COMMUTING OPERATORS OF RANK $l > 1$ WITH PERIODIC COEFFICIENTS

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1. This note is devoted to the following problem. Suppose that an ordinary linear differential operator of the form

$$L = \sum_{i=0}^{n} a_i(x) \frac{d^{n-i}}{dx^{n-i}} \tag{1}$$

is given, where the $a_i(x)$ are smooth periodic (possibly matrix-valued) functions, $a_0 = \text{const.}$, and the eigenvalues of the matrix a_0 are all different. What spectral property is characteristic of the fact that L is "algebraic", i.e., that there exists a nontrivial operator

$$A = \sum_{i=0}^{n} b_j(x) \frac{d^{m-j}}{dx^{m-j}}$$

such that $[A, L] = 0$?

We recall the important concept of the *rank* of a commuting pair A, L (see [4]). According to a classical lemma (Burhnal–Chaundy) the operators A and L are connected by an algebraic relation

$$P(A, L) = \sum d_{ij} A^i L^j = 0, \quad d_{ij} = \text{const.} \tag{2}$$

A common eigenvector, a solution of the equations

$$L\phi = \lambda\phi, \quad A\phi = \mu\phi \tag{3}$$

depends on a point on a Riemann surface Γ:

$$P(\mu, \lambda) = 0, \quad Q = (\mu, \lambda) \in \Gamma, \quad \phi = \phi(x, Q). \tag{4}$$

The number l of linearly independent solutions of (3) for a fixed point $Q \in \Gamma$ is called the *rank of the pair* A, L. We can choose a basis for the solutions of (3) at a point $Q \in \Gamma$ in general position (for the scalar case) as follows:

$$\phi(x, Q) = (\phi_1, \phi_2, \dots, \phi_l), \quad \frac{d\phi_q}{dx^{k-1}}\Big|_{x=x_0} = \delta_{kq}, \quad k, q = 1, 2, \dots, l. \tag{5}$$

Everything is analogous for the matrix case. The analytic properties of the vector $\phi(x, x_0, Q)$ were described in [4] for operators in "general position". We will call ϕ a *one-parameter Baker–Ahiezer vector of rank* l, i.e. it depends on one variable, x. A classification of commuting pairs A, L in general position was obtained in [4] for any rank l. The problem of the effective calculation of the coefficients for $l > 1$ has

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not been solved. It is not clear, for example, for what values of the parameters L is periodic (quasiperiodic).

The method of "deformation of Turin parameters" was developed in [5]–[7], permitting the effective calculation of the coefficients of the operators A and L in some instances The calculation was carried out for the simplest nontrivial case: genus $g = 1$, rank $l = 2$ (see [6]). It was also pointed out in [6] that it is possible to obtain a closed solution for $g = 1$, $l = 3$, but it turns out to be complicated. For $g = 1$, $l = 2$, these results were repeated later in [9]. A classification of commuting pairs with rational coefficients for $g = 1$ and $l = 2$ was obtained in [2] by the use of the methods of [6].

2. P. G. Grinevič and the author have noted that among commuting pairs A, L of genus $g = 1$ and rank $l = 2$, it is possible to distinguish efficiently those for which the operator L of fourth order is a real selfadjoint positive operator with smooth periodic coefficients and that there are many such pairs (see [2], the Appendix). The spectral theory of periodic operators of rank l in the Hilbert space $\mathcal{L}_2(-\infty, \infty)$ is similar to the theory of an operator of order l with spectral parameter on a Riemann surface Γ.

For $l = 1$ a spectral characterization of algebraic operators with periodic coefficients was given in [1] and [8], if L is a second-order operator. The general case of rank $l = 1$ was studied in [3].

We denote the period of L by T, its order by n, and the vector dimension by q. We recall that a *Bloch eigenfunction* (*Floquet function*) is a solution of the equation $L\psi_j = \lambda\psi_j$ such that

$$\psi_j(x + T, \lambda) = \hat{T}\psi_j = \mu_j(\lambda)\psi_j(x, \lambda). \tag{6}$$

For fixed λ in general position we have precisely $N = nq$ eigenvalues μ_j of $\hat{T}$, i.e., $j = 1, \dots, N$. By the same token, the Bloch function $\psi_j(x, \lambda)$ is single-valued on an N-sheeted covering $\hat{\Gamma}$ over the λ-plane

$$p\colon \hat{\Gamma} \to CP^1 \setminus \infty, \quad \psi = \psi(x, Q), \quad Q \in \hat{\Gamma}.$$

The points Q consist of pairs $Q = (\lambda, j)$. We will assume that $\hat{\Gamma}$ is a nonsingular Riemann surface. The ordinary spectrum in $\mathcal{L}_2(-\infty, \infty)$ is found, where $|\mu_j(\lambda)| = 1$ for at least one j.

3. The principal result of this note is the following general proposition.

Theorem. *The operator L in general position with periodic coefficients is algebraic of rank l (i.e., belongs to a commuting pair of rank l) if and only if the N-sheeted Riemann surface Γ of the Bloch eigenfunction ψ over the λ-plane is an l-sheeted covering over the algebraic curve Γ (which is itself Nl^{-1}-sheeted over the Riemann sphere CP^1), $N > l \geq 1$:*

$$\hat{\Gamma} \xrightarrow{p_1} \Gamma \xrightarrow{p_2} CP^1 \setminus \infty, \quad p = p_2 \circ p_1, \quad p(\lambda, j) = \lambda.$$

Example 1. For $l = 1$ this means that ψ is meromorphic on a Riemann surface of finite genus. In this case it has been proved that the Bloch function reduces to the classical Gordon–Clebsch–Baker–Ahiezer "function of exponential type" on the surface $\hat{\Gamma} = \Gamma$ and coincides with the common eigenvector ϕ of the operators L and A, which is the basis of the formal-algebraic (local with respect to x) Burhnal–Chaundy theory of commuting pairs of rank $l = 1$ (see the surveys [1] and [3]).

The function $\psi(x, Q)$ and the coefficients of the operators are expressed in terms of θ-functions, and the operators in general position are periodic (quasiperiodic).

Example 2. *The case of scalar operators of rank $l = 2$.* In this case the monodromy matrix $\hat{T}(\lambda)$ reduces to a 2×2 matrix $\hat{T}(Q)$ on the surface Γ. The spectrum in $\mathcal{L}_2(-\infty, \infty)$ for real selfadjoint case can be described as follows: given the anti-involution $\sigma\colon \Gamma \to \Gamma$, $\sigma^2 = 1$, the stability (and instability) zones are on the fixed ovals $\sigma(Q) = Q$ and are characterized by the condition $\operatorname{Tr} \hat{T} \le 2$. There is one point at infinity on all of Γ, and it is on the "infinite" oval. The entire spectrum is contained in the image of these ovals on the real λ-axis (see the appendix of [2], where the results of the author and Grinevič are described).

4. We now sketch the proof of the theorem.

1) Suppose that L is algebraic of rank l. The operator A, which acts on a basis of solutions of $L\eta_j = \lambda\eta_j$, turns out to be a polynomial matrix $\Lambda(\lambda)$ in λ:

$$\begin{aligned} A\eta_j(x, \lambda) &= \sum_i \alpha_{ij}(\lambda)\eta_i(x, \lambda), \\ \Lambda &= (\alpha_{ij}), \quad i, j = 1, 2, \dots, N. \end{aligned} \tag{7}$$

The identity

$$[\hat{T}, \Lambda] = 0, \tag{8}$$

taken from the theory of equations of Korteweg–de Vries type in [8], can be established just as for $l = 1$.

According to our assumption, the eigenvectors of Λ are singular with multiplicity l. Therefore under a change of basis $\hat{T}(\lambda)$ reduces to a block matrix with the matrices $\hat{T}(Q_s)$, $Q_s = (\lambda, s)$, $s = 1, 2, \dots, Nl^{-1}$ along the diagonal; Q_s is a point of the Riemann surface (4). The eigenvectors $\psi_j(x, \lambda)$ of $\hat{T}(\lambda)$ turn out to be eigenvectors for all of the matrices $\hat{T}(Q_s)$. Therefore the surface $\hat{\Gamma}$ covers Γ.

2) Conversely, suppose that $\hat{\Gamma}$ covers the l-sheeted algebraic curve

$$\Gamma \setminus \infty\colon \hat{\Gamma} \xrightarrow{p_1} \Gamma \setminus \infty \xrightarrow{p_2} CP^1 \setminus \infty, \quad p = p_2 \circ p_1.$$

Let $(\mathcal{P}_1, \dots, \mathcal{P}_l) = p_1^{-1}(Q)$. In the linear space of $\psi(\mathcal{P}_1), \dots, \psi(x, \mathcal{P}_l)$ we choose a new basis $\phi_q(x, x_0, Q)$ such that (scalar operators, for example)

$$\begin{aligned} \phi_q &= \sum a_{qk}(Q, x_0)\psi(x, \mathcal{P}_k), \\ \frac{d\phi_q}{dx^{k-1}}\Big|_{x=x_0} &= \delta_{kq}. \quad k, q = 1, 2, \dots, l. \end{aligned} \tag{9}$$

Lemma. *In the case of general position, the basis ϕ_q has the analytic properties of the one-parameter Baker–Ahiezer function of rank 1 on the algebraic curve Γ described in [4] and [7].*

Following [4], we can derive without difficulty a construction for the operator A which commutes with L.

Remark. The theorem is probably also true for the quasiperiodic case if the Bloch function exists. In addition, some of the periods can become infinite, and the coefficients can have poles. All commuting pairs are like that for $l = 1$. Finally, for $l > 1$, this does not hold. We recall that the Bloch function is determined by the condition that $(\ln \psi)_x$ have the same group of periods as the operator L.

Conjectures. 1) Suppose that the arbitrary functions $(u_0(x), \dots, u_{l-2}(x))$ which enter into the parameters classified in [4] are periodic in x (or even quasiperiodic?), that the curve Γ is nonsingular, and the Turin parameters are in general position. Then the operators L and A have, in general, quasiperiodic coefficients.

2) General periodic operators L (including algebraic operators of arbitrary rank) can be approximated by algebraic operators of rank $l = 1$: $L_i \to L$ (but it is not possible to approximate every commuting pair A, L). The situation is probably similar for two-dimensional Schrödinger operators of the form

$$L = -\left(\frac{\partial}{\partial x} - ieA_1\right)^2 - \left(\frac{\partial}{\partial y} - ieA_2\right)^2 + u(x,y),$$

where the flux through an elementary cell is zero (i.e., the coefficients of L are periodic) (cf. [7], §3).

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