ALGEBROTOPOLOGICAL APPROACH TO THE REALITY PROBLEM. REAL ACTION VARIABLES IN THE THEORY OF FINITE-ZONE SOLUTIONS OF THE SINE-GORDON EQUATION

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Abstract. The paper develops an algebrotopological approach to the problem of effective selection of real finite-zone solutions of the sine-Gordon equation, which uses the so-called γ-representation on the Riemann surface, in which "action" variables can be computed explicitly. This approach is general and applies to many systems for which the reality problem has not yet been solved.

Introduction

Starting with the works of Gardner and of Zakharov and Faddeev [1] it became clear that the fundamental nonlinear evolution systems of soliton theory, integrable by the method of the inverse problem, are field-theoretic completely integrable Hamiltonian systems. Their modern theory is developed in various functional spaces of smooth, rapidly decreasing, periodic or quasiperiodic in x fields which met appropriate reality requirements (see [2]). In the theory of periodic and quasiperiodic solutions a key role is played by a family of finite-dimensional submanifold, the so-called "finite-zone" (finite-band, or finite gap) solutions (see [3]) in the functional space of fields (as it turned out, this family is everywhere dense). On these finite-zone phase manifolds the dynamics of the system induces finite-dimensional Hamiltonian systems that arc integrable in the sense of Liouville. It is interesting that a surface level of a collection of commuting integrals becomes, after appropriate compactitication, a complex Abelian torus which is the Jacobi variety of a certain hyperelliptic Riemann surface. In the final analysis, the solution itself is expressible in terms of the θ-functions of this torus with the argument depending linearly on the coordinate x and time t. The linear coordinates on the Jacobi torus are the (complexified) "angles" appearing in Liouville's theorem. To isolate a real torus within the complex a separate discussion is needed. Note, however, that the "action" variables canonically conjugate to the indicated angles are a subject of the real theory only, and they may not be described in the language of θ-functions; this important circumstance leads to the type of problems considered in this paper. This is a good place to draw reader's attention to the fact that the general formalism of "finite-zone integration" yields readily θ-functional formulas for the general complex solutions; now it was trivial to single out from them the values of the parameters which lead to smooth real solutions for KdV and the Toda lattice (see [3]), but this turned out to be a difficult task for, say, the SG equation and other systems for which the corresponding Lax operator is of order higher than

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two, or for matrix systems, even of first order, if the matrix is of dimension higher than two. Effective conditions for the reality of the θ-functional formulas for the SG equation were first obtained in [4]. We recall that the variables in terms of which the θ-functions are written correspond to the complexified Liouville "angles" of the original Hamiltonian system; this was already remarked in [5]. Later, it was remarked in [6] that the "angle" variables for the KdV and the Toda lattice (so important in applications) which are canonically conjugate to the angles with respect to the usual field-theoretic Poisson bracket, may be naturally described in a different representation in terms of certain integrals over the forbidden zones (lacunas) on the Riemann surface of the Bloch function. This observation was further developed in [7] and [8]. It was shown that this fact is common to a wide class of Poisson brackets on finite dimensional manifolds of solutions, for which all higher KdV equations are Hamiltonian systems. One can say that this is a universal property of all nontrivial classical integrable Hamiltonian systems the integration of which was reduced to Abelian integrals and θ-functions.

In [9] the program outlined above was carried out for the sine-Gordon (SG) equation. Despite the fact that the problem of reality in the θ-functional formulas was settled in a relatively effective manner [4], no information can be extracted from such formulas on the so-called "topological charge," not to mention the "action" variables. The present work is basically an account of the result of paper [9], where for the first time it was developed what we shall refer to as the "algebrotopological approach" to the problem of effective selection of real solutions, achieved in the so-called γ-representation on the Riemann surface in which "action variables" are computed explicitly. This approach is extremely general and undoubtedly applies to many systems for which the reality problem has not yet been solved (and these are the majority!).

1. Complex Finite-Zone Solutions. Hamiltonian Formalism in the γ-Representation

The SG-equation has the form

$$u_{tt} = u_{xx} - \sin u$$

and, as is known, admits a commutation representation (see [2]) AKHC

$$\left[\frac{\partial}{\partial x} + A, \frac{\partial}{\partial t} + B\right]$$

where

$$A = \sqrt{\lambda}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{i}{4}(u_t + u_x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{16\sqrt{\lambda}}\begin{pmatrix} 0 & e^{iu} \\ e^{-iu} & 0 \end{pmatrix}$$

(for the explicit form of $B(\lambda)$ see [2]; we do not need it here); this representation follows from the investigations of Ablowitz–Kaup–Newell–Segur. It Is required that function $\exp(iu)$ be periodic or quasiperiodic. The solutions of SG are termed "physical" if $u(x,t)$ is real-valued. The quantity

$$\bar{e} = \frac{1}{2\pi} \lim_{L\to\infty} \frac{1}{L} \int_0^L u_x \, dx$$

is called the "mean topological charge density." In the periodic case with period T the quantity $\bar{e}$ has the form

$$2\pi\bar{e} = \frac{m}{T},$$

where m is an integer known as the topological charge. In the periodic case one can define the operator of translation by the period $x \to x + T$ and the Bloch (or Floquet) function for the operator $L(\lambda) = \partial_x + A$

$$\psi_\pm(x + T, t, \lambda) = \exp\{\pm ip(\lambda)T\}\psi_\pm(x, t, \lambda).$$

The quantity $p(\lambda)$ is called the "quasimomentum." An important remark is that $\psi_\pm$ depends only on λ, whereas the operator L depends on $\sqrt{\lambda}$. For an appropriate normalization, the quasimomentum has the asymptotics

$$p(\lambda) = \sqrt{\lambda} + 2\pi\bar{e} + c_+(16\sqrt{\lambda})^{-2} + \dots, \quad \lambda \to \infty,$$
$$p(\lambda) = -(16\sqrt{\lambda})^{-1} + \pi\bar{e} - c_1\sqrt{\lambda} + \dots, \quad \lambda \to 0,$$

In the finite-zone case the Riemann surface Γ of the Bloch function $\psi_\pm(x, \lambda)$, which is a two-sheeted covering of the λ plane, is by definition nonsingular and of finite genus $n < \infty$. It may be represented in the form

$$y^2 = \prod_{j=0}^{2n}(\lambda - \lambda_j), \quad \lambda_0 \cdot \lambda_1 \cdot \dots \cdot \lambda_{2n} = 0,$$

and $\lambda = 0$ and $\lambda = \infty$ are ramification points. The quasimomentum is defined as the integral along a path in Γ of the 1-form dp,

$$dp = dp_+ + dp_-$$

where

a) $dp_+ = dz\left(-\frac{1}{z^2} + O(1)\right), \ z = \lambda^{-1/2} \to \infty,$

$dp_- = dw\left(\frac{1}{16w^2} + O(1)\right), \ w = \lambda^{1/2} \to 0,$

and

b) if $(a_1, \dots, n, b_1, \dots, b_n)$ is a canonical base of cycles on Γ, such that $a_i \circ a_j = b_i \circ b_j = 0$, $a_i \circ b_j = \delta_{ij}$, then

$$\oint_{a_j} dp_\pm = 0, \quad j = 1, 2, \dots, n.$$

Therefore, the quasimomentum is meaningful for all finite-zone operators L even if $exp(iu)$ is a quasiperiodic rather than a periodic function. Note that the definition of the quasimomentum depends on the semibasis $(a_1, \dots, a_n)$, the choice of which will be discussed later.

An important role in the whole theory is played by the zeros of the first component, of the Bloch function, i.e., of the vector $\psi(x, t, P)$, where $P = (\lambda, \pm)$ is the generic point on surface Γ. We denote these zeros, which are exactly n points on Γ, by $\gamma_j(x, t)$, $j = 1, 2, \dots, n$. For a general complex solution $u(x, t)$, the zeros are arbitrary points on surface Γ. The arrangement of these zeros for real solutions $u(x, t)$ is one of the subjects discussed below. We see that the finite-zone families of complex solutions lie on functional manifolds with the following parametrization (the collection of data of the inverse problem in the γ-representation);

$$(\lambda_0, \dots, \lambda_{2n}, \gamma_1, \dots, \gamma_n), \quad \lambda_0 \cdot \dots \cdot \lambda_{2n} = 0,$$

where λ_j are points in the λ plane and γ_q are points on the surface Γ, given by the equation $y^2 = \prod(\lambda - \lambda_j)$. The parameters γ_q must be specified only for $x = x_0$.

In papers [7, 8] were introduced and studied general algebrogeometric and analytic Poisson brackets on such complex manifolds. These are defined by a 1-form $Q(\Gamma, \lambda)\, d\lambda$ on surface Γ, or one of its coverings. The form $Q\, d\lambda$ depends on Γ as a parameter. The Poisson brackets must possess the following properties (which, in the case of the KdV-equatlon and for the simplest examples of brackets, were established in the γ-representation back in 1976, see [6, 10]); the universality of these properties for all algebrogeometric integrable cases was remarked in [7]:

$$\{\lambda_j, \lambda_i\} = \{\gamma_q, \gamma_p\} = 0,$$
$$\{Q(\gamma_q), \gamma_p\} = \delta_{qp},$$
$$\{Q(\gamma_q), Q(\gamma_p)\} = 0,$$

here λ_j and γ_q designate the projection of the phase manifold on the corresponding factors. The symplectic 2-form is given by

$$\Omega_Q = \sum dQ(\Gamma, \gamma_i) \wedge d\gamma_i.$$

Moreover, the annihilator of the Poisson bracket must be specified by a collection of functions of the arguments $(\lambda_0, \dots, \lambda_{2n})$ only, i.e., functions which do not depend on $(\gamma_1, \dots, \gamma_n)$. The level surfaces of this annihilator are $2n$-dimensional. It is further required that the derivatives of the form $Q(\Gamma, \lambda)\, d\lambda$ in directions tangent to the level surfaces of the annihilator be meromorphic 1-forms on Γ. We call a Poisson bracket "compatible with the SG dynamics" if all higher analogs of the SG equation, when restricted to finite-zone solutions, are Hamiltonian with respect to this bracket. The complex (i.e., over the complex field) theory of such brackets for the KdV equation was completed in [8]; the complex theories of the KdV and SG equations are completely similar (which is by no means true for the real theories). Here are the main examples.

Example 1. The standard field Poisaon bracket for the SG equation has the form

$$\{u(x), u(y)\} = \{\pi(x), \pi(y)\} = 0,$$
$$\{u(x), \pi(y)\} = \delta(x - y), \quad \pi = u_t.$$

upon restricting it to finite-zone families one obtains brackets which are algebrogeometric, analytic, etc., and for which function Q is given by

$$Q = Q_1(\Gamma, \lambda) = 4ip(\lambda)\lambda^{-1}.$$

Example 2. For the other Poisson bracket on a finite-zone family, obtained when the latter is characterized as a set of stationary points of "higher analogs of the SG equation," function Q has the form

$$Q = Q_2(\Gamma, \lambda) = q_i \left(1 + 16\sqrt{\prod_{\lambda_i \neq 0} \lambda_j}\right) \sqrt{\prod_{j=0}^{2n} (\lambda - \lambda_j)\, \lambda^{-2}}.$$

In the first case the role of the annihilator is obviously played by the group of periods as functions of $(\lambda_0, \dots, \lambda_{2n})$,

$$T_j(\gamma_0, \dots, \lambda_{2n}), \quad j = 1, \dots, n,$$
$$\{T_j, f(\lambda_0, \dots, \lambda_{2n}, \gamma_1, \dots, \gamma_n)\}_1 \equiv 0.$$

In the second case the annihilator is given by the following functions

$$f_p = \sigma_p(\lambda_0, \dots, \lambda_{2n}) = \sum_{j_1 < \dots < j_p} \lambda_{j_1}\lambda_{j_2}\dots\lambda_{j_p}, \quad p = 1, 2, \dots, n-1,$$
$$f_n = \sqrt{\sigma_{2n}},$$
$$\{f_q, f(\lambda_0, \dots, \lambda_{2n}, \gamma_1, \dots, \gamma_n)\}_2 = 0.$$

Note that the 1-form $Q_1\, d\lambda$ is defined and single-valued only on a covering $\hat{\Gamma}$ or Γ, whereall a-cycles remain closed, because $p(\lambda)$ is defined and single-valued only on this covering of Γ.

In the second example the form $Q_2\, d\lambda$ is meromorphic and single-valued on surface Γ itself.

Although paper [8] is written for the KdV equation, an identical reasoning leads to the following theorem: The algebrogeometric, analytic Poisson brackets $Q(\Gamma, \lambda)\, d\lambda$ are compatible with the SG-dynamics if and only if the derivatives of the form $Q(\Gamma, \lambda)\, d\lambda$ in the moduli space in the directions tangent to the level surfaces of the annihilator of the given bracket form a basis of holomorphic 1-forms (of first kind) on the surfaces Γ.

Abel's transformation linearizes the dynamics of every Hamiltonian of the form $H(\Gamma)$ for such brackets and introduces "angle coordinates." Following Abel's transformation we obhtain the collection of "data of the inverse problem in the θ-representation; this can be regarded as a new set of coordinates on the finite-zone manifold $(\lambda_0, \dots, \lambda_{2n}, \eta^1, \dots, \eta^n)$, where the complex vector η of angle coordinates is given modulo the lattice spanned in C^n hy the vectors $[e_1, \dots, e_n, e'_1, \dots, e'_n]$, with $e'_j = \sum_i b_{ji} e_i$; e_j the vector of the (standard) basis in C^n, and (b_{ij}) the Riemann matrix

$$b_{ij} = \oint_{b_j} \Omega_i, \quad \frac{1}{2\pi i} \oint_{a_j} \Omega_i = \delta_{ij},$$

in which Ω_j is a normalized basis of the 1-form on Γ.

Hereafter we shall work only in the γ-representation, in which, as it turns out, it is possible to calculate effectively the action variables and develop an algebrotopolugical an proach to conditions ensuring the reality of solutions.

2. Algebrotopological Approach to the Reality Problem in the γ-Representation

The Abel transformation from the γ- to the θ-representation is defined by the collection of holomorphic 1-forms on Γ (n is the genus of Γ)

$$\eta_k = A_k(\gamma_1, \dots, \gamma_n) = \sum_{j=1}^{n} \int_{\infty}^{\gamma_j} \Omega_k,$$

where the coordinates $(\eta_1, \dots, \eta_n)$ are defined by the indicated integrals modulo vectors belonging to the aforementioned lattice in C^n.

The reality problem may be stated as follows: find effective conditions on the collection $(\Gamma, \gamma_1, \dots, \gamma_n)$ ensuring that the solution $u(x,t)$ of the SG equation is real. The conditions on Γ are found easily: it is necessary and sufficient that the surface $y^2 = \prod_{j=0}^{2n} (\lambda - \lambda_j) = R(\lambda)$ be defined by a real polynomial $R(\lambda)$ of degree $2n+1$ which has $\lambda = 0$ as a root, has no real strictly positive roots, and has no multiple real nonpositive roots. We shall consider only the "general position" case in which all complex roots $R(\lambda_j) = 0$ are simple. In this case surface Γ is nonsingular. This result may be extracted from papers [11] and [12], although no precise formulation is given in these references. Papers [13] and [14] also deal with the reality problem. Paper [14] gives an algebrogeometric reformulation of the problem, which allowed the author to obtain a noneffective proof of the theorem on the number of real components for solutions of the SG equation for a fixed surface Γ. In [13] the useful notion of "number of oscillations" m_j for small perturbations of the trivial operators L (see below) was introduced. This notion was subsequently used and developed In [9].

The algebrogeometric reformulation of the reality conditions for the solutions of the SG equation shows that after Abel's transformation, on the complex torus $J(\Gamma)$ of complex dimension n there arises the antiholomorphic involution $\delta\colon J(\Gamma) \to J(\Gamma)$, $\delta^2 = 1$. The (-1)-fixed points $\delta z = -z$ of this involution correspond to real solutions of the SG equation. Paper [4] is devoted precisely to the effective computation of this involution in the θ-representation.

Note, however, that involution δ is "collective" in variables γ. On going back to the original γ-representation via the inverse Abel transform we see that the coordinate γ_i of the collection $(\gamma_1, \dots, \gamma_n) = A^{-1}(\eta)$ with $\delta\eta = -\eta$ may, in principle, lie at any point ot the Riemann surface Γ. We remind the reader that in the KdV case the point γ is beforehand allowed to lie only in the lacuna (forbidden zone) with number j, provided that the solution is smooth and real, i.e., the anticomplex involution on the torus $J(\Gamma)$ is generated by an involution of surface Γ itself. For the SG equation this is no longer true.

Thus, in contrast to the KdV, for the SG equation there is no point in seeking the position of the individual zeros $\gamma_j(x)$ of the first component of the Bloch function ψ. The idea of the algebrotopological approach ([9]) may be described as follows. By analogy with ergodic theory and the definition of the so-called rotation numbers, we must consider a large interval $[x_0, x_\alpha]$ such that

$$\gamma_j(\alpha) \to \gamma_j(x_0), \quad \alpha \to \infty, \ |x_\alpha| \to \infty.$$

It is readily checked that the points $\gamma_j(x)$ never lie at the special branching points 0 or ∞ on Γ. Closing the long segment $[\gamma_j(x_0), \gamma_j(x_\alpha)]$ by a short geodesic on Γ we attain a sequence of cycles $Z_{\alpha j}$, $\alpha \to \infty$ on surface $\Gamma \setminus (0 \cup \infty)$. It is not hard to see that the following limit exists

$$w_j = \lim_{\alpha\to\infty} \frac{[Z_{\alpha j}]}{x_\alpha - x_0} \in H_1(\Gamma \setminus (0 \cup \infty), R),$$

where $[Z_{\alpha j}]$ is the cohomology class of the cycle $Z_{\alpha j}$. We also write w_j to denote the image of w in $H_1(\Gamma, R)$. In ergodic theory one often encounters such situations; moreover, in the limit $\alpha \to \infty$ the element w_α can be written, generally speaking, as a linear combination of integral cohomology classes ("geometric cycles") with real coefficients. As we shall see below, in our case these elements could have been a linear combination of n cycles $a_1, \dots, a_n$, with $a_i \circ a_j = 0$. However, the situation turns out to be considerably brighter. We have the following set of assertions (as we shall indicate below, these assertions are proved not for all values of the parameters). The element w_j (more precisely, its image in the group $H_1(\Gamma, R)$) is a real multiple of a unique indivisible integer cycle $a_j \in H_1(\Gamma, \mathbb{Z})$:

$$w_j = m_j a_j, \quad m_j \in R.$$

Moreover, the collection of cycles $(a_1, \dots, a_n)$ is half of the canonical basis

$$a_i \circ a_j = 0.$$

Assertion 2. *If the differential of the quasimomentum dp* (see Sec. 1) *is normalized with respect to the basis $(a_1, \dots, a_n)$, then*

$$2\pi m_j = U_j = \oint_{b_j} dp, \quad a_i \circ b_j = \delta_{ij}.$$

Strictly speaking, we must add here that this equality determines the direction of the cycle a_j. Lemma 2 of [9] contains an inexact statement: the quantities U_j and m_j are not necessarily positive.

Assertion 3. *Suppose that the Riemann surface Γ has exactly $2k$ ramification points $\lambda_0 < \lambda_1 < \dots < \lambda_{2k-2} < \lambda_{2k-1} < 0 = \lambda_{2k}$, and that among them are exactly $2n - 2k$ distinct complex-conjugate pairs*

$$\lambda_{2k+1} = \bar{\lambda}_{2k+2}, \ \dots, \ \lambda_{2n-1} = \bar{\lambda}_{2n}.$$

Suppose further that the numbers m_j are all different: $m_j \neq m_s$, $j \neq s$.

Then the homology classes a_j can be realized on surface $\Gamma \setminus (0 \cup \infty)$ by the closed, pairwise disjoint non-self-intersecting curves M_j specified by the following properties:

a) *their projection N_j on the λ plane are either non-self-intersecting curves or smooth non-self-intersecting segments* (*doubly covered by the original curves) with the end-points at pairs or complex-conjugate ramification points, moreover, the projections λ on the λ plane are pairwise disjoint and invariant under complex conjugation $\lambda \to \bar{\lambda}$.*

b) The closed projections N_j go once around the point $\lambda = 0$, and intersect the positive (negative) real half-axis once at points $\mu_j > 0$ (respectively, $\mu_j < 0$), where $\lambda_{2j-2} < \mu_j < \lambda_{2j-1}$, $j = 1, 2, \dots, k$.

c) The segment-type projections N_j end at the ramification points $\lambda_{2j-1} = \bar{\lambda}_{2j}$, $j = k+1, \dots, n$, and intersect the positive semiaxes at points $\mu_j > 0$.

d) If $m_j > m_p$, then $\mu_j > \mu_p$; for $p, j = 1, \dots, k$ we always have $m_j > m_p$, if $j > p$.

These properties of the curves M_j and of their projections N_j determine completely the cohomology classes a_j, and the classes w_j in $H_1(\Gamma)$.

Assertion 4. *For a fixed collection of projections N_j and classes $a_j \in H_1(\Gamma, \mathbb{Z})$ one can choose curves M_j on $\Gamma \setminus (0 \cup \infty)$ in more than one way. More precisely, there are 2^k distinct possible selections. These selections distinguish the components of the real solutions of the SG equations corresponding to one and the same surface Γ. For $j \leq k$ each curve N_j has exactly two preimages on Γ, on the two distinct sheets, and any of these may be taken as M_j on the corresponding component of the set of real solutions. These two curves, denoted hereafter by M'_j and M''_j are taken one into the other by the following antiinvolution of surfaces Γ:*

$$\tau \colon (y, \lambda) \to (-\bar{y}, \bar{\lambda}), \quad \tau^2 = 1,$$
$$\tau(M'_j) = M''_j; \quad \tau_* a_j = a_j \in H_1(\Gamma, \mathbb{Z}).$$

Antiinvolution τ changes the direction of the projections $N'_j = N_j \to N_j^{-1} = N''_j$, $j = 1, 2, \dots, k$. The mean topologlcal charge density is calculated as follows

$$\bar{e} = \frac{1}{2\pi i} \lim_{L\to\infty} \frac{1}{L} \int_0^L d\ln(\gamma_1(x)\gamma_2(x)\dots\gamma_n(x)),$$

because $u(x,t) = \ln(\gamma_1 \cdot \dots \cdot \gamma_n) + \mathrm{const}$. *Consequently,*

$$\bar{e} = \sum_{j=1}^{k} \sigma_j m_j = \frac{1}{2\pi} \sum_{j=1}^{k} \sigma_j U_j, \quad \sigma_j = \pm,$$

where the sign σ_j depends on the connected component in the set of real solutions of the SG equation, i.e., on which of the curves M'_j or M''_j we pick on $\Gamma \setminus (0 \cup \infty)$. Thus, connected components are indexed by collections

$$\sigma = (\sigma_1, \dots, \sigma_k), \quad \sigma_j = \pm, \quad j = 1, \dots, k,$$

where $2k$ is the total number of negative ramification points. We remark that the subgroup of γ-cycles spanned by $a_1, \dots, a_n$ in $H_1(\Gamma, \mathbb{Z})$ can be readily defined by algebrotopological means. In fact, the homology group of the whole complex torus $H_1(J(\Gamma))$ can be identified with $H_1(\Gamma)$. Any real Liouville torus $T^n \subset J(\Gamma)$, i.e., any component of the set of (-1)-fixed points of the anti-involution

$$\delta\colon J(\Gamma) \to J(\Gamma), \quad -\delta/T^n \equiv 1,$$

which singles out the real solutions in the θ-representation, is embedded in $J(\Gamma)$ and thus induces a monomorpnism (isomorphic embedding) $H_1(T^n) \to H_1(J(\Gamma))$ of the one-dimensional homology group. It is precisely this image-subgroup that is generated by the γ-cycles $(a_1, \dots, a_n)$. This subgroup is readily found using [4]. In the recent preprint [15], written at the beginning of 1983, the problem of computing this subgroup was solved for genus $n = 2$. Unfortunately, the authors of this preprint were seemingly not acquainted with the already published papers [4] and [9].

The selection of the basis of γ-cycles corresponding to the individual curves $\gamma_j(x)$ contains considerably more information. In particular, the formula for the topological charge holds exactly in this basis.

The principle behind the proof of these assertions, proposed by B. A. Dubrovin and the author, may be described as follows. Consider first small perturbations of the trivial operator where $u = \mathrm{const}$. For such perturbations all our assertions are verified with no difficulty. In this case $k = 0$. Next, we obtain all finite-zone operators by deformation, avoiding all codimension-two singularities in the parameter space. The emergence of real negative bifurcation points is a singularity of codimension one in the space of parameters on which the real surfaces depend. This bifurcation is easily dealt with. Therefore, all algebrotopological properties of the families of operators obtained from a trivial operator, through the indicated deformations, are undoubtedly valid for the potentials attainable by deformations which do not involve other codimension-one singularities. However, this is still not completely founded. In particular, the emergence of equalities $m_i = m_j$, with $i \neq j$, may turn out to be such a bifurcation, although up to both its sides one can, apparently, manage with deformations of the indicated type. Thus, for the moment the properties formulated above are rigorously proved for only a part of the entire real finite-zone family.

In the note [16] was found the asymptotic behavior of the analytic properties of the operators L for $\to\infty$. B. A. Dubrovin showed that every smooth periodic operator L can be approximated by finite-zone ones.

Among other problems to which it would be interesting to apply the algebrotopological approach in the study of reality (Hermitianness) conditions, the two-dimensional operators L with double periodic (quasiperiodic) coefficients hold a special position.

I. a) Nonstationary Schrödinger operator

$$i\psi_y = -\psi_{xx} + u(x,y)\psi, \quad L = i\frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} + u,$$

b) parabolic operator

$$\psi_y = \psi_{xx} + u(x,y)\psi, \quad L = \frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} + u.$$

II. Two-dimensional stationary Schrödinger operator in electric and magnetic fields

$$L = \left(i\frac{\partial}{\partial x} - A_1(x,y)\right)^2 + \left(i\frac{\partial}{\partial y} - A_2(x,y)\right)^2 + u(x,y).$$

In all these cases one studies the Bloch solution of the equation

$$L\psi = 0,$$

where $\nabla(\ln\psi)$ has the same group of periods as the operator L_0.

In the "analytically admissible" case the Bloch solutions of the equation $L\psi = 0$ form, after an appropriate complexification, a one-parameter family $\psi(x,y,P)$, where P runs over the points of a Riemann surface Γ. The operator L is called "finite-zone" if the Reimann surface Γ has finite genus. One distinguishes points "at infinity," a single point ∞ in the case I and two points, ∞_1 and ∞_2, in the case II, at which one has the asympotics

$$\text{I}\quad \text{a)}\ \psi(x,y,P) = e^{kx+k^- y}(1+O(k)^{-1}), \quad P\to\infty,$$

$$\text{b)}\ \psi(x,y,P) = e^{ikx+k^2 y}(1+O(k)^{-1}), \quad P\to\infty,$$

$$\text{II}\qquad \psi(x,y,P) = \begin{cases} e^{ik_1 z}(1+O(k^{-1})), & P\to\infty_1, \\ ce^{ik_2\bar z}(1+O(k_2^{-1})), & P\to\infty_2, \end{cases}$$

where $z = x+iy$, k^{-1}, k_1^{-1} and k_2^{-1} are local parameters in the neighborhoods of the points ∞, ∞_1 and ∞_2, respectively, and $c(x,y)$ is a function which does not depend on P. Next, function ψ has exactly n poles independent of (x,y) and n zeros $\gamma(x,y),\dots,\gamma_n(x,y)$, where n is the genus of Γ. Situation I was first considered in [17], and situation II, in [18], for every collection of data, Riemann surface Γ points at infinity on it, local parameters in the neighborhood of these points, and collection of poles or zeros. When $x\to x_0$ the poles and zeros merge and $\psi(x_0,P)\equiv 1$. In the complex domain cases I, a) and I, b) are indistinguishable. In the real theory one must settle the following problems:

1. How to select a class of admissible surfaces Γ, points at infinity, and local parameters k^{-1}, k_α^{-1}, $\alpha = 1,2$?
2. Once problem 1 is solved, how to select collections of zeros $\gamma_j(x)$?

Both problems are still open, although beginning with papers [14, 19] we are in possession of a nontrivial class of examples, some in the most difficult case II, in which all coefficients (u, A_1, A_2) are real. In the case I, a) the answer is, probably, simple: surface Γ must be endowed with an antiinvolution $\delta\colon \Gamma \to \Gamma$, $\delta^2 = 1$, which has exactly $n+1$ fixed ovals $S^1_1, S^1_2, \dots, S^1_{n+1}$, $\delta(S^1_q) = S^1_q$. The point ∞ must lie on the oval S^1_{n+1}, while the points $\gamma_j(x)$ must lie on the ovals S^1_j, $j = 1, \dots, n$. The local parameter k^{-1} in the vicinity of ∞ must also be invariant $\overline{\delta^*(k)} = k$. These conditions guarantee that the quantity $u(x, y)$ and the corresponding solution $u(x, y, t)$ of the KP equation are smooth and real.

In cases I, b) and II also the Riemann surface Γ must very likely be real, i.e., possess an antiinvolution $\delta\colon \Gamma \to \Gamma$, $\delta^2 = 1$, such that $\delta(\infty) = \infty$ in case I, b) and $\delta(\infty_1) = \infty_2$ in case II. Also, it is very likely that the local parameters k in case I, and k_1, k_2 in case II, must be left invariant and, respectively, be sent one into another by this antiinvolution. The admissible class of real curves Γ, however, could be nontrivial.

In any case, if this is a correct answer to problem 1, we must still solve problem 2 concerning the admissible collections of zeros $(\gamma_1, \dots, \gamma_n)$ in the study of which it is natural to apply the algebrotopological approach in cases I, b) and II.

In the general complex case II for operators L of second order in (x, y) we can also isolate a less interesting subcase analogous to I, a). This is the problem of isolating not the Hermitlan, but rather the purely real operators L in which A_1 and A_2 are purely imaginary. Such a problem is discussed in [18]. Apparently, for the operator to belong to this class it is necessary that the zeros $\gamma_j(x)$ lie on distinct ovals of antiinvolution δ and that the points ∞_1, ∞_2, and the local parameters be permutated by L. The problem whether these conditions are necessary is, as in case I, a), difficult.

Another problem, namely that of isolating the Hermitian operators L with A_1 and A_2 real, is both more interesting and intricate. As shown in [14], when the previous requirements on surface Γ are met, one can also give sufficient conditions for the collections $\gamma_1, \dots, \gamma_n$, where n is the genus of Γ. Let K be the canonical class of surface Γ of degree $2n-2$. Consider the divisor $D = \gamma_1 + \dots + \gamma_n - \infty_1$. If the condition (unfortunately, noneffective) $\delta(D) \equiv K - D$ is satisfied (here L designates linear equivalence), then the operator L is Hermitian.

As indicated in [18], in the complex-field theory of the two-dimensional Schrödinger operator L, there is a simple subcase which contains the theory of the SG equation; it corresponds to hyperelliptic surfaces Γ of the form $y^2 = \prod_{j=0}^{2n}(\lambda - \lambda_j)$, $\prod_j \lambda_j = 0$ with $\infty_1 = 0$, $\infty_2 = \infty$, though here the reality conditions are different. With case II are related also "semicommutative" algebras generated by the operator $L_0 = L$, L_1, L_2 with relations $[L_\alpha, L_\beta] = c_{\alpha\beta} L_0$, $\alpha, \beta = 0, 1, 2$, where $c_{\alpha,\beta}$ are differential operators (see [13, 21]).

A number of interesting problems arise also for one-dimensional operators L of order higher than two or for matrix systems of dimension higher than two. The reality conditions for the corresponding Riemann surfaces Γ were recently unraveled by Dubrovin [20]. The other problems in this case are still open, and the algebrotopological approach seems also a long-range objective.

3. Real Action Variables for the SG Equation

Action variables are defined in the case where the level surface of the commuting integrals of motion is compact and is a torus T^n with canonical angular coordinates $\phi_1, \dots, \phi_n$, each normalized and varying from 0 to 2π. The Poisson bracket has, according to Liouville, the form

$$\{\phi_i, \phi_j\} = 0.$$

If $I_1, \ldots, I_n$ are commuting integrals, then the angles ϕ_i are specified up to transformations

$$\phi_i' = \phi_i + c_i(I)$$

where matrix $q_{ij} = \{\phi_i, c_j\}$ is symmetric. Also, we are allowed to subject the set of angles to linear transformations with integral coefficients and determinant ± 1. Now fix a set of angles and define the action variables

$$J_i = \frac{1}{2\pi} \oint_{a_i'} p\, dq,$$

where a_i' is the cycle on T^n, given by the equations $\phi_s = \text{const}$, $s \neq i$. Here $p\, dq$ is the standard 1-form so that $d(p\, dq) = \Omega$ gives the symplectic form on the $2n$-dimensional phase space and hence the Poisson bracket. For the Hamiltonian formalisms of concern here (see Sec. 1) the form $p\, dq$ in the γ-representation is identical with the form

$$p\, dq = \sum_{i=1}^{n} Q(\Gamma, \gamma_i)\, d\gamma_i$$

on the level surface of the annihilator of the Poisson bracket, which is given by equations on the moduli (branching points) of surface Γ only. We assume that the Abel transformation A is a diffeomorphism on the connected components of the real solutions in question. This is the case for finite-zone solutions of SG with nonsingular Γ and the constraint $m_n \neq m_q$ $(j \neq q)$. Thanks to this circumstance, we can select the set of basis cycles (a_j') on the real torus T^n corresponding to the angles ϕ_j $(0 \leq \phi_j \leq 2\pi)$ so that they will have the form of the γ-cycles indicated in Sec. 2; more precisely, we take

$$a_i' = A_*^{-1}(a_i),$$

where A_*^{-1} designates the embedding of the homology group $H_1(T^n)$ in the group $H_1(\Gamma)$. Comparing this with the expression of the form $p\, dq$ given above, we get

$$J_i = \frac{1}{2\pi} \oint_{a_i'} p\, dq = \frac{1}{2\pi} \oint Q(\Gamma, \lambda)\, d\lambda,$$

where $a_i \in H_1(\Gamma \setminus (0 \cup \infty), \mathbb{Z})$ are the basic γ-cycle on surface Γ.

In the KdV case, and in any other case where there is exactly one real torus, the computation of the action variables ends here, provided that the cycles a_i on surface Γ are indicated explicitly. As a result of these arguments, the computation of the algebrogeometric analytic Poisson brackets introduced in Sec. 1 is reduced to the computation of the periods of the 1-form $Q(\Gamma, \lambda)\, d\lambda$ on surface Γ with respect to certain 1-cycles a_j.

For the finite-zone solutions of SG there are, generally speaking, several distinct real tori T^n_σ, $\sigma = (\sigma_1, \ldots, \sigma_k)$, $\sigma_s = \pm$ (see Sec. 2) for a fixed surface Γ. As σ varies, the collection of cycles $a_i(\sigma)$ varies too in the group $H_1(\Gamma \setminus (0 \cup \infty), \mathbb{Z})$; more precisely, only the first k cycles—those which intersect the negative real semiaxis—change. There arises the natural question: how do the action variables change when a connected component of the real solution varies?

This problem is investigated in [9]. For forms $Q\, d\lambda$, meromorphic on Γ and holomorphic in the complement of $\lambda = 0, \infty$, the answer is rather simple. At the component variation $\sigma' \to \sigma''$ some of the γ-cycles M_j remain unchanged (among them, all M_j with $j > k$ and some of the M_j's with $j \le k$), whereas the remaining γ-cycles are affected by the antiinvolution $\tau(y, \lambda) = (-\bar{y}, \bar{\lambda})$ as indicated in Sec. 2. The curves M_j' and M_j'', where $M_j' = M_j''$, are homotropic on Γ but not on $\Gamma\backslash(0\cup\infty)$. The next equality is readily verified:

$$\frac{1}{2\pi}\oint_{M_j'} Q\,d\lambda - \frac{1}{2\pi}\oint_{M_j''} Q\,d\lambda = \operatorname*{res}_{\lambda=0}[Q\,d\lambda].$$

Since the passage to another component amounts exactly to replacing the curves M_j' by M_j'' (or conversely) for some indices $j \le k$, the last equality settles the question of how the action variables change:

$$J_j(\sigma') = J_j(\sigma''), \quad j > k,$$
$$J_j(\sigma') - J_j(\sigma'') = \frac{1}{2}(\sigma_j' - \sigma_j'')\operatorname*{res}_{\lambda=0}[Q\,d\lambda],$$
$$\sigma' = (\sigma_1', \dots, \sigma_k'), \quad \sigma'' = (\sigma_1'', \dots, \sigma_k'').$$

Such is, for example, the Hamiltonian formalism of the stationary problem, indicated in Example 2 of Sec. 2, where

$$Q\,d\lambda = Q_2\,d\lambda = 2i\left(1 + 16\sqrt{\prod_{j\ne 0}\lambda_j}\right)\sqrt{\prod_{j=0}^{2n}(\lambda-\lambda_j)}\,\lambda^{-2}\,d\lambda.$$

However, for the Hamiltonian formalism (important in applications) generated by the restriction to a finite-zone solution of the field-general standard local Poisson bracket (in which case the role of the annihilator is played by the periods of the quasiperiodic function $\exp(iu)$), the picture is more complicated. In this case we have (see Example 1 of Sec. 2)

$$Q\,d\lambda = Q_1\,d\lambda = 4ip(\lambda)\lambda^{-1}\,d\lambda,$$

where $p(\lambda)$ is the quasimomentum. We note, first of all, that the form $Q_1\,d\lambda$ is meromorphic only on the covering $f\colon \hat\Gamma \to \Gamma$, where the image $\operatorname{Im} f_*(H_1(\hat\Gamma) \to H_1(\Gamma))$ is the subgroup spanned by the cycles $(a_1, \dots, a_n)$. The monodromy group of this covering is free Abelian: $\mathbb{Z}\times\cdots\times\mathbb{Z}$ (n copies) and is generated by the motions

$$\kappa_i\colon \hat\Gamma \to \hat\Gamma, \quad i = 1, \dots, n.$$

On the covering surface $\hat\Gamma$ the form $f^*(dp)$ is exact and the function $p(\lambda)$ is single-valued. Moreover, at all preimages of the points $(\lambda = 0)$ and $(\lambda = \infty)$ the residue of the form $Q_1\,d\lambda$ vanishes, as follow from the expansion of $p(\lambda)$ at $\lambda = 0$, indicated in Sec. 1. At the points $(4i\bar{e})$ this residue equals $f^{-1}(0)$, where $\bar{e}$ is the mean topological charge density (see Sec. 1). According to Sec. 2, we have

$$\bar{e} = \sum \sigma_j m_j, \quad \sigma_j = \pm,\ m_j = \frac{1}{2\pi}U_j,$$

where $\sigma = (\sigma_1, \dots, \sigma_k)$ is the "index" of the connected component of the real solutions, and $U_j = \oint_{b_j} dp$. However, this formula is meaningful only for the real components described by the collection σ, whereas the topological charge density makes sense for smooth complex-valued $u(x)$ whenever function $\exp(iu)$ is quasiperiodic. In the monodromy group (group of motions) of the covering $f\colon \hat\Gamma \to \Gamma$, isomorphic to $\mathbb{Z}^n$, we single out the subgroup $\mathbb{Z}^k \subset \mathbb{Z}^n$, generated by the first motions $\kappa_1, \dots, \kappa_k$ according to the indexation of a-cycles indicated in Sec. 2. Under motions we have

$$\kappa_j^* p(\lambda) = p(\lambda) + U_j.$$

Consider the collection of vertices of the k-dimensional cube in R^k with coordinates $(\pm 1/2, \pm 1/2, \dots, \pm 1/2)$. Let κ_j $(j = 1, \dots, k)$ act on R^k by adding $(+1)$ to the j-th coordinate. Then the vertices of the cube represent symbolically the real components, in such a way that the vertex with coordinates $(1/2, \dots, 1/2)$ represents the component $\sigma' = (+, \dots, +)$ and corresponds to the "initial" point of the total preimage: $Q_0 \in f^{-1}(O)$. The other vertices are obtained from this particular one by transformations $\kappa_1^{\varepsilon_1} \cdot \dots \cdot \kappa_k^{\varepsilon_k}$, where $\varepsilon_i = 0$, or (-1). On passing to another sheet, $Q_0 \to \prod_i \kappa_i^{\varepsilon_i}(Q_0)$, the regular part of function $p(\lambda)$ changes according to the rule

$$p(\lambda) \to p(\lambda) + \sum_{j=1}^{k} \varepsilon_j U_j.$$

This preserves the form of the expansion of $p(\lambda)$ at $\lambda = 0$, indicated in Sec. 1, thanks to the equalities

$$\pi \bar{e} = \sum_{j=1}^{k} \sigma_j U_j/2, \quad p(\lambda) = -\frac{1}{16\sqrt{\lambda}} + \pi \bar{e} + O(\sqrt{\lambda}).$$

Thus, the transition to another components $\sigma'' = (\sigma_1'', \dots, \sigma_k'')$, $\sigma_j'' = (-1)^{\varepsilon_j}$, of real solutions means, firstly, the transition to another "initial" point $O_{\sigma''} \in f^{-1}(O)$, and hence to another branch of the multivalued function $p(\lambda)$. Secondly, as indicated in Sec. 2, it is necessary to replace the curves M_j, $j \le k$, representing the cycles (a_j) for those indices j for which $\sigma_j'' \ne \sigma_j'$, by $M_j'' = \tau M_j'$; here τ designates the antiinvolution $(y, \lambda) \to (-\bar{y}, \bar{\lambda})$. At the transition to a new real component the collection of γ-cycles is modified through two operations: antiinvolution on the curves M_j' $(j \le k)$ with index j such that σ_j' changes: $\sigma_j'' \ne \sigma_j'$, and the shift of the quasimomentum to another branch. This induces a change of all components $J_j(\sigma)$ for $j \le k$. Computing the change of the action integral we obtain the formulas describing the change of variables J_j for the standard Poisson bracket:

$$J_j(\sigma'') = J_j(\sigma'), \quad j > k,$$

$$J_j(\sigma'') - J_j(\sigma') = 8\pi \sum_{s=1}^{k} m_s \frac{\sigma_s' \sigma_j' - \sigma_s'' \sigma_j''}{2}, \quad j \le k.$$

In the particular case where $\sigma_j'' = -\sigma_j'$ for all $j \le k$, we get $J_j(\sigma'') = J_j(\sigma')$. This is natural because the transformation $\sigma \to -\sigma$ corresponds to the trivial transformation $u \to -u$, $\bar{e} \to -\bar{e}$ thanks to the oddness of function $\sin u$.

Action variables are useful in a series of important applications. First, these variables are needed in quasiclassical quantization. In soliton theory quasiclassical quantization was started (for classes of functions of rapid decrease) by Fadeev and Takhtadzhyan [22]. Second, action variables play an important role in Hamiltonian theory of slow modulations—field analogs of the Witham-type averaging method of Bogolyubov et al. or of the nonlinear analog of the WKB method. We shall not pursue further this question; for a discussion of its profound differential-geometric nature the reader is referred to [23].

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