, sufficient conditions were obtained in [19] for the realizability of an integral k-dimensional homology class as the continuous image of a closed smooth oriented manifold (the Steenrod problem). For example, it is sufficient for realizability that in the homology groups of dimension less than k in a manifold there should be no p-torsion for primes $p > 2$. A precise form of this result was indicated in [20]: it is sufficient for the realizability of a cycle that there should be no p-torsion in the homology of the manifold in dimensions of the form $k - 2q(p-1) - 1$ for $q \geqslant 1$ and $p > 2$. We recall that for mod 2 cycles the problem was solved by Thom [58].

In the period 1960–1965 the important new concept of "generalized" homology (and cohomology), also called "extraordinary" homology, arose in algebraic topology and bore important fruit. This development began from work of Grothendieck, Atiyah and Hirzebruch on K-theory, which had its origin in algebraic geometry. For any polyhedron X, the elements of the group $\tilde{K}(X)$ are the stable equivalence classes of complex vector bundles over X. Let $K^0(X) = \mathbb{Z} + \tilde{K}(X)$. By definition we set $K^{-i}(X) = \tilde{K}(E^i X^*)$, $i > 0$, where $E^i X^*$ is the i-fold suspension of the disjoint union $X^* = X + \text{point}$. Then by the Bott periodicity theorem we have $K^j(X) = K^{j+2}(X)$. This extends the definition of the collection of groups $K^j(X)$ to all positive numbers j. The "relative" groups $K^j(X, Y) = K^j(X/Y, \mathcal{P})$, where $\mathcal{P}$ is a point, are introduced analogously.

According to a theorem of Eilenberg and Steenrod, ordinary cohomology (homology) theories are uniquely determined axiomatically as homotopy invariant contravariant (covariant) functors $H^j(X, Y)$ of pairs X, Y of complexes such that a) $H^j(X, Y) = H^j(X/Y, \mathcal{P})$ and $H^j(X) = H^j(X^*, \mathcal{P})$, b) the exact sequence of a pair X, Y holds and is functorial, and c) if X is contractible, then $H^j(X) = 0$ for $j \neq 0$ (the "axiom on homology of a point", as X is homotopy equivalent to a point).

For K-theory $K^j(X, Y)$ all these conditions hold except the last one; the groups K^j of a point $\mathcal{P}$ are nontrivial in nonzero dimensions:

$$K^j(\mathcal{P}) = \begin{cases} \mathbb{Z}, & j \text{ even;} \\ 0, & j \text{ odd.} \end{cases}$$

The full collection $K^*(X, Y) = \sum_j K^j(X, Y)$ is even a cocommutative graded associative ring, and a module over $K^*(\mathcal{P})$, where $K^*(\mathcal{P}) = \mathbb{Z}[t, t^{-1}]$.

The dimension of the generator t is equal to 2. It is known as the "Bott periodicity operator".

As a consequence of work of Atiyah and Adams, the methods of K-theory yielded a succession of results in homotopy theory. There began to be general consideration of other possible and useful "extraordinary" homology or cohomology theories with complicated homology of a point (see the survey [21]).

The author developed a series of algebraic-topological methods in bordism theory, where the bordism of a point coincides with one of the cobordism rings Ω_G considered above for $G = SO, U, SU, \mathrm{Sp}$. The theory of "unitary" bordism, with $G = U$, proved to be the most successful and effective as a new tool in algebraic homotopy theory.

The definition of bordism groups of arbitrary complexes is a natural combination of the Steenrod problem and the concept of bordism. Bordism is a very simple and important form of "nonlocalized" homology, which cannot be calculated as the homology of a chain complex.

By a "closed G-bordism" (or cycle) in a complex X we shall mean a pair consisting of a closed manifold M^n with normal G-structure, as above, together with a map $f\colon M^n \to X$.

Films [nonclosed G-bordisms] are defined analogously: they are manifolds W^{n+1} with boundary, normal G-structure and a map $g\colon W^{n+1} \to X$.

The sum of cycles, the notion of boundary and so on are defined in a natural way. So there arises the G-bordism group

$$\Omega_n^G = \text{cycles}//\text{boundaries}.$$

If X lies in the sphere S^N and $Y = S^N \setminus X$, then by definition (Alexander duality) we set

$$\Omega_G^k(Y) = \Omega_{N-k}^G(S^N, X).$$

This gives a definition of "cobordism" which is dual to "bordism". The cobordism and bordism $\Omega_G^*(X,Y)$ and $\Omega_*^G(X,Y)$ of pairs are defined in a natural way using manifolds with boundary, where the image of the boundary lies in Y. Thus there arises an important extraordinary homology theory for which, when $G = SO$ or U, the bordism groups $\Omega_*^G = \sum_j \Omega_j^G$ of a point were calculated in the above-mentioned theorems of Milnor and the author. For example, Ω_*^U is a graded polynomial ring having one polynomial generator in every even positive dimension. For the case $G = O$ the groups $\Omega_*^O(X)$ reduce to ordinary homology. The O and SO bordism theory was first considered as a homology theory by Atiyah. Some beautiful applications to a problem about fixed points of smooth maps of finite order were discovered by Conner and Floyd (see [44]). For smooth closed quasicomplex manifolds M^n an operation of "intersection" of cycles is defined, converting the full bordism group into a ring $\Omega_*^U(M^n) = \sum \Omega_j^U(M^n)$. For arbitrary spaces X the object dual to bordism, which is the cobordism $\Omega_U^*(X)$, always forms a graded ring, and the cobordism of a point is a ring of polynomials $\mathbb{Z}[t_1, t_2, \dots]$ having one generator in every negative even dimension, $\dim t_j = -2j$. The Poincaré–Atiyah duality law for quasicomplex manifolds of real dimension $2n$ states that, as usual

$$\Omega_j^U(M^{2n}) = \Omega_U^{2n-j}(M^{2n}).$$

(For a point, we have $n = 0$.)

Among the natural operations in bordism theory (where naturality means commutativity with continuous maps, i.e. functoriality) there arise some whose analogues in classical homology theory are completely trivial. These are the operations of "multiplication by a scalar": $x \to \lambda x$, $x \in \Omega_*(X)$, $\lambda \in \Omega_*(\mathcal{P}) = \Lambda$, where $\mathcal{P}$ is a point. Here, however, the role of the ring of "scalars" is played by the nontrivial bordism ring Λ of a point.

The general name for arbitrary additive maps $\theta\colon \Omega_*(X) \to \Omega_*(X)$, which are defined for all complexes and which commute with continuous maps and with the operation of suspension, is "stable cohomology operations" or "endomorphisms of the stable theory" Ω_* as an additive group. The algebra of all such operations θ for the bordism theory Ω_*^G is denoted by A^G. The similar object in the theory of classical $\mathbb{Z}_p$-homology was calculated by Serre and Cartan, and was called the "Steenrod algebra" A_p: it played a very important part in the machinery of algebraic homotopy theory considered above. The analogues of natural operations in K-theory are constructed from series of representations of unitary groups, which enable one to construct new stable classes of vector bundles from old ones: they were first considered by Grothendieck and Atiyah. The most important of these were discovered and actively applied by Adams.

A program for constructing a sequential algebraic-topological method, based upon extraordinary homology, for calculating stable homotopy groups was formulated in the first instance by Adams and others for the case of K-theory in about 1962. However, this program based upon K-theory simply failed to become reality. It is now clear that ordinary K-theory will, in principle, not do for this purpose.

A program of this kind, based upon unitary bordism theory, was completely realized by the author in [22] and [23]. The ring A^U of stable cohomology operations for this theory was completely calculated, and an "extraordinary" analogue of the above-mentioned Adams spectral sequence was discovered. This made is possible to calculate stable homotopy groups by proceeding from methods of homological algebra, and incorporated (as it turned out) everything attainable by K-theory. In comparison with its previous analogue based upon ordinary $\mathbb{Z}$-homology, this method gave a whole sequence of new results in a series of problems, and in particular in the classical problem of computing this stable homotopy groups of spheres, though it quickly became clear that this method offers no chance of a complete solution of that.

Lying at the foundation of these results is the construction of a unitary analogue, with values in $\Omega_U^*(X)$, of the ordinary Chern classes of complex vector bundles with base X, and of an analogue of Steenrod operations. The full ring S of the latter, together with the ring of multiplications by "scalars" $\lambda \in \Lambda = \Omega_U^*(\mathcal{P})$ mentioned above, generates the whole ring of operations A^U: if $a \in A^U$, then $a = \sum \lambda_i s_i$, so $A^U = \overline{\Lambda S}$ (the factors do not commute, and this difference from the classical theory has fundamental significance). Without going into the somewhat involved concepts and properties which arise here, we mention one very interesting matter which was first revealed and applied by the author and A. S. Mishchenko (see [22], Applications 1 and 2). One-dimensional complex bundles with fiber C^1 and group U_1 have a unitary analogue of the Chern class

$$\sigma_1(\eta) \in \Omega_U^2(X),$$

where X is the base of the bundle η. How can we calculate the class $w = \sigma_1(\eta \otimes \xi)$ in terms of $u = \sigma_1(\eta)$ and $v = \sigma_1(\xi)$? In classical cohomology theory, the class $c_1(\eta \otimes \xi)$ is simply the sum; in algebraic geometry, when X is an algebraic variety and η is a holomorphic bundle, the "holomorphic" classes c_1 of these form an abelian Lie group—the Picard variety.

In U-bordism theory a so-called commutative "formal group" arises. A formula

$$w = f(u, v) = u + v + \sum_{i \geqslant 1, j \geqslant 1} \lambda_{ij} u^i v^j$$

holds where $f(u, v) = f(v, u)$, $f(d(u, v), w) = f(u, f(v, w))$, and the elements λ_{ij} lie in the coefficient ring Λ, the cobordism ring of a point $\Lambda = \Omega_U^*(\mathcal{P})$.

Mishchenko obtained the following expression for $f(u, v)$:

$$f(u, v) = g^{-1}(g(u) + g(v)),$$

where

$$g(u) = \sum_{n \geqslant 0} \frac{[CP^n]}{n+1} u^{n+1}, \quad [CP^0] = 1$$

and $[CP^n]$ are the unitary cobordism classes of the complex projective spaces. These facts and a series of applications of them were published in [22] and [23].

Important progress was made in a paper of Quillen [53], which substantially developed the technique introduced in [22] of applying formal groups to homotopy theory. In a series of papers of the author, Kasparov, Mishchenko, Bukhshtaber, Bogomolov, Guseĭn-Zade, Krichever, Musin, and others, applications of formal groups were developed to problems in the theory of finite and compact groups of smooth transformations of manifolds (see [10] and [11]). In a joint paper of the author and Bukhshtaber [10], examples of some new "many-valued" analogues of formal groups [9] were discovered. A deep theory of two-valued formal groups was created later by Bukhshtaber [6], [7], and a series of topological applications of this was developed by him in collaboration with Panov, Nadiradze, and Shokurov [8].

§3. Simply-connected differential manifolds. The problem of classifying these in dimensions $n \geqslant 5$

The application of contemporary methods of algebraic topology to a profound investigation of the properties of smooth manifolds was begun in the work of Thom, as mentioned in §2, and then was continued by Hirzebruch in connection with problems arising from algebraic geometry (theorems of Riemann–Roch type). Important formulas were obtained for the signature $\tau(M^{4k})$ of the quadratic form defined by "intersection of cycles" in the middle homology group $H_{2k}(M^{4k}, R)$: this extended the original insight gained by Thom and Rokhlin in the early 1950's. It follows from bordism theory that $\tau(M^{4k})$ can be expressed in terms of the higher-dimensional Pontryagin characteristic classes (the Pontryagin numbers) of the closed smooth manifold. In particular, for $k = 1$ there was the well-known Thom–Rokhlin formula $\tau = \frac{1}{3}p_1(M^4)$; for $k = 2$ the following formula holds:

$$\tau = \frac{1}{45}(p_1^2 - 7p_2)$$

where p_i is the $4i$-dimensional Pontryagin class. We shall not state Hirzebruch's general formula. The structure of this formula for $k = 2$ stimulated Milnor to an important discovery in 1956: among the fiber bundles in the sense of Steenrod with base S^4, fiber S^3 and structural group $SO(4)$ there is a family of "fibrations of Hopf type", depending upon an integer α, in which the total space M^7_α of the fibration is homotopy equivalent to the sphere S^7 and bounds a smooth manifold E^8_α with boundary M^7_α. The manifold E^8_α is itself a fibration, but with a different fiber, namely the disc D^4.

The Pontryagin class of the total space E^8_α of the fibration can be any multiple having the form $p_1 = (4\alpha+2)a$, $\alpha = 0, \pm1, \pm2, \dots$, of the cohomology class a which generated $H^4(E^8_\alpha, \mathbb{Z}) = \mathbb{Z}$. By definition of the "Euler class", the selfintersection index is such that $a^* \circ a^* = 1$, where $(a^*, a) = 1$. Starting from the formula

$$p_2 = \frac{1}{7}(p_1^2 - 45\tau),$$

one can deduce that if the manifold M^7_α is diffeomorphic to the sphere S^7 then the expression $\frac{1}{7}(p_1^2 - 45\tau)$ must necessarily be an integer. Here $\tau = 1$ and $p_1^2 = (4\alpha + 2)^2a^2$, where the cohomology class a^2 generates $H^8(E^8_\alpha, M^7_\alpha; \mathbb{Z})$. In fact, if the manifold M^7_α is diffeomorphic to S^7, then one can glue the manifold E^8_α to the disc D^8 and obtain a smooth closed manifold with the same quantities (p_1, τ), but necessarily having an integral class p_2. By constructing a smooth function f which clearly had two nondegenerate critical points on M^7_α, Milnor further showed that M^7_α was topologically (and even piecewise linearly) homeomorphic to S^7. From this follows the remarkable result that different smooth structures can exist on manifolds: for example, M^7_α for $\alpha = 1$ gives a new smooth structure on S^1. After this work, the field known as differential topology took definitive shape [51].

In the early 1960's, as a result of work of Milnor and Kervaire, a theory of smooth structures on manifolds homotopy equivalent to spheres of dimension $n \geqslant 5$ (the theory of homotopy spheres) was developed. All of these are homeomorphic to ordinary spheres (Smale, Stallings, Wallace). Important general theorems were also proved about other simply-connected manifolds, such as Smale's theorem on the existence of a specific function for which the Morse inequalities become equalities (1960). These theorems of Smale soon proved extremely important in justifying the classification theory of closed smooth manifolds of dimension $n \geqslant 5$ (devised by the author at the end of 1961 and first published in [24]; detailed exposition in [25]). In fact, the method developed by the author allows one to classify manifolds up to what is known as h-cobordism, and in this sense some of the results hold also for $n = 4$; however, for $n \geqslant 5$ this is the exact classification. According to the author's method, one considers a class of manifolds having the same homotopy type and the same homotopy class of stable tangential (or normal) fibrations. This means that for any manifolds M^n_1 and M^n_2 of the given class, there exists a "tangential" or "normal" homotopy equivalence

$$f\colon M^n_1 \to M^n_2, \quad f^*\nu(M^n_2) = \nu(M^n_1),$$

transforming the stable tangential fibration $\nu(M^n_2)$ into $\nu(M^n_1)$. Such a class of manifolds can be absolutely effectively described by means of a homotopy group of an auxiliary space which we shall now describe. We take one of the manifolds M^n in this class, and embed it in a Euclidean space of sufficiently large dimension: $M^n \subset R^{n+N}$, $N > n+1$. For small $\varepsilon > 0$, a smooth ε-neighborhood U_ε of this manifold forms the total space of a normal fibration over M^n with the N-dimensional disc D^N as fiber. For large N, this fibration is uniquely defined by the manifold itself, since all smooth embeddings $M^n \subset R^{n+N}$ are equivalent.

We denote by $T_N(M^n)$ the quotient space in which the entire complement of the neighborhood U_ε in R^{n+N} is identified together into a single point:

$$T_N = T_N(M^n) = R^{n+N}/\bar{V}_\varepsilon, \quad \bar{V}_\varepsilon = R^{n+N} \setminus U_\varepsilon.$$

It is easy to see that the manifold M^n itself defines a certain "preferred" element in the homotopy group $\pi_{n+N}(T_N)$. In fact, a mapping of the sphere $g\colon S^{n+N} \to T_N$ is defined by the very process of identifying to a point the complement $\bar{V}_\varepsilon$ of the neighborhood U_ε, which occurs in the definition of the space T_N.

It is easy to see, using results of Cartan and Serre, that the group $\pi_{n+N}(T_N)$ has the form $\pi_{n+N}(T_N) = \mathbb{Z} + A$, where A is a finite abelian group. Here the element $1 = [g] \in \pi_{N+n}(T_N)$ constructed from the original manifold M^n, gives a generator of the group $\mathbb{Z}$. We consider the elements of the form $1 + x \in \pi_{N+n}(T_N)$, where $x \in A$. Otherwise stated, these are the homotopy classes of all possible maps of the sphere $g_x\colon S^{N+n} \to T_N$ which are homologous (but possibly not homo topic) to the map $g = g_0$, $x = 0$, constructed above.

We can now apply some very important ideas which evolved as bordism theory was developed. According to the definition of the space $T_N(M^n)$, the manifold M^n itself lies therein, together with a smooth neighborhood: $T_N \supset M^n$.

We shall assume that the map $g_x\colon S^{N+n} \to T_N$ is transverse regular to the submanifold M^n. The full inverse image $g_x^{-1}(M^n)$ is a smooth closed submanifold W^n of the sphere S^{n+N}, and a normal fibration $\bar\nu(W^n)$ is embedded with it. On account of its transverse regularity, the map g_x induces a mapping $g_x^*\colon \bar\nu(W^n) \to \bar\nu(M^n)$ of normal fibrations. The other important fact is that the map $g_x\colon W^n \to M^n$ has degree $+1$. These are all "trivial" inferences which can be deduced from transverse regularity. In fact, suppose given an arbitrary closed manifold $W^n \subset S^{n+N}$ with normal fibration $\bar\nu(W^n)$, and a mapping $\tilde g\colon W^n \to M^n$ of degree $+1$ which extends to a map $\tilde{\tilde g}$ of normal fibrations $\bar\nu(W^n) \to \bar\nu(M^n)$. Any such entity determines a mapping $S^{n+N} \to T_N(M^n)$ of the sphere, and the homotopy class of this mapping has the form $1 + x \in \pi_{n+N}$, where x is an element of finite order.

However, the properties of these "normal" maps of degree 1 prove to be extremely fruitful. One can say that in some intuitive sense such maps of degree 1 behave very much like retractions, or projections onto a direct summand, in which the complementary kernel is "like" a parallelizable manifold. This opens the prospect of adapting an analogue of Milnor and Kervaire's technique to the kernels of maps.

In every case, the following theorem can be proved:

1. *In dimensions $n \neq 4k+2$, $n \geqslant 5$, every homotopy class of the form $1+x$ in the group $\pi_{n+N}(T_N)$ has a t-regular representative, which is a map $g_x\colon S^{N+n} \to T_N$ having the property that the full inverse image $g_x^{-1}(M^n) = W^n$ is a manifold which is normal homotopy equivalent to the original manifold M^n.*

For $n = 4k + 2$, this holds either for all the elements of the form $1 + x$ or for half of them, i.e. for a subgroup of index 2 in the finite component A of the group $\pi_{n+N}(T_N) = \mathbb{Z} + A$.

For $n = 6$ or 14 exactly half of the elements are of this form.

2. *Suppose we already have two homotopic maps, $g_x^{(1)}$ and $g_x^{(2)}$ for which the full inverse images ${g_x^{(i)}}^{-1}(M^n) = W_i^n$ are normal homotopy equivalent to the manifold M^n. Then*:

a) *W_1^n is diffeomorphic to W_2^n if n is even and $n \geqslant 6$; and*

b) *W_1^n is obtained from W_2^n by the addition of a Milnor "exotic" sphere which bounds a parallelizable manifold, for odd values of $n \geqslant 5$.*

There are a finite number of non-trivial Milnor spheres of this kind; there is none for $n = 5$ or 13; there is never more than one for $n = 4k+1$; there are rather a large number of them for $n = 4k + 3$, which has been calculated by Milnor and others; this number is equal to 27 for $n = 7$.

3. *On the homotopy group $\pi_{n+N}(T_N)$ there is a natural action of the group of homotopy classes of automorphisms of the pair consisting of the manifold M^n and its normal fibration $\bar\nu$ in R^{n+N}. The orbits of this group on the set of elements of the form $1 + x$ correspond to manifolds which are tangentially homotopy equivalent to M^n, under the conditions detailed in statements* 1 *and* 2.

For $n = 4$, in statement 2 one has to change the word "diffeomorphic" into "h-cobordant." In addition, the analogue of the result of W. Browder and the author, which is formulated below and which gives conditions under which a simply-connected complex K is homotopy equivalent to a smooth manifold, is known not to be true for $n = 4$.

The method of proving this theorem consists essentially of an elaboration of the technique, initiated by Milnor and Kervaire, of "killing" homotopy groups of manifolds by means of Morse modifications. However, even in the special case of manifolds of the homotopy type of the sphere S^n, which had been previously investigated by Milnor and Kervaire, the essential geometrical idea of the author's approach differs from the geometrical basis of theirs in that it identifies the smooth structures on spheres by means of the homotopy groups of another space.

One of the most significant consequences of this theorem is the following. A simply-connected smooth manifold of dimension $n \geqslant 5$ is determined to within a finite number of possibilities (for which one can easily give an upper bound) by its homotopy type and the integrals of its Pontryagin classes over cycles.

By using the same technique, one can settle the problem of determining in which cases a pair consisting of a simply-connected complex K and a vector bundle $\bar{\nu}$ over K with fiber R^N is homotopy equivalent to a closed smooth manifold $f\colon M^n \to K$ such that the bundle $f^*\bar{\nu}$ over M^n is the normal bundle to M^n in R^{n+N} (due to the author, when K is a smooth or PL-manifold, and to W. Browder in a much more general form: see the author's short communication to the International Congress of Mathematicians at Stockholm, August 1962; and Browder's report to the Aarhus colloquium on algebraic topology, August 1962).

An improved exposition of all the proofs, together with various applications and a series of further problems, appeared in [25]. The generalization of this technique to the case of piecewise linear (PL) manifolds, which presented no problems, is also discussed there. The generalization to manifolds with boundary was given by Golo [12] and Wall [63].

Subsequently, this technique was applied by various authors to a series of problems. It was also generalized to non-simply-connected manifolds (the author [27], [28], Wall [61], [62], and others). In particular the reader will find important applications of this technique, generalized to manifolds with free abelian fundamental group, in the following section. The subject there will be the proof of the topological in variance of the rational Pontryagin classes.

§4. Pontryagin classes and the fundamental group. Topological invariance of the rational classes. Hermitian K-theory for rings with involutions

As we know, for complexes and in particular for closed smooth manifolds there are four fundamental equivalence relations which enter into problems of topology. They are: smooth homeomorphism with smooth inverse, or diffeomorphism; piecewise-smooth (PL)-homeomorphism; continuous homeomorphism; and, finally, homotopy equivalence. The appearance of homotopy equivalence, and its role in topology, are occasioned by the fact that all the classical topological invariants—homology and the fundamental group, and all the homotopy groups as well—turned out to be homotopy invariants. The only exception was the so-called Reidemeister torsion, discovered in the 1930's. This is a strange invariant which is associated with the fundamental group and allows one to prove that certain non-simply-connected homotopy equivalent manifolds (for example, lens spaces) are not smoothly or piecewise linearly equivalent. In the three-dimensional case, Moise proved in the 1950's the so-called Hauptvermutung der Topologie: topological manifolds always admit one and only one PL structure (and even smooth structure, as became clear later). Therefore examples were known of three-dimensional closed smooth manifolds (lens spaces) which had been proved topologically nonhomeomorphic although they were homotopy equivalent. For simply-connected manifolds, however, this question (the Hurewicz problem) remained open. Since the topological invariance of Reidemeister–Whitehead (et al.) torsion had not been established in dimensions $n > 3$, the question of the relationship between homotopy type and topological homeomorphism of closed manifolds was also open in the mid-1960's for all non-simply-connected manifolds for $n > 3$.

Again, in the mid-1950's, Thom and Dold showed that as a consequence of Serre's theorem on the finiteness of homotopy groups of spheres there exist very many simple homotopy equivalent but nondiffeomorphic smooth closed manifolds even among fiber bundles with spheres of unequal dimension as fiber and base, these examples having different Pontryagin classes. Thus the failure of homotopy invariance for the integrals of Pontryagin classes over cycles (or, as one usually says, for the rational classes) was generally known. In 1957 Thom, Rokhlin and Shvarts proved the combinatorial (or PL) invariance of the rational classes (see [59] and [30]). Since the method of these papers was applied by the author in 1965 to prove the topological invariance of the integrals of Pontryagin classes over cycles, we shall explain the idea of it here.

According to the general formula of Hirzebruch for the signature $\tau(M^{4k})$ of the natural quadratic form on the homology group $H_{2k}(M^{4k}, R)$, which we considered before at the beginning of §3, there exists for each number $k = 1, 2, \dots$ a canonical graded "Hirzebruch polynomial" $L_k(p_1, \dots, p_k)$ in the Pontryagin classes, with rational coefficients, such that

$$\tau(M^k) = (L_k, M^{4k}) = \int_{M^{4k}} L_k,$$

where $L_1 = \frac{1}{3}p_1$, $L_2 = \frac{1}{45}(p_1^2 - 7p_2)$, ... Here the class p_k occurs in the expression L_k with nonzero coefficient (this is important!), $L_k = \alpha_k p_k + \dots$, $\alpha_k \neq 0$. We note that the Pontryagin–Hirzebruch number (L_k, M^{4k}) is a homotopy-invariant expression in the rational Pontryagin classes, in view of its equality with the signature $\tau(M^{4k})$. This number turns out to be the unique homotopy-invariant expression in the rational Pontryagin classes for all simply-connected manifolds of dimension $4k$.

We now transfer from the basis $(p_1, p_2, \dots)$ for the rational classes to the multiplicative basis $(L_1; L_2, \dots)$. This is possible in view of the properties of the polynomials L_k indicated above.

Let us consider a smooth manifold M^N and a cycle $z \in H_{4k}(M^N, \mathbb{Z})$ therein. It is easy to reduce matters to the cycles of codimension $N - 4k > N/2$ which can be realized by smooth submanifolds with trivial normal bundle $M^{4k} \times R^{N-4k} \subset M^N$, $[M^{4k}] = z$. Using the Hirzebruch formula, we define the class $L_k(M^N)$ to be the cohomology class whose integral over the cycle z is equal to $\tau(M^{4k})$:

$$(L_k(M^N), z) = \tau(M^{4k}).$$

It is easily proved that such a definition is combinatorially invariant. That is the idea of Thom, Rokhlin and Shvarts. It is curious that the analogue of this construction of the classes p_k in mod q cohomology for prime numbers q is not valid for all q, but only for sufficiently large $q > q_0$, where q_0 depends upon k. This analogue was discovered later, in 1966, by Rokhlin and the author (see the author's survey [29]). It is based upon an extension of the concept of the signature $\tau(M^{4k})$

to oriented manifolds with boundary in which the boundary $\partial M^{4k} = V_1 \cup \cdots \cup V_q$ is nonempty, since mod q cycles are realized by manifolds with $V_1 = \cdots = V_q$. Of course, in the foregoing, the quantity $\tau(M^{4k})$ is the difference between the numbers of positive and negative squares in the "intersection of cycles" form on the space $H_{2k}(M^{4k}, R)$. Now, however, the form may be degenerate; but that has little significance. What is most important from our point of view is the additivity property of the signature for manifolds with boundary:

$$\tau\left(M_1^{4k} \cup_w M_2^{4k}\right) = \tau(M_1^{4k}) + \tau(M_2^{4k}),$$

where the gluing takes place along a whole connected component of their boundaries: $\partial M_1^{4k} = W \cup \tilde{W}$ and $\partial M_2^{4k} = W \cup \tilde{\tilde{W}}$. This is necessary for the validity of the definition of the classes. The additivity property of the signature in this sense, for $4k$-dimensional manifolds with boundary, under the operation of gluing along a whole component of the boundary, links the signature with the Euler characteristic. This property attracted the attention of a number of authors. It can be made into the basis of an axiomatization which singles out these two invariants.

We note in passing that for $k = 2$ we have the formula

$$p_2 = \frac{1}{7}(9L_1^2 - 45L_2).$$

As Milnor and Kervaire showed in 1962, the factor $\frac{1}{7}$ is there "by no accident": the Pontryagin class p_2 as an integral cohomology class (that is, its 7-torsion), or the class p_2 in modulo 7 cohomology, turns out not to be a PL invariant (see Milnor's plenary address to the International Congress of Mathematicians in Stockholm, August 1962, [64]).

Thus the class $L_k(M^{4k})$ is homotopy invariant, and there are no other homotopy invariant expressions for closed simply-connected manifolds. However, the class $L_k(M^{4k+1})$ is only of significance for non-simply-connected manifolds, for which $H_1(M^{4k+1}, R) = H^{4k}(M^{4k+1}, R)$. In the autumn of 1964 the author obtained a homotopy invariant formula for the integral of the class $L_k(M^{4k+1})$ over an integer homology cycle z in terms of the cohomology ring of an infinite-sheeted covering $\hat{M} \xrightarrow{p} M^{4k+1}$, $(\operatorname{Im} p_*, z) = 0$. The construction of the covering space $\hat{M}$ is this: the cycle z is realized by a closed submanifold $M^{4k} \subset M^{4k+1}$. By cutting the manifold M^{4k+1} along the cycle M^{4k}, we obtain a manifold W with two identical boundary components isomorphic to M^{4k}:

$$\partial W = M_1^{4k} \cup M_2^{4k}.$$

The covering space $\hat{M}$ is obtained by gluing an infinite number of copies of W to one another: $\hat{M} = \bigcup W_i$, $W_i = W$, where the second boundary component of the copy W_i is glued to the first boundary component of the copy W_{i+1}. There is a cycle $\hat{z} \in H_{4k}(\hat{M})$ which is realized by the same manifold $M^{4k} \subset \hat{M}$, the first boundary component of the copy W_0. On the linear space $H^{2k}(\hat{M}, R)$ (which is infinite-dimensional in general) there is a form of finite rank which depends upon the cycle $\hat{z}$:

$$\langle x, y\rangle = (xy, \hat{z}), \quad x, y \in H^{2k}(\hat{M}, R).$$

The signature of the form $\langle x, y\rangle$, which is denoted by $\tau(\hat{z})$, is finite and coincides with the integral of the class $L_k(M^{4k+1})$ over the cycle z (see [26]). The number $\tau(\hat{z})$ is clearly a homotopy invariant of the manifold M^{4k+1}. Thus in non-simply-connected manifolds there sometimes are cycles over which the integrals of the

Pontryagin–Hirzebruch classes L_k are homotopy invariant. The cycles of codimension one are the simplest cycles of this kind. Cycles of codimension two are generally not of this kind: for instance, in simply-connected manifolds they never are, even for the very simple manifold $S^2 \times S^4$.

The following approach to proving the topological invariance of the rational classes p_k or L_k, which uses the fundamental group as an auxiliary device, was proposed and completely carried out by the author. Suppose that, as previously, we are considering cycles $z \in H_{4k}(M^N, \mathbb{Z})$ which are realized by smooth submanifolds with trivial normal bundle $M^{4k} \times R^{N-4k} \subset M^N$, $[M^{4k}] = z$. In the original smooth structure we have a direct product and a formula

$$(L_k(M^N), z) = \tau(M^{4k}).$$

Let us change to a completely arbitrary new smooth structure on the manifold M^N. A new smooth structure on the whole manifold directly induces a smooth structure on each open set in M^N, and in particular on the set $M^{4k} \times R^{N-4k} \subset M^N$. In $M^{4k} \times R^{N-4k}$ it is possible to find a "toric" open subset

$$M^{4k} \times T^{N-4k-1} \times R \subset M^{4k} \times R^{N-4k},$$

where T^j is a torus of dimension j. Here $T^{N-4k-1} \subset R^{N-4k}$ is any standard embedding whatsoever of the torus in Euclidean space of dimension one greater. Without loss of generality, it can be assumed that the manifold M^{4k} is simply-connected, and that $k > 1$. The set $M^{4k} \times T^j \times R$ (where $j = N - 4k - 1$) has a free abelian fundamental group. In the new smooth structure, it is a priori not necesesarily a direct product in the smooth sense.

Nevertheless, by generalizing to non-simply-connected manifolds a technique of smooth topology which had been worked out during the development of manifold classification theory (see §3) and subsequent progress, by W. Browder and J. Levine in particular, the author proved the following [27], [28].

1. *A smooth manifold W of high dimension, which is topologically homeomorphic to the direct product of a closed manifold V with a line* ($W \approx \tilde{V} \times R$), *is actually diffeomorphic to a smooth direct product if $\pi_1(W)$ is a free abelian group $W = \tilde{V} \times R$, where $\tilde{V}$ is homotopy equivalent to V.*

2. *If a smooth closed manifold W of high dimension is homotopy equivalent to a product with a circle, i.e. $W \sim V \times S^1$, then the natural $\mathbb{Z}$-covering $\hat{W} \to W$ is diffeomorphic to a direct product with a line*: *$\hat{W} = \tilde{V} \times R$, where $\tilde{V}$ is homotopy equivalent to V. It is also assumed here that $\pi_1(W)$ is a free abelian group.*

By applying these assertions directly to the non-simply-connected open set $M^{4k} \times T^{N-4k-1} \times R \subset M^N$ which we introduced in an apparently artificial way, we obtain in any smooth structure an equality expressing the integral of the L-class over the cycle z in terms of a homotopy invariant of the open set $M^{4k} \times T^{N-4k-1} \times R$:

$$(L_k(M^N), z) = \tau(M^{4k}).$$

This proves the topological invariance of the rational Pontryagin classes. The general idea of this argument recalls the idea of "étale topology", which was introduced by Grochendieck at the end of the 1950's in order to define the homology of algebraic varieties over fields of finite characteristic by using a novel structure involving coverings of non-simply-connected open sets in the Zariski topology.

We note that from the above lemmas one can also obtain the homotopy invariance of certain integrals of the classes L_k over cycles. If the manifold W^{4k+m} is closed and is homotopy equivalent to a product $M^{4k} \times T^m$, then the inner product (integral) $L_k(W)$ with the cycle $z = [M^{4k}]$ coincides with $\tau(M^{4k})$.

In a series of papers of the author, Rokhlin, Kasparov, Hsiang and Farrell, topological methods have been used to prove a theorem on the homotopy invariance of the integrals (L_k, z) over all cycles which have the form of an intersection of cycles of codimension one. An analytical proof was found by Lusztig (see the survey [16]).

The general "higher signatures conjecture" consists of the following. Let us consider a closed manifold M^n and a cycle $z \in H_{n-4k}(M^n, \mathbb{Z})$. Let $\pi = \pi_1(M^n)$. We consider the natural mapping into an Eilenberg–Mac Lane complex which induces the identity isomorphism of fundamental groups

$$f\colon M^n \to K(\pi, 1), \quad f^*\colon H^*(\pi, R) \to H^*(M^n, R).$$

Suppose that the cycle z lies in the image of the mapping f^* (i.e. it "lives" on the fundamental group, as one says, and is induced from its homological algebra).

Conjecture. *The integrals $(L_k(M^n), z)$ over such cycles are homotopy invariants. There are no other homotopy invariant integrals of Pontryagin classes.*

Results proving this conjecture in particular cases of nonabelian groups π have been obtained by Lusztig, Mishchenko, Yu. P. Solov'ev, Cappell, Kasparov and others (see [16] and [14]). In its general form, however, the conjecture has as yet been neither proved nor disproved. The generalization of classification techniques to non-simply-connected manifolds is closely connected with the theory of stable invariants of Hermitian or skew-Hermitian forms $V^* = \pm V$ which are nondegenerate (i.e. are given by invertible matrices V) and have elements in the group rings $\mathbb{Z}[\pi]$, regarded as rings with involution. The involution is defined by the equation $\pm\bar{\sigma} = \sigma^{-1}$ where the sign depends upon the orientation of $\sigma \in \pi$. This theory was first considered in 1966 by the author [28], in connection with problems on the invariance of Pontryagin classes, and by Wall [61], working independently; a generalization of this kind, in the form of a reduction of the surgery problem to algebraic objects of an unimaginable degree of complexity, was completed by Wall in 1970 (see [62]). The invariants of these forms, as the author showed in his papers on the topological invariance problem, are closely connected with characteristic classes. It was in this connection that "Hermitian algebraic K-theory" over rings with involution was later developed in [30]–[32]. It turned out that the Hermitian analogue of K-theory, in particular, possessed remarkable formal properties.

From a purely classical point of view, this algebraic theory is required to give an effective classification of the stable invariants of nondegenerate Hermitian or skew-Hermitian forms, i.e. of invertible Hermitian matrices V such that $\pm V^* = V$, whose elements belong to some given ring with an involution $a \to \bar{a}$, $\overline{ab} = \bar{b}\bar{a}$, such as for example the group ring in the theory of non-simply-connected manifolds, with $\bar{\sigma} = \pm\sigma^{-1}$, $\sigma \in \pi$. In creating this branch of stable algebra the author made use of the algebraic analogies, which proved to be extremely natural, with the well-known Hamiltonian formalism of analytical mechanics and symplectic geometry. It emerged that in this particular theory, but not in the general algebraic K-theory of Quillen, Volodin and others, there is (if it is permissible to divide by two in the ring) an analogue of the so-called Bott periodicity of classical K-theory; thus the topology of manifolds is to some extent echoed in the case of group rings, though the algebrization of these "echoes" proved difficult. In fact, this theory must contain the key to the conjectured relationship between the Pontryagin classes of closed manifolds and the homological algebra which is associated with the fundamental group (see [32]). Lusztig, Mishchenko, Solov'ev (see [16]) and Kasparov [15] applied the methods of functional analysis to solve the problem in particular cases, but the latent algebraic mechanism of the connections which exist here still remains obscure.

Returning to problems in topology, we note that it is strange how in the subsequent work of Kirby, Siebenmann and others, which is devoted to the solution of a series of classical problems in the topology of continuous homeomorphisms, they continue to use and develop the seemingly artificial method of introducing "toric" open sets and reducing purely continuous problems to techniques in the smooth topology of manifolds with free abelian fundamental group. The only exception is Reidemeister–Whitehead torsion, the topological invariance of which was later successfully proved by more direct and elementary means (Edwards and Chapman).

Also closely related to the family of questions under consideration is the "Browder–Levine problem": under what conditions is an open manifold the interior of a compact manifold with boundary? If every component of the boundary is simply-connected (this is easily formulated in terms of the open manifold itself) then the problem is amenable to the classification technique of Browder and the author (see §3), as was first shown by Browder, Levine and others in 1964. A special case of the general boundary problem is the question about decomposing a manifold as a smooth product $W = V \times R$ of a closed manifold with a line, which was also solved for simply-connected manifolds by Browder and Levine. We recall (see above) that a generalization of this last result to the case of a free abelian group π_1 played a vital technical role in the proof of the topological invariance of the rational Pontryagin classes. In this, unlike the papers of Browder and Levine written in 1964, the author was obliged to take steps to search out effective algebraic tests for coverings of compact manifolds: if $n \geqslant 6$, the discrete group $\mathbb{Z}$ acts by smooth or topological transformations on the manifold W^n with compact quotient $W^n//\mathbb{Z}$, the group $K^0(\pi_1)$ is trivial and the manifold W^n has the homotopy type of a finite complex, then $W^n = V \times R$. For example, the group $K^0(\mathbb{Z} \times \cdots \times \mathbb{Z}) = 0$ on account of a classical theorem of Hilbert. This program of enquiry was carried on by V. L. Golo, who constructed beautiful examples of manifolds W^n not admitting a product decomposition, with cyclic $\pi_1 = \mathbb{Z}_p$, where $K^0(\mathbb{Z}_p) \neq 0$, by using classical results of Kummer and others in the theory of numbers [13]. A. M. Brakhman investigated the general problem of the "compactification", by means of a boundary, of manifolds W^n which are covering spaces of compact closed manifolds: he obtained a number of substantial results (see [4]). The general problem of boundaries for open manifolds was investigated, from 1965 onwards, by Siebenmann. His results, first published in [56] and [48], seem to us to be insufficiently effective.

§5. Qualitative theory of foliations. Critical points and level surfaces of many-valued functions

By definition, a foliation (without singularities) consists of an integrable distribution which assigns, in a smooth fashion, to every point of an n-dimensional manifold some k-dimensional plane tangential at that point. Integrability of the distribution means that through every point of the manifold one can construct a k-dimensional surface (a leaf) to which the distribution is tangential at every one of its points. The integrability conditions for a foliation with $k = n - 1$ are called the "conditions for solvability of the Pfaffian equations" in classical textbooks (the

Frobenius condition). For $k = 1$, on the other hand, there are no integrability conditions: they are always satisfied. One-dimensional foliations ($k = 1$) again arise because many properties in the qualitative theory of trajectories in dynamical systems (for example, existence of periodic trajectories, integral manifolds, the property of remaining indefinitely in a compact set, boundary sets, etc.) depend upon the dynamical system only through its "foliation" and do not depend upon the parametrization of the trajectories. The case $k = 2$ arises for holomorphic (complex) dynamical systems with complex time, where the integrability conditions is again always satisfied. A series of beautiful and important examples of foliations arises even on three-dimensional manifolds, and they play a large part in the ergodic and qualitative theory of so-called strongly hyperbolic (Anosov) systems. For every trajectory $\gamma(t)$ in such a system there are two families of trajectories $\kappa_\pm(t)$ such that

$$\begin{aligned} \kappa_+(t) &\to \gamma(t), \quad t \to +\infty \\ \kappa_-(t) &\to \gamma(t) \quad\ \ t \to -\infty \end{aligned}$$

(the speed of approach of $\kappa_\pm(t)$ to $\gamma(t)$ as $t \to \pm\infty$ is exponential). The families $\kappa_\pm(t)$ lie on the leaves of foliations $\alpha_\pm$ of dimensions $k_\pm$, where $k_+ + k_- = n + 1$. The intersections of leaves $\alpha_+ \cap \alpha_-$ consist of trajectories: a pair of leaves intersects in precisely one trajectory. In addition, an arbitrary perturbation of the system has similar foliations (Anosov [2]).

The simplest examples of such systems are:

a) The geodesic flow on n-dimensional manifolds of negative curvature, which gives a dynamical system on the $(2n-1)$-dimensional manifold of unit linear elements: here $k_+ = k_- = n$.

b) We consider the linear mapping of the torus $T^n \to T^n$ given by a unimodular matrix A with integer elements whose eigenvalues $\lambda_j(A)$ are such that $|\lambda_j| \neq 1$. Let $k_+ - 1$ be the number of values of j for which $|\lambda_j| < 1$, and $k_- - 1$ the number of values of j for which $|\lambda_j| > 1$. We can construct a dynamical system with continuous time from the mapping A by a standard method: we consider the manifold

$$M^{n+1} = T^n \times R / \{(x, p) \sim (A(x), p + 1)\},$$

where $x \in T^n$ and $p \in R$. We note that M^{n+1} is a twisted product, with base S^1 and fiber T^n. The trajectories of the dynamical system are (x_0, t), $-\infty < t < \infty$. One says that systems have "discrete time" if they are induced by a single map, as the system in the case just given is induced by the map $A\colon T^n \to T^n$.

A series of nontrivial examples of C^∞-smooth foliations with $k = 2$ on the three-dimensional sphere S^3 was constructed by Reeb [55]. It was observed by Zieschang and the author in [33], and also by Lickorish, that Reeb's methods allow one to construct C^∞-smooth foliations on an arbitrary three-dimensional manifold. Later, direct constructions of this kind (and more complicated ones) were brought to a virtuoso standard of perfection, and after important intermediate results by various authors they enabled Thurston and others to prove the existence of foliations without singularities on arbitrary closed manifolds, in any homotopy class of distributions.

It seems that the first observations of the qualitative topological properties of foliations are due to Reeb. For example, the concept of a "limit cycle" generalizes in a natural way to foliations, and links up with the differential-geometric concept of holonomy. Similar ideas were evolved independently and at the same time by

I. G. Petrovskiĭ, who wished to apply them to the well-known problem of the number of limit cycles of systems in the plane with rational right-hand side: however, this program of research, which he and E. M. Landis were jointly attempting to carry out, proved not to be feasible.

It is interesting to note that research into the theory of foliations in the USSR was to a large extent initiated through the influence of this very work. In 1961 V. I. Arnol'd took the initiative by reviewing the work of Petrovskiĭ and Landis in his seminar, although he did not succeed in making much progress with it then. Through his influence, and also that of Smale's ideas on dynamical systems, discussion and propaganda began concerning certain problems in the theory of foliations, and propaganda about them. There soon appeared papers of Anosov and Sinaĭ about structural stability and the metric theory of dynamical systems which possess pairs of foliations as considered above. The author's work on the closed leaf in foliations of codimension one was carried out at the end of 1963.

In the winter and spring of 1964 Anosov and the author, having gained experience in working with foliations, examined in detail a new text prepared by Landis. It soon became clear that there was a definite inaccuracy in the proof that the fundamental invariant which they introduced, the "genus", was unaltered by change of parameters in the system; and the calculation of it for small perturbations of concrete systems was to a lesser degree unclear. After some consideration by the author and Landis, this was acknowledged, and in 1964 Petrovskiĭ and Landis announced in print that their work was unproved. In the autumn of 1964, the celebrated mathematician and scientific administrator Ivan Georgievich Petrovskiĭ invited the present author, then a Candidate in the physico-mathematical sciences, to his rectorial office in the Lenin hills. He said persuasively and with conviction that the problem of limit cycles should be tackled and an attempt made to carry it through to a conclusion. I, however, expressed the opinion that there seemed to be no "hold" on the problem with the present range of methods, and that it was unlikely to be solved in less than 50 years. (It was typical of the rare qualities of the man that Petrovskiĭ thereafter supported all by scientific and pedagogical undertakings.) Was this an accurate estimate? A few years afterwards, in 1969, a student of Landis, Yu. S. Il'yashenko, who apparently had taken part in the analysis in 1964 of the criticisms put forward, by Anosov and the author, published a paper in which he proved rigorously that the "genus" does not have values which are bounded (with respect to the degree of the polynomials). Most importantly, he later refuted the proof given by Dulac in 1923 of the finiteness theorem for the number of cycles of an individual equation. Only most recently, through new ideas of students of Arnol'd (Varchenko and Petrov), have proper results been obtained about the finiteness of the number of cycles for small perturbations of Hamiltonian systems (a problem posed by Arnol'd in 1976).

Let us consider an arbitrary k-dimensional foliation without singularities on a manifold M^n. In this foliation we choose a leaf $\alpha \subset M^n$ and a point $x_0 \in \alpha$. At x_0 we construct D^{n-k} normal to the leaf. An arbitrary closed path $\gamma(t)$ in the leaf α, beginning and ending at x_0 ($\gamma(0) = \gamma(1) = x_0$), defines a mapping of some finite neighborhood U of x_0:

$$U \subset D^{n-k}, \quad \sigma_\gamma \colon U \to D^{n-k}.$$

This "holonomy map" does not change under homotopy of the path within its class in $\pi_1(\alpha, x_0)$. Thus there is defined a "holonomy representation" $\sigma \colon \pi_1(\alpha, x_0) \to G$, where G is the group of germs of diffeomorphisms of a neighborhood of zero $x_0 \in D^{n-k}$.

Those paths γ in the leaves α which determine nontrivial holonomy are called "limit cycles".

In the case of foliations of codimension one, in which $k = n - 1$, we have a mapping into the group of germs of diffeomorphisms of the line. In the important case of "orientable" foliations, all the transformations σ_γ for all the leaves preserve the orientation of the disc D^{n-k}. In this case, if $k = n-1$, the map σ_γ determines separately germs around zero of mappings of the half-lines, R_+ on the right and R_- on the left:

$$\sigma_+ \colon \pi_1(\alpha, x_0) \to G_+, \quad \sigma_- \colon \pi_1(\alpha, x_0) \to G_-,$$

where G_+ are the groups of germs about zero of mappings of the half-lines $R_\pm$ into themselves preserving the origin of coordinates, $0 \to 0$. Thus there are "right" and "left" limit cycles, with $\sigma_{+\gamma} \neq 1$ and $\sigma_{-\gamma} \neq 1$ respectively. For a C^∞ foliation it can happen that $\sigma_{+\gamma} \neq 1$ and $\sigma_{-\gamma} = 1$ (a one-sided limit cycle). For real analytic foliations, $\sigma_{+\gamma} = 1$ always implies $\sigma_{-\gamma} = 1$ and vice versa. Reeb conjectured that any C^∞ foliation of codimension one on S^3 has a one-sided limit cycle. In particular, on S^3 there are no analytic two-dimensional foliations without singularities. Haefliger [46] proved that conjecture in the following form: if on any manifold M^n there is a real analytic foliation without singularities of codimension one, then there is an element of infinite order in the group $\pi_1(M^n)$. Every C^∞ foliation on a simply-connected closed manifold has a one-sided limit cycle on some leaf.

The key idea of the paper [33] on the theory of foliations of codimension one concerns an application, not of limit cycles, but of the "vanishing" cycles on leaves which the author introduced. In [33] the author called them "paths which as limits are homotopic to zero".

Suppose given an element $\gamma \in \pi_1(\alpha, x_0)$ for some leaf, such that $\sigma_{+\gamma} = 1$ (a nonlimit cycle "on the right"). We say that γ is a vanishing cycle if an arbitrary displacement $\gamma_\tau(t)$ of it onto a nearby leaf on the right (for sufficiently small τ, where $\gamma_0 = \gamma$) is homotopic to zero in that leaf. Any one of the conditions listed below guarantees the presence of leaves with nontrivial vanishing cycles (we are considering either foliations on closed manifolds M^n or foliations on manifolds with boundary in which every component of the boundary is a whole leaf).

a) $\pi_1(M^n)$ is finite.

b) $\pi_2(M^n) \neq 0$, but $\pi_2(\alpha) = 0$ for any leaf α. For $n = 3$, $\pi_2(\alpha)$ implies $\alpha = S^2$ or $\alpha = RP^2$. In this case all the leaves are compact.

c) There is a leaf α such that the map $\pi_1(\alpha) \to \pi_1(M^n)$ has a nontrivial kernel. For example, the leaf α might be the boundary of M^n. An important example is $n = 3$, $M^n = D^2 \times S^1$ (the solid torus).

The closed leaf theorem is obtained from a combination of this assertion and the following theorem: if $n = 3$ and α is a leaf with a nontrivial vanishing cycle, then the leaf α is compact, is diffeomorphic to a torus T^2, and bounds a solid torus $D^2 \times S^1$ inside which the foliation is homeomorphic to a certain standard "Reeb foliation". In particular, on any three-dimensional manifold M^3 whose universal covering space is noncontractible, any C^∞ 2-foliation has a closed leaf which is a torus T^2 (or S^2).

Some other applications of the concept of vanishing cycle are the following:

1) If there exists an analytic foliation on M^3, then there also exists a smooth foliation in which the leaves do not have vanishing cycles. Consequently, either $\pi_1(M^3)$ is infinite and $\pi_2(M^3) = 0$, or all the leaves are compact and the covering of M^3 is $S^2 \times R$.

2) If on M^n we are given a system of strongly hyperbolic type in which one of the foliations has codimension 1 (see above), then π_1 is infinite and $\pi_2 = 0$. Later, Margulis proved that for $n = 3$ the group π_1 has exponential growth.

3) The topological types of analytic foliations on the solid torus $D^2 \times S^1$, in which the boundary is a leaf, are easily classified by means of so-called closed braids, which are conjugacy classes in the Artin braid groups.

Apart from these results, there is the following theorem of Sacksteder and the author: if the leaves have no limit cycles, then all the leaves are identical, and the manifold M^n is obtained as follows:

$$M^n = (\alpha \times R)//\mathbb{Z} \times \cdots \times \mathbb{Z},$$

where α is a leaf, and the group $\mathbb{Z} \times \cdots \times \mathbb{Z}$ operates freely on the covering space $\alpha \times R$.

Thus foliations without limit cycles look like the level surfaces of closed 1-forms without singularities. If the leaf is R^{n-1}, then the manifold M^n has the homotopy type of T^n. This is the actual situation for Anosov systems with discrete time, if one of the foliations has codimension one. However, here one has to consider nonsmooth foliations. The necesesary supplementary research was carried out by Brakhman [5], who eliminated thereby a deficiency from the proof of this theorem in the author's paper [33]. The situation is considered in detail by Anosov [3].

We now consider some properties of foliations which are defined on closed smooth manifolds M^n by a general closed 1-form, i.e. by a Pfaffian equation $\omega = 0$, where $d\omega \equiv 0$. If the form ω has no singular points, we obtain a foliation without singularities and without limit cycles. If k is the number of pairwise incommensurable periods (integrals of the form ω over 1-cycles) then we have, as above,

$$M^n = (\alpha \times R)//\mathbb{Z} \times \cdots \times \mathbb{Z} \quad (k \text{ factors}),$$

and α is the leaf. It is not hard to prove that in this case the leaf α is a regular covering of some compact manifold with group $\mathbb{Z} \times \cdots \times \mathbb{Z}$ ($k-1$ factors); see [35], §6. As for the topology of the manifold M, one can only say that it is a twisted product whose base is a circle, for any $k \geqslant 1$.

Far more difficult, interesting and extremely comprehensive from the viewpoint of analysis is the situation when the closed form ω has nondegenerate (and therefore isolated) critical points; these are just like the critical points of smooth functions. Foliations of this kind were formerly not studied. Some results, and a discussion of a number of problems, can be found in [35], §6. We note the simplest special case $k = 1$, when all the integrals of the form co over 1-cycles are commensurable: suppose that they are all integers. Then there is a mapping into the circle, or a complex function modulo 1:

$$f = \exp\left(2\pi i \int_{x_0}^{x} \omega\right) : M^n \to S^1.$$

The level surfaces here are all compact, but there arises the question of estimating the number of critical points of given index. Let us consider the ring of (formal)

Laurent series with integer coefficients having any finite number of negative terms:

$$q \in \hat{K}^{+}, \quad q = \sum_{i \gg -N} n_i t^i.$$

This ring is the completion of the group ring of the group $\mathbb{Z}$. We consider the Z-covering $\hat{M}$ which converts f into a single-valued real function

$$\hat{M} \to M^n, \quad \frac{1}{2\pi i} \ln f \colon \hat{M} \to R.$$

We denote by $m_i(f)$ the number of critical points of index f. The group $\mathbb{Z}$ operates on $\hat{M}$; we denote a generator of it by t. Therefore the chain complex of $\hat{M}$ becomes a $\mathbb{Z}[t, t^{-1}]$-module. We can consider the homology of $\hat{M}$ with coefficients in the ring $\hat{K}^{+} \supset \mathbb{Z}[t, t^{-1}]$. The ring $\hat{K}^{+}$ is a principal ideal ring. Therefore in the homology of $\hat{M}$ with coefficients in $\hat{K}^{+}$ we can introduce the analogues of the "Betti numbers" (the ranks) and the "torsion coefficients". We denote them respectively by $b_i(M^n, a)$ and $q_i(M^n, a)$, where $a = [\omega] \in H^1(M^n, \mathbb{Z})$ is the cohomology class of the form ω. The following inequalities have been established in [34] and [35]:

$$m_i(f) \geqslant b_i(M^n, a) + q_i(M^n, a) + q_{i-1}(M^n, a).$$

As has already been stated, the level surfaces of closed forms on finite-dimensional manifolds for $k \geqslant 2$ are extremely complicated. For $k = 2$ there are initial results; they are considered in [35]. The necessity of constructing analogues of Morse theory for the critical points of 1-forms (many-valued functions) arose initially, as was shown in [35], from problems of mechanics, on infinite-dimensional manifolds of curves.

Variational principles leading to "many-valued" functionals are of great interest in this context and in the theory of so-called chiral fields, which are considered in [35], §5. These fields play an essential role in the theory of elementary particles. The "quantization condition" is the requirement that the corresponding closed 1-form on the space of fields (the variational functional) should define an integral 1-dimensional cohomology class.

As I. Shmel'tser and the author showed [35], when one considers problems in mechanics concerning the motion of a solid body in an ideal fluid or in a gravitational field about a stationary point, one obtains after "reduction of order" variational principles involving "many-valued" or nonpositive functionals on spaces of directed closed curves or spaces of curves joining two points of the sphere S^2. A specific analogue of Morse theory for many-valued or nonpositive functionals has successfully been constructed only for periodic boundary conditions (on spaces of closed curves) (see [35] and [29]). As simple examples show, in the classical problem with two end-points there is no analogue of the principle of Morse theory (i.e. topological estimates of the numbers of critical points). In a system arising from problems in mechanics, periodic trajectories of the system after reduction of order to one degree of freedom give, in a natural way, two-dimensional invariant tori of the initial system. In fact, the existence of a large number of such 2-tori is proved in this way in our papers.

Finally, it should be noted that the author's papers [65] and [66], which were considered in detail in the survey [35], contain some inaccuracies. In particular, for problems which are not reversible with respect to time, there is no generalization of the two-dimensional Lyusternik–Schnirelmann theory of nonselfintersecting extremals. This carries over to many-valued functionals as well: they are always nonreversible. An idea of Arnol'd for proving the existence of one nonselfintersecting extremal, which was discussed in the author's survey [35], §5 appears also to be invalid in this case. The point is that in the closure of the set of nonselfintersecting curves there may be closed C^1-curves of the following kind. The curve consists of several parts, some of which are extremals of the functional itself, whilst others, alternating with these, are extremals of the symmetrization of this functional with respect to time. Each piece of the "symmetrized part" must be traversed twice, forward and back, among the components into which the curve is dissected. The different parts are connected together only in class C^1.

Such curves are "boundary" extrema of the functional when it is restricted to the subset of nonselfintersecting curves (or, more precisely, its boundary); they are not extrema amongst all smooth curves. For reversible functionals, these "semi-extrema with zero angles" are not encountered, as it is easy to see. Therefore all that remains unconditionally true is the author's geneiai theorem on the existence of one smooth periodic extremal of index 1, arising from spanning a zero-dimensional cycle, for many-valued or nonpositive functionals on the sphere S^2. A second extremal of index 3 may prove to be geometrically dependent upon this one. It is not yet possible to say that these are nonselfintersecting.

Some results on nonselfintersecting or geometrically distinct extremals can be obtained, nevertheless, essentially by applying special properties of two dimensions. They will be published in a note by I. A. Taĭmanov and the author [67].

References

[1] B. G. Averbukh, *Algebraic structure of cobordism groups*, Dokl. Akad. Nauk SSSR **125** (1959), 11–14. (Russian)

[2] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov. **90** (1967); English transl., Proc. Steklov Inst. Math. **90** (1967).

[3] ———, *Smooth dynamical systems. Introductory article*, Smooth Dynamical Systems (D. V. Anosov, editor), "Mir", Moscow, 1977, pp. 7–31; English transl. in Amer. Math. Soc. Transl. (2) **125** (1985).

[4] A. L. Brakhman, *Finiteness conditions for open manifolds and infinite complexes*, Trudy Moskov. Mat. Obshch. **36** (1978), 195–235; English transl. in Trans. Moscow Math. Soc. **1979**, no. 2(36).

[5] ———, *Foliations without limit cycles*, Mat. Zametki **9** (1971), 181–191; English transl. in Math. Notes **9** (1971).

[6] V. M. Bukhshtaber, *Two-valued formal groups. Algebraic theory and applications to cobordism.* I, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 1044–1064; English transl. in Math. USSR Izv. **9** (1975).

[7] ———, *Two-valued formal groups. Algebraic theory and applications to cobordism.* II, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 289–325; English transl. in Math. USSR Izv. **10** (1976).

[8] ———, *Topological applications of the theory of two-valued formal groups*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 130–184; English transl. in Math. USSR Izv. **12** (1978).

[9] V. M. Bukhshtaber and S. P. Novikov, *Formal groups, power systems and Adams operators*, Mat. Sb. **84(126)** (1971), 81–119; English transl. in Math. USSR Sb. **13** (1971).

[10] V. M. Bukhshtaber, A. S. Mishchenko, and S. P. Novikov, *Formal groups and their role in the apparatus of algebraic topology*, Uspekhi Mat. Nauk **26** (1971), no. 2(158), 131–154; English transl. in Russian Math. Surveys **26** (1971).

[11] I. A. Volodin, *Algebraic K-theory as extraordinary homology theory in the category of associative rings with unity*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 844–873; English transl. in Math. USSR Izv. **5** (1971).

[12] V. L. Golo, *Smooth structures on manifolds with boundary*, Dokl. Akad. Nauk SSSR **157** (1964), 22–25; English transl. in Soviet Math. Dokl. **5** (1964).

[13] ———, *On an invariant of open manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 1091–1104; English transl. in Math. USSR Izv. **1** (1967).

[14] G. G. Kasparov, *On the homotopy invariance of the rational Pontryagin numbers*, Dokl. Akad. Nauk SSSR **190** (1970), 1022–1025; English transl. in Soviet Math. Dokl. **11** (1970).

[15] ———, *Topological invariants of elliptic operators.* I: *K-homology*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 796–838; English transl. in Math. USSR Izv. **9** (1975).

[16] A. S. Mishchenko, *Hermitian K-theory. The theory of characteristic classes and methods of functional analysis*, Uspekhi Mat. Nauk **31** (1976), no. 2(188), 69–134; English transl. in Russian Math. Surveys **31** (1976).

[17] ———, *Homotopy invariants of non-simply-connected manifolds.* I: *Rational invariants*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 501–514; English transl. in Math. USSR Izv. **4** (1970).

[18] S. P. Novikov, *Cohomology of the Steenrod algebra*, Dokl. Akad. Nauk SSSR **128** (1959), 893–895. (Russian)

[19] ———, *Some problems in the topology of manifolds connected with the theory of Thom spaces*, Dokl. Akad. Nauk SSSR **132** (1960), 1031–1034; English transl. in Soviet Math. Dokl. **1** (1960).

[20] ———, *Homotopy properties of Thom complexes*, Mat. Sb. **57(99)** (1962), 407–442. (Russian)

[21] ———, *New ideas in algebraic topology* (*K-theory and its applications*), Uspekhi Mat. Nauk **20** (1965), no. 3(123), 41–66; English trans, in Russian Math. Surveys **20** (1965).

[22] ———, *The methods of algebraic topology from the viewpoint of cobordism theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 855–951; English transl. in Math. USSR Izv. **1** (1967).

[23] ———, *Adams operators and fixed points*, Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 1245–1263; English transl. in Math. USSR Izv. **2** (1968).

[24] ———, *Diffeomorphisms of simply-connected manifolds*, Dokl. Akad. Nauk SSSR **143** (1962), 1046–1049; English transl. in Soviet Math. Dokl. **3** (1962).

[25] ———, *Homotopy equivalent smooth manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 385–474; English transl. in Amer. Math. Soc. Transl. (2) **48** (1965).

[26] ———, *Homotopy and topological invariance of certain rational Pontryagin classes*, Dokl. Akad. Nauk SSSR **162** (1965), 1248–1251; English transl. in Soviet Math. Dokl. **6** (1965).

[27] ———, *Topological invariance of rational Pontryagin classes*, Dokl. Akad. Nauk SSSR **163** (1965), 298–300; English transl. in Soviet Math. Dokl. **6** (1965).

[28] ———, *Manifolds with free abelian fundamental groups and their applications* (*Pontryagin classes, smoothness, multidimensional knots*), Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 207–246; English transl. in Amer. Math. Soc. Transl. (2) **71** (1968).

[29] ———, *Pontrjagin classes, the fundamental group and some problems of stable algebra. Essays on Topology and Related Topics* (Mémoires Dédiés à Georges de Rham), Springer-Verlag, 1970, pp. 147–155.

[30] ———, *Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of Hamiltonian formalism. Applications to differential topology and the theory of characteristic classes.* I, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 253–288; English transl. in Math. USSR Izv. **4** (1970).

[31] ———, *Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of Hamiltonian formalism. Applications to differential topology and the theory of characteristic classes.* II, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 475–500; English transl. in Math. USSR Izv. **4** (1970).

[32] ———, *Analogues hermitiens de la K-théorie*, Proc. Internat. Congr. Math. (Nice, 1970), vol. 2, Gauthier-Villars, Paris, 1971, pp. 39–45.

[33] ———, *Topology of foliations*, Trudy Moskov. Mat. Obshch. **14** (1965), 248–278; English transl. in Trans. Moscow Math. Soc. **14** (1965).

[34] ———, *Many-valued functions and functionals. An analog of Morse theory*, Dokl. Akad. Nauk SSSR **260** (1981), 31–35; English transl. in Soviet Math. Dokl. **24** (1981).

[35] ———, *Hamiltonian formalism and a many-valued analog of Morse theory*, Uspekhi Mat. Nauk **37** (1982), no. 5 (227) 3–49; English transl. in Russian Math. Surveys **37** (1982).

[36] V. A. Rokhlin, *Intrinsic homology*, Dokl. Akad. Nauk SSSR **89** (1953), 789–792. (Russian)

[37] ———, *Theory of intrinsic homology*, Uspekhi Mat. Nauk **14** (1959), no. 4(88), 3–20; English transl. in Amer. Math. Soc. Transl. (2) **30** (1963).

[38] V. A. Rokhlin and A. S. Shvarts, *The combinatorial invariance of Pontryagin classes*, Dokl. Akad. Nauk SSSR **114** (1957), 490–493. (Russian)

[39] J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180–214.

[40] ———, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632.

[41] ———, *On the groups $J(X)$*. I, Topology **2** (1963), 181–195.

[42] William Browder, *Surgery on simply-connected manifolds*, Springer-Verlag, 1972.

[43] W. Browder and J. Levine, *Fibering manifolds over a circle*, Comment. Math. Helv. **40** (1966), 153–160.

[44] P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Springer-Verlag, Berlin, and Academic Press, New York, 1964.

[45] ———, *Periodic maps which preserve a complex structure*, Bull. Amer. Math. Soc. **70** (1964), 574–579.

[46] André Haefliger, *Sur les feuilletages analytiques*, C. R. Acad. Sci. Paris **242** (1956), 2908–2910.

[47] Michel A. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. **34** (1960), 257–270.

[48] ———, *Lectures on the theorem of Browder and Novikov and Siebenmann's thesis*, Lectures in Math., no. 46, Tata Inst. Fund. Res., Bombay, 1969.

[49] Michel A. Kervaire and John W. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537.

[50] John Milnor, *On the cobordism ring Ω^* and a complex analogue*. I, Amer. J. Math. **82** (1960), 505–521.

[51] ———, *On manifolds homeomorphic to the* 7*-sphere*, Ann. of Math. (2) **64** (1956), 399–405.

[52] ———, *A procedure for killing homotopy groups of differentiable manifolds*, Differential Geometry, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc, Providence, R. I., 1961, pp. 39–55.

[53] Daniel Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

[54] ———, *Cohomology of* groups, Proc. Internat. Congr. Math. (Nice, 1970), vol. 2, Gauthier-Villars, Paris, 1971, pp. 47–51.

[55] Georges Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Sci. Indust. vol. 1183, Hermann, Paris, 1952.

[56] L. C. Siebenmann, *On detecting Euclidean space homotopically among topological manifolds*, Invent. Math. **6** (1968), 245–261.

[57] R. E. Stong, *Cobordism and Stiefel–Whitney numbers*, Topology **4** (1965), 241–256.

[58] René Thom, *Quelques propriétés globules des variétés différentiates*, Comment. Math. Helv. **28** (1954), 17–86.

[59] ———, *Les classes caractéristiques de Pontrjagin des variétés triangulées*, Internat. Sympos. Algebraic Topology (1956), Univ. Nac. Autónoma de Mexico and UNESCO, Mexico City, 1958, pp. 54–67.

[60] C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311.

[61] ———, *Surgery of non-simply-connected manifolds*, Ann. of Math. (2) **84** (1966), 217–276.

[62] ———, *Surgery on compact manifolds*, Academic Press, 1970.

[63] ———, *An extension of results of Novikov and Browder*, Amer. J. Math. **88** (1966), 20–32.

[64] J. Milnor, Topological manifolds and smooth manifolds, Proc. Internat. Congr. Math. (Stockholm, 1962), Inst. Mittag-Leffler, Djursholm, 1963, pp. 132–138.

[65] S. P. Novikov and I. Shmel'tser, *Periodic solutions of the Kirchhoff equations for the free motion of a rigid body in a fluid and the extended Lyusternik–Schnirelmann–Morse theory*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 3, 54–66; English transl. in Functional Anal. Appl. **15** (1981).

[66] S. P. Novikov, *Variational methods and periodic solutions of equations of Kirchhoff type*, Funktsional. Anal. i Prilozhen. **15** (1981), no. 4, 37–52; English transl. in Functional Anal. Appl. **15** (1981).

[67] S. P. Novikov and I. A. Taĭmanov, *Periodic extremals of many-valued or not-everywhere-positive functionals*, Dokl. Akad. Nauk SSSR **274** (1984), 26–28; English transl. in Soviet Math. Dokl. **29** (1984).