POISSON BRACKETS AND COMPLEX TORI

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Abstract

This paper is devoted to a theory, introduced by the authors, of a broad class of completely integrable Hamiltonian systems and Poisson brackets on the space of finite-zone potentials or a fibering of hyperelliptic Jacobians. Analysis shows that the Poisson brackets corresponding to known examples of integrable systems on this phase space belong to the class studied. The geometry of the "action–angle" variables and the coordination condition for the structures introduced with KdV theory is investigated; examples are considered.

Bibliography: 30 titles.

Introduction

As is known, at the end of the sixties Gardner, Greene, Miura and Lax discovered the "method of the inverse scattering problem" for solving the Korteweg-de Vries equation (KdV) in the class of rapidly decreasing functions in x.

A history of this discovery and further development of the method can be found in [1]. The problem of finding solutions periodic in x led to the discovery of the so-called method of finite-zone integration created in 1974–1975 (see [2]–[6] and the survey [7] or [1], Chapter II). The method of finite-zone integration is based on a synthesis of the theory of classical, completely integrable Hamiltonian systems, the spectral theory of linear operators with periodic and quasiperiodic coefficients (further developed in connection with KdV even for the classical Hill or Schrödinger operator), and, finally, the classical algebraic geometry of Riemann surfaces and Abelian varieties, including the theory of multidimensional theta functions. Further development of the method and a number of new applications of it can be found in the surveys [8]–[11].

Already in the case of the ordinary KdV equation the so-called finite-zone solutions (including the formal complex analogue of them) form a finite-dimensional manifold of dimension $3g + 1$ in the space of functions of x for each integer $g \geqq 1$. The collection of "higher KdV" on this manifold in one realization is a commutative pencil of Hamiltonian systems in the space $\mathbb{C}^{2g}$ depending rationally on $g + 1$ parameters and having polynomial Hamiltonians. Another realization of this same pencil of phase spaces identifies it with the affine part of the $(3g + 1)$-dimensional space of a fibering of hyperelliptic Jacobians; the base of this fibering is the moduli space of hyperelliptic curves (more precisely, the symmetrized set of roots $\lambda_0, \lambda_1, \dots, \lambda_{2g}$ of the polynomial $R_{2g+1}(\lambda) = \prod_j(\lambda - \lambda_j)$ for the realization of the curve Γ in the form $y^2 = R_{2g+1}(\lambda)$; the fiber is the Jacobi variety $J(\Gamma)$ of the curve Γ, i.e., an Abelian torus of complex dimension g). The "higher KdV" in the second realization are rectilinear windings of the tori $J(\Gamma)$. As I. R. Shafarevich pointed out after a talk by S. P. Novikov in 1974 before the algebraic geometry seminar at Moscow State University, the fact of the unirationality of the space of moduli of hyperelliptic Jacobians $J(\Gamma)$ which follows from a comparison of these relations was previously not known in algebraic geometry (see the theorem in [12]). Of course, this fact could have been derived also from other systems of classical mechanics and geometry know already in the nineteenth century to be integrable in theta functions (for example, the case of Kovalevskaya for genus $g = 2$, as indicated in [7], or the geodesic flow on ellipsoids of all dimension in elliptic coordinates discovered by Jacobi for the integration of geodesics [13]). Earlier, however, before the appearance of the method of finite-zone integration in KdV theory, algebraic geometry did not exploit the connection with the theory of classical integrable Hamiltonian systems (although classical mechanicists, on the other hand, who discovered familiar integrable cases, used theta functions, especially of genus $g = 2$). Further more profound applications of the method of finite-zone integration of nonlinear systems in the theory of Abelian varieties (in particular, in the well-known Schottky problem) can be found in the survey [9]. A number of new applications of the theory of vector bundles over algebraic curves Γ and projective space $\mathbb{C}P^3$ to the integration of equations of mathematical physics can be found in the surveys [10] and [11].

This paper is devoted to further development of the theory of Hamiltonian systems completely integrable in the Liouville sense on the phase space already discussed above of finite-zone potentials (solutions of KdV) or of hyperelliptic Jacobians $J(\Gamma)$; this theory was begun in [14]. The rich collection of classical and new integrable systems, connected, as analysis shows, with precisely this phase space, leads to a multitude of distinct Poisson brackets on it which possess an interesting algebro-geometric structure that is, in a particular sense, common to all of them. It is interesting that many Poisson brackets lead to pencils of complex tori that are not even Abelian varieties (see the and of §1).

1 Phase spaces (complex). Analytic Poisson brackets. Complex tori

We prescribe some algebraic family (variety) M^n whose points are some collection of parameters determining algebraic curves which are not necessarily hyperelliptic, $\Gamma \in M^n$. The same curve up to equivalence may be encountered several times; n is the complex dimension. For any integer $k \geqslant 1$ there is the natural fibering

$$N^{n+k} \xrightarrow{S^k\Gamma} M^n, \tag{1}$$

where the fiber over a point $\Gamma \in M^n$ is the symmetric degree of the curve Γ: $(P_1, \dots, P_k) \sim (P_{i_1}, \dots, P_{i_k})$. For the special case $k = g$ (the genus of the curve) the fiber is birationally equivalent to the Jacobi variety $S^k\Gamma \cong J(\Gamma)$.

There are two most important examples in the theory of integrable systems.

Example 1. $\Gamma \in M^{2g+1}$ is given in the form

$$y^2 = \prod_{j=0}^{2g} (\lambda - \lambda_j) = R(\lambda). \tag{2}$$

Here Γ is determined by the symmetrized collection $(\lambda_0, \dots, \lambda_{2g})$. In some problems the covering $\hat{M}_g \to M^{2g+1}$ determined by the unsymmetrized collection $(\lambda_0, \dots, \lambda_{2g}) \in \hat{M}_g$ arises.

Example 2. $\Gamma \in M^{2g+2}$ is given in the form

$$y^2 = \prod_{j=0}^{2g+1} (\lambda - \lambda_j) = R(\lambda), \tag{3}$$

where the polynomial $R(\lambda)$ has degree $2g + 2$,

$$(\lambda_0, \dots, \lambda_{2g+1}) \sim (\lambda_{i_0}, \dots, \lambda_{i_{2g+1}}) \in M^{2g+2}.$$

For convenience we shall henceforth always assume that the family of algebraic curves M^n is given to us in the form of m-sheeted coverings of the λ-plane, $\Gamma \xrightarrow{\lambda} \mathbb{C}^1$, so that a "general" point P has the function $\lambda(P)$ as local coordinate. A symmetrized collection $(P_1, \dots, P_k) \in S^k\,\Gamma$ has a standard collection of local coordinates at a "general" point $\lambda(P_1) = \gamma_1, \dots, \lambda(P_k) = \gamma_k$, where $P_j = (\gamma_j, \varepsilon_j)$ and ε_j is the index of the sheet which assumes m values. In the most important hyperelliptic case of Examples 1 and 2 we have $m = 2$ and $\varepsilon_j = \pm\sqrt{1}$. Sometimes we shall somewhat carelessly denote the point P_j itself by γ_j if in the given context this causes no confusion. Thus, points of the space N^{n+k} have the form

$$N^{n+k} = \{(\Gamma; (\gamma_1, \varepsilon_1), \dots, (\gamma_k, \varepsilon_k)), (\gamma_i, \varepsilon_i) \in \Gamma, \, \Gamma \in M^n\}. \tag{4}$$

The analytic Poisson brackets of importance for us are determined as follows:

a) There is given a subsheaf of rings A in the sheaf of germs of meromorphic functions on M^n depending only on a point of the base $\Gamma \in M^n$. Subrings of the form A_U for open domains U play the role of the "annihilator" of the Poisson bracket which is actually concentrated on subvarieties $N_A \subset N^{n+k}$ where $f = \text{const}$ for all $f \in A\colon N_A \to M_A \subset M^n$.

b) There is given a meromorphic differential 1-form $Q(\Gamma)$ on the Riemann surface Γ or its covering $\hat{\Gamma} \to \Gamma$ depending on $\hat{\Gamma}$ as a parameter. In local notation we have:
$$Q(\Gamma) = Q(\Gamma, \lambda)\, d\lambda.$$
It is required that the derivatives of $Q(\Gamma)$ along all directions of the base tangent to the manifolds M_A be globally defined meromorphic differential forms on the algebraic curve Γ itself (rather than on the covering $\hat{\Gamma}$).

c) In all the most important examples it has been found that either the form Q is meromophic on Γ from the very beginning or it is meromorphic on the regular covering $\hat{\Gamma}$ with Abelian momodromy group $\hat{\Gamma} \to \Gamma$, where the image $\pi_1(\hat{\Gamma}) \to \pi_1(\Gamma) \to H_1(\Gamma, \mathbb{Z})$ is generated by a collection of cycles $a_1, \dots, a_k$ with zero pairwise intersection indices $a_j \circ a_s = 0$.

Remark. We have $k = g$ (the genus) for the KdV and sine-Gordon equations and the Toda lattice, and $k = g + 1$ for the NS equation (the nonlinear Schrödinger equation) (see [18]). For general matrix systems of order $l + 1$ we have $k = g + l$.

Definition. If the closed 2-form

$$\Omega_Q = \sum_{i=1}^{k} dQ(\Gamma, \gamma_i) \wedge d_{\gamma_i} \tag{5}$$

is nondegenerate at a "general" point, i.e. in a Zariski-open region of the manifold N_A where the pair (A, Q) possesses properties a), b), and c), then we say that an *analytic Poisson bracket with annihilator* A is given on an open region of the manifold N^{n+k}. In this case the dimension of the manifold N_A must be equal to $2k$, $\dim M_A = k \leqslant n$.

By definition, the Poisson bracket (5) is given by the properties

$$\begin{aligned}
&1) \quad \{\gamma_i, \gamma_j\} = 0 \quad (i, j = 1, \dots, k),\\
&2) \quad \{Q(\gamma_i), Q(\gamma_j)\} = 0 \quad (i, j = 1, \dots, k),\\
&3) \quad \{Q(\gamma_j), \gamma_i\} = \delta_{ji} \quad (i, j = 1, \dots, k),\\
&4) \quad \{f, \gamma_j\} = \{f, Q(\gamma_k)\} = 0, \quad f \in A,
\end{aligned} \tag{6}$$

where the annihilator A contains only functions of the form $f(\Gamma)$, $\Gamma \in M^n$. The following straigthforward result holds.

Proposition 1. *Any analytic Poisson bracket possesses the following property: any two functions g, h depending only on a point of the base $\Gamma \in M^n$ (i.e., only on the Riemann surface) are in involution:*

$$\{g(\Gamma), h(\Gamma)\} = 0. \tag{7}$$

Proof. Let $\lambda_1, \dots, \lambda_k$ be coordinates on the manifold M_A; then it follows easily that the form Ω_Q contains in its expansion only terms of the form $d\lambda_i \wedge d\gamma_j$, i.e., the matrix of the form has block form:

$$\omega = \begin{Vmatrix} 0 & A \\ -A^t & 0 \end{Vmatrix}, \quad A_{ij} = \frac{\partial Q}{\partial \lambda_i}(\gamma_j).$$

The Poisson bracket is determined by the inverse matrix which has the same block structure, whence the involutivity of λ_i and (7) follow. The proposition is proved.

Remark 1. Below it will be very important (see the end of this section) that the Poisson bracket is given only on an open region in the manifold N^{n+k}. It will be clear that the "true" phase space of the "action-angle" variables, which we shall call the complex phase space of Liouville variables, if it has a natural compactification may lead to fiberings different from (1) where the fiber is bimeromorphically (birationally) nonisomorphic to the variety $S^k\,\Gamma$.

Remark 2. The rest of the theory is not altered in an essential way if γ_j is replaced by $\tilde{\gamma}_j = \gamma_j + f_j(\Gamma)$ in the definition of the bracket. Although this is apparently precisely the case in some concrete systems, we shall not investigate this case.

We further consider pairs (A, Q) such that all derivatives of Q along tangential directions $\tau_1, \dots, \tau_k$ to the manifolds M_A at a "general" point $\Gamma \in M^A$ constitute a collection of meromorphic 1-forms $\nabla_{\tau_i} Q$ on Γ with the following properties.

1. The forms $\nabla_{\tau_i} Q$ can be presented in the form

$$\nabla_{\tau_i} Q = \omega_i + \tilde{\omega}_i + \sum_{t=1}^{l} \tilde{\tilde{\omega}}_{it},$$

where the forms ω_i are holomorphic on Γ, and the forms $\tilde{\omega}_i$ are meromorphic with zero residues at all poles; the forms $\tilde{\tilde{\omega}}_{it}$ have a pair of poles of first order (P_t', P_t'') at which the residues differ only in sign.

2. If $k \geqslant g$, then it is required that the forms ω_i generate the one dimensional cohomology group $H^{1,0}(\Gamma) = \mathbb{C}^g$ and $w_i = 0$, $i > g$.

For the distinguished canonical basis of cycles

$$(a_1, \dots, a_g, b_1, \dots, b_g), \quad a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}$$

we require the normalization

$$\oint_{a_j} \omega_i = \delta_{ij} \quad (i \leqslant g), \qquad \oint_{a_j} \tilde{\omega}_q = \oint_{a_j} \tilde{\tilde{\omega}}_{qt} = 0.$$

The case where the pairs of poles of first order (P_t', P_t'') do not intersect pairwise or with the other poles of higher orders we shall call the case of "general position".

The residues of the form Q whose gradients on M_A are the residues of the forms $\tilde{\tilde{\omega}}_{it}$ at the poles of first order P_t' depend only on Γ. We denote them by $\varphi_t(\Gamma)$. The transcendence degree of the collection of functions $\varphi_t(\Gamma)$ on a "general" manifold M_A (i.e., the functional dimension modulo the annihilator A) we denote by κ.

According to Proposition 1, all Hamiltonians of the form $H(\Gamma)$ commute with one another. The standard Liouville procedure for introducing formal "angle" variables (complex for the time being) which have linear time dependence leads to the following assertion for Hamiltonians of the form $H(\Gamma)$.

Proposition 2. *If a Poisson bracket of general position is given by a pair* (A, Q) *satisfying conditions* 1 *and* 2 *(see above), then the following complex variables at a point* $(\Gamma = \Gamma_0, (\gamma_1, \varepsilon_1), \dots, (\gamma_k, \varepsilon_k))$ *of general position are independent and have linear dynamics in time:*

$$\Psi_j = \sum_{i=1}^{k} \int_{P_0}^{(\gamma_i, \varepsilon_i)} \nabla_{\tau_j} Q. \tag{8}$$

The variables Ψ_j are defined up to the lattice in the space $\mathbb{C}^k$ generated by the $2g + l$ vectors $e_1, \dots, e_g, e'_1, \dots, e'_g, \eta_i, \dots, \eta_k$ of the form

$$e^i_q = \oint_{a_q} \nabla_{\tau_i} Q = \delta^i_q, \ \ e'^i_q = \oint_{b_q} \nabla_{\tau_i} Q = b^i_q, \ \ \eta^i_s = \oint_{\delta_s} \nabla_{\tau_i} Q. \tag{9}$$

The cycles δ_s are small circular contours enclosing the poles of first order $P'_s, s = 1, \dots, l$. Among these vectors at a point of general position $(\Gamma_0, \gamma_1, \dots, \gamma_k)$ there are precisely $2g+\kappa$ which are linearly independent over $\mathbb{Z}$. The vectors τ_i are tangent to the surface of the constant annihilator $M_A \subset M^n$ in the base of the fibering (1) and generate the tangent space to M_A. The following inequality holds:

$$2g + \kappa \leqslant 2k, \quad \kappa \leqslant l.$$

The proof of the proposition follows immediately from the classical Liouville procedure (see, for example, [15], §50).

Remark 1. It makes sence to discuss the special choice of the vectors τ_i corresponding to differentiation of Q along the so-called action variables canonically conjugate to the "angles" on the tori varying from 0 to 2π only for the real theory in the case where the level surfaces are compact, i.e., represent tori T^k (see §2).

Remark 2. The compact complex torus T^{2k} can arise for the variables Ψ in the case where $2g+\kappa=2k$. The most important case is $\kappa=0$ ($k=g$). An Abelian torus is by no means always obtained (on the contrary, very rarely). We point out that the present situation can arise for systems whose original phase space was an algebraic variety, while the integrals were polynomials or rational functions. Therefore, the level surfaces in the complex sense are algebraic varieties. The replacement (8) may possess ambiguity and singularities. Therefore, the complex space of Liouville variables, generally speaking, is bimeromorphically nonequivalent to the original phase space. Their real parts may nonetheless be analytically diffeomorphic. If the torus T^{2k} is non-Abelian, then there is no theory of theta functions connected with it, and there are no good formulas for solutions in terms of meromorphic special functions on the torus where the argument varies linearly with time. It is interesting that only tori of the form $J(\Gamma)$ arise from the method of the inverse problem.

We point out that in correspondence with the preceding results we construct a theory of integration only for Hamiltonians of the form $H(\Gamma)$. We shall first consider a system given only on an open region of points of general position (the Poisson bracket may have singularities on a special submanifold). Thus, for example, on the fibers $S^k\,\Gamma$ there are distinguished only subregions $V\subset S^k$ on which the replacement (8) is one-to-one.

Conclusion. The family of algebraic curves M^n with 1-form $Q(\Gamma)$ and annihilator A makes it possible, using (8), to construct a completely different complex fibering distinct from (1), where the fiber is the factor $J_Q(\Gamma)=\mathbb{C}^k/D$ and D is the lattice of (9).

It is possible that the fibering (10) can be constructed nonsingularly only over "general" level surfaces of the annihilator A:

$$N_{(Q,A)}\xrightarrow{J_Q(\Gamma)} M_A. \tag{10}$$

We shall consider the important case $\kappa=0$ or $k=g$ of general position when the variables (8) are actually defined on the compact complex torus $J_Q(\Gamma)$. The following result is a summary of the foregoing considerations.

Theorem 1. *The usual Abel transformation $S^g\,\Gamma\to J(\Gamma)$ linearizes the dynamics of all Hamiltonians of the form $H(\Gamma)$ for Poisson brackets defined by pairs (A,Q) of general position possessing properties* 1 *and* 2 *(see above) if and* *only if the derivatives $\nabla_{\tau_i} Q$ in all directions tangent to the "general" manifold of the constant annihilator M_A provide a basis of the holomorphic 1-forms of the Riemann surface Γ.*

The proof follows immediately from the Propositions 1 and 2 by comparing (8) with the Abel transformation.

2 The real structure. "Action-angle" variables

We shall assume that the family of algebraic curves M^n is given as a manifold with an antiholomorphic involution $\sigma\colon M^n \to M^n$, $\sigma^2 = 1$, extended to an anti-involution of the fibering with fiber Γ: $N^{n+1} \to M^n$. The fixed manifold M_A^σ thus consists only of curves Γ with an antiholomorphic involution $\sigma_\Gamma\colon \Gamma \to \Gamma$, $\sigma_\Gamma^2 = 1$. The form $Q = Q(\Gamma, \lambda)\, d\lambda$ and the annihilator A here must also be coordinated in the natural way with the involutions σ and σ_Γ of the fibering $M^{n+1} \to M^n$ with fiber Γ:

$$\begin{aligned} &\text{a)}\quad \sigma_\Gamma^* Q = \overline{Q} \text{ on } N^{n+1}, \\ &\text{b)}\quad \sigma^* A = \overline{A} \text{ on } M^n. \end{aligned} \tag{11}$$

The simplest class of real structures, which we shall call "elementary" for the Hamiltonian systems of interest to us, can be described as follows. We consider M- or $(M-1)$-curves Γ where the number of fixed ovals of the anti-involution σ_τ is equal either to $g+1$ (M-curves) or g ($(M-1)$-curves). For M-curves $k = g$ or $k = g+1$ is possible. For $(M-1)$-curves only $k = g$ is possible (we assume that $k \geqq g$; see §1; the case $k < g$ is also interesting, but we do not consider it for the time being). The Hamiltonian $H(\Gamma)$ must be real, i.e., $H(\sigma\Gamma) = \overline{H}(\Gamma)$.

Lemma. *If the Poisson bracket (A, Q) is coordinated with the elementary real structure, i.e., it possesses property (11), then the real collections $\gamma_1, \dots, \gamma_k$ are invariant under the dynamics generated by $H(\Gamma)$, where the γ_i all lie on pairwise distinct, fixed ovals of the anti-involution:*

$$\gamma_j \in a_j,\ a_i \cap a_j = \Phi \quad (i = i_1, \dots, i_g \text{ or } i = 1, 2, \dots, g+1). \tag{12}$$

For M-curves and $k = g$ the possible collections $\gamma_i \in a_i$ form the $g+1$ connected component which is isomorphic to real torus T^g. For M-curves and $k = g+1$ or $(M-1)$-curves and $k = g$ there is only one connected component—the real torus T^{g+1} or T^g.

The lemma is proved by an obvious verification. We have the real structure with the tori (12) in the case $k = g$ for KdV and the case $k = g+1$ for NS_+ (the nonlinear Schrödinger equation with attraction).

Elementary real structures of more general type, by definition, represent a transformation $S^k\,\sigma_\Gamma\colon\ S^k\,\Gamma \to S^k\,\Gamma$ generated by the original anti-involution σ_Γ on the curve; the "real" subset (in the complex case the torus T^k) may be distinguished among the fixed points $S^k\,\sigma_\Gamma(x) = x$ in some manner distinct from (12).

Nonelementary real structures for $k = g$ arise in the important example of the very familiar sine-Gordon equation and for $k = g+1$ for the NS_- equation (the nonlinear Schrödinger equation with repulsion). For all systems admitting general Lax matrix representations of order $l+1$ we have $k = g + l$ (see [7], Chapter III, and [16]); if the realness conditions are such that the so-called monodromy matrix after a period (in the periodic problem) for the corresponding linear Lax L-operator belongs to a compact Lie group (similar to the group SU(2) for the sine-Gordon and NS equations), then the real structure is nonelementary.

Suppose the Poisson bracket is given by a pair (A, Q) and there is given an anti-involution

$$\sigma\colon M^n \to M^n, \quad \sigma_\Gamma\colon N^{n+1} \to N^{n+1},$$

while the Hamiltonian is real: $H(\sigma x) = \overline{H}(x)$, $x \in M^n$, X a Riemann surface. According to §1, we have the fibering (10)

$$N_{(A,Q)} \xrightarrow{J_Q(\Gamma)} M_A$$

over a "general" level surface of the annihilator A.

Definition. A *nonelementary real structure* is an arbitrary anti-involution τ: $N_{(A,Q)} \to N_{(A,Q)}$ commuting with the fibering (10), i.e., with the usual involution on the family of Riemann surfaces: σ, σ_Γ. The realness conditions for A and Q remain the same (11). On fibers $J_Q(\Gamma)$ the transformation τ must be a superposition of translation and an isomorphism of the real commutative groups $J_Q(\Gamma)$.

Real submanifolds in phase space are distinguished proceeding from conditions (13) or (14):

$$\sigma(\Gamma) = \Gamma, \quad \tau(\eta) = \eta; \tag{13}$$

$$\sigma(\Gamma) = \Gamma, \quad \tau(\eta) = -\eta + \eta_0^{(\alpha)}. \tag{14}$$

The case (14), which is encountered for the sine-Gordon and NS$_-$ equations, is especially interesting. For a given curve the constant vector $\eta_0^{(\alpha)}$ may assume a finite number of values $\alpha = 1, \dots, m$. Calculation of these quantities for the sine-Gordon equation was first carried out in [17]; the problem of explicitly distinguishing real solutions in the language of theta-function formulas on $J(\Gamma)$ was thus solved.

However, the involution τ is badly described in the variables $\gamma_1, \dots, \gamma_k$; apparently, in the cases of NS$_-$ and sine-Gordon it is not possible to explicitly and effectively distinguish the collections $\gamma_1, \dots, \gamma_k$ for real solutions.

In the joint work of Dubrovin and Novikov the homological image in

$$H_1(\Gamma \setminus (P_1 \cup P_2), \mathbb{Z})$$

was determined and computed corresponding to "real" collections $\gamma_1, \dots, \gamma_k$ for these two equations, where P_1 and P_2 are the infinitely distant points for NS$_-$ (the curve Γ has the form $y^2 = R_{2g+2}(\lambda)$); for the sine-Gordon equation Γ is given in the form $y^2 = \prod_{j=0}^{2g}(\lambda - \lambda_j)$, $\lambda_0 = 0$; $P_1 = 0$, $P_2 = \infty$. The homological image is a collection of cycles $a_1, \dots, a_k$, where $a_j \circ a_i = 0$, "corresponding" to real tori T^k.

The next assertion, which shows the importance of this collection of elements in $H_1(\Gamma, \mathbb{Z})$, follows from the proof of the Liouville theorem (see [14]).

Proposition 3. *For analytic Poisson brackets satisfying elementary and nonelementary realness conditions the action variables J_j canonically conjugate to the angular coordinates on the tori T^k varying from 0 to 2π are given by*

$$J_j = \frac{1}{2\pi} \oint_{a_j} Q(\Gamma, \lambda)\, d\lambda. \tag{15}$$

Proof. Formula (15) represents, by definition, the quantity $J_j = \oint_{\tilde{a}_j} p\, dq$, where the $\tilde{a}_j$ are the basis 1-cycles on the tori T^k. For the class of Poisson brackets we are studying the variables J_j acquire an important interpretation not only as integrals over the elements $\tilde{a}_j$ from the group $H_1(T^k, \mathbb{Z})$ according to Liouville, but also as integrals over the elements a_j of the group $H_1(\Gamma \setminus P, \mathbb{Z})$ where P is the set of poles of the form Q or the forms $\nabla_{\tau_j} Q$ for its derivatives along M_A; this proves the proposition.

Lemma. *Compact real tori T^k are possible only when*

$$k \leqslant 2g + \kappa, \tag{16}$$

where κ is the number of essential residues of the poles of the form Q modulo the annihilator A.

Proof. As indicated in §1, the number $2g+\kappa$ coincides for "general" brackets given by the pair (A, Q) with the rank of the lattice D in the space $\mathbb{C}^k$ of the variables (8). Since the real anti-involution τ is an automorphism of the real group $J_Q(\Gamma)$ and the dimension of the torus over $\mathbb{R}$ is equal to k, inequality (16) means simply that the dimension of the torus cannot be greater than the rank of the lattice. The lemma is proved.

We shall call $\delta = 2k - 2g - \kappa$ (the corank of the lattice) the number of variables of "phase type", if the real parts are compact tori.

The varieties $J_Q(\Gamma)$ are fibered in the form

$$J_Q(\Gamma) \xrightarrow{\mathbb{C}^s/\mathbb{Z}^\delta} \tilde{J}_Q^{k-s}(\Gamma), \quad s \geqslant \delta,$$

where $\tilde{J}_Q(\Gamma)$ is a compact complex torus. Under the conditions of the lemma we have $s = \delta$.

3 Poisson brackets coordinated with KdV theory. The most important examples

Definition. A Poisson bracket on the phase space (4), where Γ are hyperelliptic curves of the form $y^2 = \prod_{j=0}^{2g}(\lambda - \lambda_j) = R(\lambda)$ is said to be *coordinated with KdV theory* if all the higher KdV are Hamiltonian in this bracket. Here $k = g$.

In [14] a number of examples were given of brackets coordinated with KdV theory (see below). For such brackets $J_Q(\Gamma) = J(\Gamma)$ always, i.e., the usual Abel transformation defined by means of the collection of basis holomorphic differential forms linearizes the dynamics of virtue of Hamiltonians of the form $H(\Gamma)$.

Theorem 2. *a) An analytic Poisson bracket is coordinated with KdV theory if and only if all the forms $\nabla_\tau Q$—the derivatives in directions τ tangent to a level surface of functions from the annihilator A—are holomorphic and generate the group $H^{1,0}(\Gamma)$.*

b) If the bracket is coordinated with KdV, then the coefficients of the expansion

$$Q(\Gamma,\lambda) = \sum_{k=-N}^{\infty} \left(\frac{z}{2}\right)^k q_k(\Gamma), \quad z = \lambda^{-1/2},$$

are such that $q_{2l+3}(\Gamma) = h_l(\Gamma)$ are Hamiltonians of the higher KdV with index $l \geqslant 0$, while the remaining coefficients q_k belong to the annihilator A.

Remark 1. Part b) was established by the authors in [14].

Proof. The usual Abel transformation linearizes the dynamics of all higher KdV; the necessity of the condition is therefore obvious. Suppose now that there is given a bracket (A, Q) such that $\nabla_\tau Q = \omega_\tau$ are holomorphic differentials generating the entire space $H^{1,0}(\Gamma)$. Since $z = \lambda^{-1/2}$, from the holomorphicity of the derivatives $\nabla_\tau Q$ along tangent directions to M_A it follows that the expansion for large λ has the form

$$Q(\Gamma,\lambda) = \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{2k+3} H_k(\Gamma) + \mathrm{Ann},$$

where Ann is a series with coefficients in the annihilator. We shall prove that $H_k(\Gamma)$ for $k = 0, 1, \dots, g-1$ are the Hamiltonians of precisely the first g higher KdV. We consider the standard set of holomorphic forms $\omega_1, \dots, \omega_g$ normalized by the conditions

$$\oint_{a_j} \omega_s = 2\pi i \delta_{js},$$

where a_j, b_j is the basis of cycles on Γ such that

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}.$$

We define two collections of variables φ_i $(i = 1, \dots, g)$ and ψ_j $(j = 0, \dots, g-1)$:

$$\varphi_s = \sum_{j=1}^{g} \int_{P_0}^{(\gamma_j,\varepsilon_j)} \omega_s, \ \psi_s = \sum_{j=1}^{g} \int_{P_0}^{(\gamma_j,\varepsilon_j)} \alpha_s, \quad \alpha_s = \left.\frac{\partial Q}{\partial H_s}\right|_A.$$

The forms α_s are holomorphic and have at $\lambda = \infty$ the expansion ($z = \lambda^{-1/2}$)

$$\alpha_s = \left[2\left(\frac{z}{2}\right)^{2s} + O(z^{2g})\right] dz \quad (s = 0, \dots, g^{-1}).$$

The transition matrix $\omega_s = A_{sj}\alpha_j$ is formed from the coefficients of the expansion of the forms ω_s, namely

$$\omega_s = \sum_{j=0}^{\infty} c_{sj} z^{2j}\, dz,$$

at the point $\lambda = \infty$, so that

$$A_{sj} = 2^{1-2j} c_{sj}.$$

For Abelian differentials of second kind Ω_j with zero a-periods and having a single pole at $\lambda = \infty$ with asymptotics $\Omega_j = -d(z^{2j-1}) + O(1)$ the following equality holds (see [19], §10–3):

$$\oint_{b_s} \Omega_j = c_{sj} = 2^{2j-1} A_{sj}.$$

From this it follows that the columns of the matrix A_{ij} coincide with the vectors which on the Jacobians $J(\Gamma)$ give the direction of the flows of the higher KdV with the corresponding indices $j = 0, 1, \dots, g-1$. On the other hand, these columns, by definition, are the direction vectors of Hamiltonian flows with Hamiltonians $H_0, \dots, H_{g-1}$. Thus, for the first g flows coordination has been proved. For the remaining flows it is automatically satisfied, since they can be expressed in terms of the first g on our phase space.

The proof of the theorem is complete.

We now list the most important examples.

Example 1. *The standard bracket* [20], [21]. *From the results of* [22] *we derive*

$$Q = 2ip(\lambda)\, d\lambda, \quad A = \left\{T_1, \dots, T_g, \bar{u} = \lim_{T\to\infty} \frac{1}{T}\int_0^T u\, dx\right\},$$

where the T_i are the periods of the quasiperiodic potential $u(x)$, $p(\lambda)$ is the quasimomentum, and $dp(\lambda)$ is a differential of second kind with a single pole at $\lambda = \infty$,

$$dp(\lambda) = d(z^{-1}) + O(1), \quad z = \lambda^{-1/2},$$

$$\oint_{a_j} dp(\lambda) = 0, \quad j = 1, \dots g.$$

Example 2. *The pencil of local brackets for all higher KdV* (see [23]), where the Hamiltonian systems have the form

$$\dot{u} = \left(al + b\frac{\partial}{\partial x}\right)\frac{\delta H}{\delta u}, \quad l = \frac{1}{2}\frac{\partial^3}{\partial x^3} + u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u.$$

Here we have

$$Q = 2ip(\lambda)(a\lambda + b)^{-1}\, d\lambda\,.$$

If I_k are the Kruskal integrals, then

$$l\frac{\delta I_k}{\delta u} = \frac{\partial}{\partial x}\frac{\delta I_{k+1}}{\delta u}\,.$$

There is a relation among the integrals I_k — the stationary higher KdV equation which the potential $u(x)$ satisfies

$$\sum_{k=0}^{g+1} c_k \frac{\delta I_{g-k}}{\delta u} = 0, \quad c_0 = 1.$$

The quantity $J = \sum_0^g c_k I_{g-k-1}$ belongs to the annihilator of the bracket for $b = 0$, since

$$l\frac{\delta J}{\delta u} = \frac{\partial}{\partial x}\frac{\delta}{\delta u}\left(\sum_{k=0}^{g} c_k I_{g-k}\right) = \frac{\partial}{\partial x}\,\mathrm{const} = 0.$$

We thus obtain (for $b = 0$)

$$A = (T_1, \dots, T_g, J).$$

Example 3. *The Hamiltonian formalism of the stationary problem for higher KdV generated by the variational calculus.* Coordination with KdV theory is proved in [24]; the Hamiltonians of the higher KdV are computed in [25]. Here

$$Q = \sqrt{-R(\lambda)\, d\lambda}, \quad R(\lambda) = \prod_{i=0}^{2g}(\lambda - \lambda_i).$$

This result can be derived from [26] and [27]. As is known, the annihilator is generated by the first $g + 1$ symmetric polynomials in λ_i'.

Example 4. *The Hamiltonian structure generated by the so-called latent isomorphism of Moser-Trubowitz* (see [28]) and [13]) between the KdV dynamics on the space of finite-zone potentials and the classical Neumann systems describing the dynamics on the sphere S^g in $\mathbb{R}^{2g+1}$,

$$S^g = \left\{x\colon |x^2| = 1, x \in \mathbb{R}^{g+1}\right\},$$

under the action of the quadratic potential

$$U(x) = \frac{1}{2}\sum_{i=0}^{g} a_i x_i^2.$$

The coefficients a_i, representing the set of end points of the zones λ_i, here play the role of the annihilator. Using [28] and [13], we obtain (see also [27])

$$Q = \sqrt{-R(\lambda)}\prod_{j=0}^{g}(\lambda - \lambda_{2j})^{-1}\,d\lambda, \quad A = \{\lambda_0, \lambda_2, \ldots, \lambda_{2g}\}.$$

Example 5. *The integrable case of Goryachev and Chaplygin in the dynamics of a rigid body with a fixed point* [30]. Here

$$Q(\Gamma, \lambda) = \arcsin\frac{1}{\mu}\left(\frac{\lambda^2}{2} - \frac{1}{2}H - \frac{2G}{\lambda}\right),$$

where H is the energy of the gyroscope, G is the Goryachev-Chaplygin integral, and μ is a parameter of the problem. The curve Γ is given by the equation

$$y^2 = 4\mu^2\lambda^2 - (\lambda^3 - H\lambda - 4G)^2.$$

Example 6. In the well-known Kovalevskaya case the action variables were not computed previously. Let I_1, I_2 and I_3 be, respectively, the energy, area, and Kovalevskaya integrals, and let $\gamma_i = s_i$ be the Kovalevskaya variables (see [30]). Calculations lead to the result

$$Q(\Gamma, \lambda) = \frac{1}{2\sqrt{-\lambda}}\ln\left[\sqrt{-\lambda}\left((\lambda - I_1)^2 - I_3^2\right) - \frac{\nu^2}{2\sqrt{-\lambda}}(\lambda - 2I_2^2) + \sqrt{-R(\lambda)}\right],$$
$$R(\lambda) = \left(\lambda\left((\lambda - I_1)^2 + \nu^2 - I_3^2\right) - 2\nu^2 I_2^2\right)\left((\lambda - I_1)^2 - I_3^2\right),$$

while the curve Γ is given by the equation $y^2 = R(\lambda)$. Integrating $Q = Q(\Gamma, \lambda)\,d\lambda$ over the "real" cycles a_j on which the γ_i lie, we obtain the action variables J_j. In correspondence with the result of Kovalevskaya the Abel transformation linearizes the dynamics (we note that the replacement of the time and Hamiltonian structure proposed by G. V. Kolosov (see [29]) leads for this system to non-Abelian tori).

The integrable Steklov case for a solid body in a fluid is also of major interest.

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