FINITE-ZONE, TWO-DIMENSIONAL SCHRÖDINGER OPERATORS. POTENTIAL OPERATORS

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1. **Complex theory.** Two-dimensional Schrödinger operators finite-zone with respect to one energy level $\varepsilon = \varepsilon^0$ were first defined and studied in [1]. We shall consider a collection of data for the inverse problem somewhat more general than in [1]: a non-singular Riemann surface Γ of genus $g \geqslant 0$, a collection of $k+1$ labelled pairs of points (Q'_j, Q''_j), $j = 0, 1, \ldots, k$, on it, local parameters w', w'' near the points $\infty_1 = Q'_0$, $\infty_2 = Q''_0$, $k' + (w')^{-1}$, $k'' = (w'')^{-1}$, and a divisor $D = P_1 + \cdots + P_{g+k}$ consisting of $g+k$ distinct points not coinciding with the labelled points. We seek a function $\psi(P, x, y)$ meromorphic on $\Gamma \setminus (\infty_1 \cup \infty_2)$, depending on (x, y) as parameters, having a set of simple poles at the points of D, and possessing the following properties:

$$
\begin{gathered}
w' \to 0, \quad \psi(P, x, y) = \exp(ik'z)(1 + \xi_1(x, y)w' + O(w'^2)), \\
w'' \to 0, \quad \psi(P, x, y) = c(x, y)\exp(ik''\bar{z})(1 + \xi_2(x, y)w' + O(w''^2)), \\
z = x + iy, \ \bar{z} = x - iy;
\end{gathered} \tag{1}
$$

$$
\psi(Q'_j, x, y) \equiv (Q''_j, x, y), \quad j = 1, \ldots, k \tag{2}
$$

For a divisor D of general position a unique such function ψ exists and satisfies the Schrödinger equation; for the formulas for ψ, c and V see [1] for $k = 0$:

$$
L\psi = 0, \quad L = \partial^2/\partial z\, \partial \bar{z} + A\, \partial/\partial \bar{z} + V, \quad A = \partial \ln c/\partial z, \ V = -\partial \ln \xi_1/\partial \bar{z}. \tag{3}
$$

Generalization to the case $k > 0$ presents no technical difficulties. The case $k = 1$ is especially interesting; here the Bloch function ψ which occurs of the energy level $\varepsilon^0 = 0$ has analytic properties characteristic of a "critical level" of general position, for example, of a base state. For $k = 0$ the coefficients of the operator L are quasiperiodic. We shall discuss later smoothness, real, and (quasi) periodicity conditions for the coefficients of the operator L.

The function ψ with analytic properties (1) and (2) can be constructed explicitly as a linear combination of ordinary functions ψ_j with pole divisors D_j of degree g, where $(P_1, \ldots, P_g) = D_1, \ldots, (P_1, \ldots, P_{g-1}, P_{g+j-1}) = D_j$, $j = 1, \ldots, k+1$, and with asymptotics (1) on Γ. Formulas for ψ_j follow from [1]. We have

$$
\begin{gathered}
\psi = c_1\psi_1 + \cdots + c_{k+1}\psi_{k+1}, \quad c_j = \text{const}, \\
\sum_{j=1}^{k+1} c_j = 1, \quad \sum_{j=1}^{k+1} c_j\psi_j(Q'_s) = \sum_{j=1}^{k+1} c_j\psi_j(Q'_s), \quad s = 1, \ldots, k.
\end{gathered} \tag{2'}
$$

The following result holds.

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Theorem 1. 1) *Let there be given a nonsingular surface* Γ *of genus* g *with holomorphic involution* $\sigma\colon \Gamma \to \Gamma$, $\sigma^2 = 1$, *and a collection of data of the inverse problem such that*

$$\begin{aligned} &\sigma(Q'_j) = Q'_j, \quad \sigma(Q'_j) = Q''_j, \quad j = 0, 1, \dots, k, \\ &\sigma(w') = -w', \quad \sigma(w'') = -w'', \quad Q'_0 = \infty_1, \quad Q''_0 = \infty_2, \end{aligned} \tag{4}$$

where the divisor $D = P_1 + \dots + P_{g+k}$ *satisfies the condition of linear equivalence on the singular algebraic curve*

$$D + \sigma(D) \approx \sum_{j \geqslant 0} (Q'_j + Q''_j) + K; \tag{5}$$

here $\Gamma' = \Gamma/\{Q'_j = Q''_j\}$, $j = 1, \dots, k$, *and* K *is the canonical divisor of differential forms on* Γ'. *Then the operator* L *corresponding to this collection of data of the inverse problem is a potential operator*:

$$L = -\delta + u(x, y). \tag{6}$$

2) *If the collection of pairs* $\{Q'_j, Q''_j\}$ *consists of all fixed points of the involution* σ, *then equation* (5) *for the divisor* D *is solvable. The collection of admissible divisors* D, *or, equivalently, of admissible operators* L, *forms an Abelian subvariety* $\Pi_\sigma(\Gamma)$ *of the Jacobi variety* $J(\Gamma)$ *of dimension* $g - \tilde{g} = \tilde{g} + k$ *passing through the image of the point* $\sum_0^k (Q'_j + Q''_j) + K$ *under the Abel mapping in the direction where the involution* σ *on the torus* $J(\Gamma)$ *has eigenvalues* (-1)—*the shifted "Prym" variety. Here* $\tilde{g}$ *is the genus of the surface* $\tilde{\Gamma} = \Gamma/\sigma$. *The coefficients of the operator* L *are quasiperiodic and can be expressed in terms of theta functions of the Abelian variety* $\Pi_\sigma(\Gamma)$.

Remark. As I. R. Shafarevich and V. V. Shokurov have indicated to the authors, equation (5) is not solvable if the involution a has "extra" fixed points aside from those contained in (5).

The idea of the proof of part 1) of Theorem 1 is akin to [2] where conditions that a general operator of the form (3) for $k = 0$ be Hermitian were investigated. On Γ we consider a collection of normalized differentials of third kind $\Omega_0, \Omega_1, \dots, \Omega_k$ having a pair of poles of first order at the points $(Q'_0, Q''_0), (Q'_1, Q''_1), \dots, (Q'_k, Q''_k)$ respectively with residues ± 1.

The divisor of zeros of the differential $\sum_0^k \Omega_j$ is linearly equivalent on Γ' to the divisor $K + \sum_{j \geqslant 0} (Q'_j + Q''_j)$. Therefore, there exists an algebraic function f on Γ such that $f(Q'_j) = f(Q''_j)$, $j = 1, \dots, k$, and the differential $\Omega = f(\sum_0^k \Omega_j)$ has the same simple poles at the points Q'_j and Q''_j with equal residues for $j \geqslant 1$, while the zeros of Ω are simple and lie at the points $D + \sigma(D)$. This follows directly from (5). We consider the product

$$\psi \sigma(\psi) \cdot \Omega = \Omega'. \tag{7}$$

From the analytic properties (1) and (2) it follows that the differential Ω' has only poles of first order at the points (Q'_j, Q''_j), $j \geqslant 0$, and residues at them of the form

$$(8)\qquad \begin{aligned} &[a_0, -b_0 c^2(x,y)], \quad j=0, \quad Q_0' = \infty_1, \quad Q_0'' = \infty_2,\\ &a_0 = f(Q_0'), \quad b_0 = f(Q_0''), \quad a_0 = b_0, \end{aligned}$$

$$(8')\qquad \begin{aligned} &[a_j\psi^2(Q_j'), -a_j\psi^2(Q_j'')], \quad j = 1,\dots,k,\\ &a_j = f(Q_j') = f(Q_j''), \quad j \geqslant 1. \end{aligned}$$

From (8) and (8′) it follows that $c^2 \equiv b_0/a_0$, considering the absence of other poles; hence $A = \partial_{\bar z} \ln c = 0$.

$$L = -\delta + u(x,y), \quad A = \partial \ln c/\partial \bar z \equiv 0.$$

The proof of part 2) follows in a straightforward manner from algebraic considerations.

In the theory of solitons conditions of the type (2) for $k > 0$ have been encountered repeatedly. In [3], Appendix 2, for example, the results of Krichever on one-dimensional multisoliton Schrödinger potentials on a finite-zone background are presented; the result of degeneration of the Bloch function is just such a condition (but there is one infinitely distant point). Here combinations of trigonometric (hyperbolic) functions and theta functions of genus g arose. For two-dimensional operators $L = -\delta + u(x,y)$ the situation changes in an essential way: the potential $u(x,y)$ turns out to be quasiperiodic and is expressed in terms of theta functions of the torus $\Pi_\sigma(\Gamma)$ by the restriction of algebraic functions on the torus to rectilinear windings. This is almost obvious for the case $k = 0$. For $k > 0$ the pairs (Γ, σ) can be thought of as the result of degeneration $(\Gamma_\varepsilon, \sigma_\varepsilon) \mapsto (\Gamma, \sigma)$, $\varepsilon \to 0$, where all involutions $\sigma_\varepsilon^2 = 1$ have only the two fixed points $\infty_{1\varepsilon} \to \infty_1$ and $\infty_{2\varepsilon} \to \infty_2$ together with local parameters; the genus g_ε is equal to $g_\varepsilon = g + k$; the divisor D_ε of degree g_ε is such that $D_\varepsilon + \sigma_\varepsilon \approx K_\varepsilon + \infty_{1\varepsilon} + \infty_{2\varepsilon}$. As $\varepsilon \mapsto 0$ this equation degenerates into (5). The Prym variety for $\varepsilon \mapsto 0$ has no degeneracies; its limit is regular. The prelimit theta-function formulas go over smoothly and without degeneracies into the limit formula. From this we obtain Theorem 1.

2. **Conditions for realness.**

Theorem 2. *Let* $\{\Gamma, \infty_1 = Q_0', \infty_2 = Q_0'', w', w'', Q_j', Q_j', j = 1,\dots,k, D = P_1 + \dots + P_{g+k}\}$ *be a collection of data of the inverse problem, where on* Γ *there is given the group* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *of transformations generated by the holomorphic involution* $\sigma\colon \Gamma \to \Gamma$ *and the antiholomorphic involution* $\tau\colon \Gamma \to \Gamma$, $\sigma^2 = \tau^2 = 1$, $\sigma\tau = \tau\sigma$. *If the collection of data of the inverse problem satisfies conditions* (4) *and* (5) *and the divisor* D*, the local parameters, and the pairs* Q_j', Q_j'' *are* τ*-invariant,*

$$D = \tau(D), \quad \tau(\infty_1) = \infty_2, \quad t(Q_j') = Q_j'', \ j \geqslant 0, \quad \tau(w') = -\bar w'',$$

then the potential $u(x,y)$ *of the operator* $L = -\delta + u(x,y)$ *is real.*

The idea of the proof of this theorem reduces to comparing conditions for pure realness of the operator L (see [1]) with the condition for the potential property obtained in Theorem 1.

The following assertions can be proved under the hypotheses of Theorem 2 by considering the selfadjointness of a real potential operator L in the Hilbert space $\mathcal{L}_2(x,y)$.

Theorem 3. *The function $\psi(P,x,y)$ for points P such that $\sigma\tau(P)=P$ is bounded on the plane R^2, $|\psi|<\text{const}$. If the potential is doubly periodic, then the collection of such functions for all points P on the real ovals of the anticomplex involution $\sigma\tau$ (the Fermi curves $\varepsilon(p_1,p_2)=\varepsilon^0=0$) forms a complete collection of Bloch functions of the zero energy level with real quasimomenta (p_1,p_2):*

$$L\psi=0,\quad \psi(P,x+T_1y)\equiv e^{ip_1T_1}\psi,\quad \psi(P,x,y+T_2)\equiv e^{ip_2T_2}\psi.$$

For $k>0$ the points Q_j' and Q_j'', $j=1,\dots,k$, form degenerate, isolated, fixed ovals of the involution $\sigma\tau(Q_j')=Q_j''$ and $\sigma\tau(Q_j'')=Q_j'$ on the quotient curve $\Gamma'=\Gamma/(Q_j'=Q_j'')$, since $\psi(Q_j',x,y)=\psi(Q_j'',x,y)$.

Theorem 4. *If the anticomplex involution τ_0 on $\tilde\Gamma=\Gamma/\{P=\sigma(P)\}$ generated by the involutions τ and σ on Γ has exactly $\tilde g+1$ smooth, real ovals $a_1,\dots,a_{\tilde g},b$ (i.e., is a M-curve), then the condition of realness of u distinguishes $2^{\tilde g+k}$ connected components of admissible collections of poles in the shifted Prym variety $\Pi_\sigma(\Gamma)$ of the real tori $T^{\tilde g+k}$; one of the components gives operators L with a smooth, real potential $u(x,y)$ which is quasiperiodic with a group of $\tilde g+k$ periods.*

Theorem 5. *Under the hypotheses of Theorem 4 suppose that $k=0$; if the covering anti-involution τ on Γ has exactly $g+1=2\tilde g+1$ real covering ovals, then these ovals break down into g pairs $a_1',\dots,a_{\tilde g}',a_1'',\dots,a_{\tilde g}''$ and one connected oval b', where $\sigma(a_q')=a_q''$ and $\sigma(b')=b'$; if the poles lie one each on the ovals (a_q',a_q''), then the operator $L=-\delta+u(x,y)$ with smooth coefficients is strictly positive:*

$$(L\varphi,\varphi)>\delta_0\|\varphi\|^2,\quad \varphi\in\mathcal{L}_2(R^2),\ \delta_0>0.$$

In the limit case $k=1$ obtained from this by degeneration the image of the oval b' on $\Gamma'=\Gamma/(Q_1'=Q_1'')$ reduces to a single point. The operator L is nonnegative:

$$(L\varphi,\varphi)>0,\quad \varphi\in\mathcal{L}_2(R^2).$$

The function $\psi_0=\psi(x,y,Q_1')=\psi(x,y,Q_1'')$ here is smooth, quasiperiodic with the same group of periods as L, and coincides with the base state $L\psi_0$. For $\tilde g=2$, $k=0$ and $\tilde g=1$, $k=1$ the potential $u(x,y)$ is double periodic.

Remark 1. Operators of the form L_0 with potential $u(x)+v(y)$ give a family of codimension 1 for $\tilde g+k=2$.

Remark 2. Under the hypotheses of Theorem 4, if $(\tilde\Gamma,\tau_0)$ is an M-curve there are d connected ovals $b_1',\dots,b_d'$, $d\equiv k+1 \pmod 2$, and $n=\tilde g-d$ pairs $(a_1',a_1''),\dots,(a_n',a_n'')$ on Γ where the part $(b_1',\dots,b_m',a_1',\dots,a_i',a_1'',\dots,a_l'')$ is fixed for $(\sigma\tau)$ while the contemporary part is fixed for τ. Here $\sigma(b_t')=b_t'$ and $\sigma(a_p')=a_p''$. In all cases if there is given a pair of commuting anti-involutions with pairwise nonintersecting compatible collection of smooth ovals, then the sum of the number of their ovals does not exceed $g(\Gamma)+1$.

Conjecture. Any smooth, real, doubly periodic potential $u(x,y)$ can be approximated by potentials which are finite-zone with respect to the zero level. If the involution $\sigma\tau$ has no fixed ovals on Γ (i.e., the zero energy level is located in a lacuna or on its boundary), then the number $(d+k-1)/2$ gives a lower bound for the number of dispersion laws below zero if they have multiplicity one and do not intersect.

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